Ensuring Good Behaviour of the Discrete Hilbert Transform

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ABSTRACT: This paper studies the generation of stable transfer functions for which the real or imaginary part takes prescribed values at discrete uniformly spaced points on the unit circle. Formulas bounding the error between a particular interpolating function and any function consistent with the data are presented, which have the desirable property that the error goes to zero exponentially fast with the number of interpolating points. The paper also examines generation of stable minimum phase transfer functions for which the magnitude takes prescribed values at uniformly spaced points on the unit circle, and presents error bounds for this problem. Connection with the discrete Hilbert transform is made. The effect of uncertainty in the original data is also examined.

1. INTRODUCTION

This paper's concern is with the following style of problem. Suppose we have the real part of a (discrete−time) transfer function at n discrete points on the unit circle; moreover, suppose that the function is stable. How can one reconstruct the function, both real and imaginary parts, on the whole of the unit circle? A related problem is how to reconstruct, from values of the transfer function magnitude given at only n discrete points on the unit circle, the complete transfer function − magnitude and phase − on the entire unit circle. In addressing this problem we combine two ideas: interpolation and the Hilbert transform.

Apart from theoretical interest this problem has also practical motivation. Measurement data on a transfer function, either of the real part or of the magnitude, is unlikely to exist for every frequency. Much more probable is that measurements yield the real part or magnitude at only discrete frequencies: one must then recover the complete transfer function at not only the frequencies given but at all points on the unit circle.

Given real part data on the entire unit circle, recovery of the imaginary part (and vice−versa) is a more−or−less standard problem, tackled via the Hilbert transform, see e.g.[1]. Under the presumption that a transfer function is stable (i.e. free of singularities in $|z|>1$) and that the real part is known at all frequencies (i.e. on $|z|=1$), the Hilbert transform constructs the imaginary part to within an additive constant. If the transfer function corresponds to a real impulse response, it must be real at $z=1$, and this will identify the constant. Similarly, if the transfer function is stable and its imaginary part is known at all frequencies, one can construct the real part to within an additive constant.

Recovery from phase or magnitude data given on the entire unit circle is almost as simple. Under the presumption that a transfer function is stable and minimum phase (all singularities of the function and its inverse lie in $|z|<1$), so that its logarithm is stable, the logarithm of the gain and the phase, being the real and imaginary part of the logarithm of the function, are related by the Hilbert transform, and so can be determined from each other, to within an arbitrary constant. Alternatively, given the magnitude squared, spectral factorization yields the whole function.

In considering use of the conventional Hilbert transform, the question of how errors in the data translate into errors in the result is an important one, the solution of which is not as widely known as it should be. Consider for example constructing the imaginary part from the real part of a stable transfer function. Then an arbitrarily small perturbation in the real part can induce an arbitrarily large change in the imaginary part. For suppose $G(e^{j\omega}) = R(e^{j\omega}) + jI(e^{j\omega})$ with G stable. Then we have, see [1, p. 344],

$$I(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(e^{j\theta})\cot\frac{\theta-\omega}{2}\,d\theta \qquad (1.1)$$

(The Cauchy principal value of the integral is to be calculated). Suppose there is a perturbation in $R(e^{j\theta})$ by a constant amount ϵ over $(-\omega_0, \omega_0)$. Then the corresponding perturbation in $I(e^{j\omega})$ is

$$\Delta I(e^{j\omega}) = \frac{1}{2\pi}\,\epsilon\int_{-\omega_0}^{\omega_0}\cot\frac{\theta-\omega}{2}\,d\theta$$

Take $\omega = -\omega_0$; then the integral diverges.

The same sort of discontinuity can arise in going between magnitude and phase of a minimum phase stable transfer function, see e.g. [2].

It is this discontinuity which underpins the comparative difficulty of doing Hilbert transforms with data at only discrete points. Suppose that one is given the values of the real part of a stable function at frequencies $\omega_i = \exp(2\pi i/n)$ $i=0,1,\ldots,n-1$. A naive approach to reconstruction of the function, or an approximation to it, might involve fitting a curve through the interpolation data, and then Hilbert transforming as in (1.1). Because of the lack of continuity of the Hilbert transform, one could not expect, at least *a priori* that the reconstructed object is close to the true object.

The approach we follow exploits the "Hilbert transform for the discrete Fourier transform" [1, p.353]. Suppose we have data $\mathrm{Re}G(e^{j2\pi k/n})$, $k=0,1,\ldots,n-1$, and we wish to find $G(e^{j2\pi k/n})$, $k=0,1,\ldots,n-1$. There is a unique finite impulse response $g(t)$, $t=0,1,\ldots,n-1$ with $g(t)=0$ for $[n/2]<t\leq n-1$ such that its discrete Fourier transform $G(e^{j2\pi k/n})$ has a real part that coincides with the given data. Uniquely determined, therefore, is the transfer function

$$P(z) = \sum_{i=0}^{n-1} g(i)z^{-i}$$

While this guarantees interpolation in that

$$\mathrm{Re}[P(e^{j2\pi k/n})] = \mathrm{Re}[G(e^{j2\pi k/n})] \qquad k = 0,1,\ldots,n-1$$

it is not part of our problem specification that the given data corresponds to an FIR transfer function: in fact the "true" transfer function G(z) may be any member of the infinite class of functions which take the prescribed real values at certain discrete points on the unit circle. We hope that P(z) is "close" to the true function G(z): it is important to quantify the approximation's accuracy. Section 2 of this paper gives the construction P(z) and bounds the error between P(z) and G(z).

The situation is less clear again when gain/phase data is involved. In [1, p.357], the following idea is advanced. Suppose $G(z)$ is the z−transform of a finite length sequence of $g(t)$, and let $\hat{G}(z)=\log G(z)$. Choose n at least equal to twice the length of $g(t)$, and suppose that $\hat{G}(\exp 2\pi jk/n)$ is known for $k=0,1,\ldots,n-1$. Using the Hilbert transform of the discrete Fourier transform, a collection of values of argument is obtained. These are not exactly the values of $\arg G(\exp 2\pi jk/n)$, basically because $\hat{G}(z)$ does not correspond to a finite impulse response. Further, when $G(z)$ does not correspond to a finite length sequence, the validity (even in an approximate sense) is yet more difficult to pin down. Error formulas presented in Section 3 do however serve to display the validity. Also in Section 3 is an alternative approach, where we find an FIR, non−causal system which interpolates $|G(e^{j2\pi k/n})|^2$, $k=0,1,\ldots,n-1$; then spectral factorization yields an FIR approximation to $G(e^{j\omega})$.

Our approach to all these problems rests on two broad ideas. First, we describe an algorithm for interpolating the given data to come up with a stable, or stable and minimum phase transfer function. Then we establish an error bound between the function we construct and the class of functions which could have generated the data. The error becomes vanishingly small as the number of interpolating points becomes larger.

Never is one given perfectly accurate data. In Section 4, we analyse the effect of uncertainty in the original (i.e. discrete frequency) data. (This is quite different to defining the maximum error between the interpolating function we construct, and any function which could have generated error free initial data). Our analysis includes two different models for the data uncertainty, and focusses upon the effect of increasing the number of interpolation points.

Illustrative examples follow in Section 5; the paper provides concluding remarks in Section 6.

2. FUNCTION CONSTRUCTION FROM REAL PART

2.1 *Problem Statement*

For an unknown transfer function $G(z)$, we are given the following information:

(i) $G(z)$ is analytic in $|z|>\rho^{-1}$ for some $\rho>1$

(ii) Values of $\mathrm{Re}[G(\exp j2\pi k/n)]$, $k=0,1,\ldots,n-1$

(iii) $G(z)$ corresponds to a real impulse response, so that $\mathrm{Re}[G(e^{j\omega})]$ and $\mathrm{Im}[G(e^{j\omega})]$ are respectively even and odd in ω.

From this data, we seek to find a

$$P(z) = \sum_{i=0}^{v} b_i z^{-i},$$

with $n=2v$ or $n=2v+1$ such that

$$\mathrm{Re}P(\exp j2\pi k/n) = \mathrm{Re}G(\exp j2\pi k/n) \qquad k=0,1,\ldots,n-1 \tag{2.1}$$

and we seek a value of K such that

$$\|G-P\|_\infty = \max_\omega \|G(e^{j\omega})-P(e^{j\omega})\| \leq K \tag{2.2}$$

with the property that $K\to 0$ as $n\to\infty$ in a predictable fashion.

2.2 *Construction of interpolating function* $P(z)$

We proceed in two steps. We first construct a two−sided (non−causal) FIR $Q(z)$ with values interpolating $\mathrm{Re}G(\exp j2\pi k/n)$. Then from $Q(z)$ we find a (causal) FIR $P(z)$ satisfying (2.1). We treat the cases of n odd and n even separately.

(a) **n odd**, $n = 2v+1$.

By DFT methods, it is straightforward to construct $Q(z)$, of the form

$$Q(z) = b_v z^v + b_{v-1}z^{v-1} + \ldots + b_0 + \ldots + b_{-v+1}z^{-(v-1)} + b_{-v}z^{-v} \tag{2.3}$$

such that

$$Q(\exp j2\pi k/n) = \mathrm{Re}[G(\exp j2\pi k/n)] \qquad k=0,1,\ldots n-1 \tag{2.4}$$

Of course, with the FIR constraint of (2.3), the function $Q(z)$ is unique. Because $\mathrm{Re}G(e^{j\omega})$ is even in ω,

$$Q(\exp -j2\pi k/n) = \mathrm{Re}[G(\exp j2\pi k/n)]$$

so that $Q(z^{-1})$ also interpolates the data. Hence, by uniqueness, $Q(z) = Q(z^{-1})$ and $b_i = b_{-i}$. Also, if $\bar{Q}(z)$ denotes $Q(z)$ with complex conjugations of every coefficient, a similar argument gives $\bar{Q}(z) = Q(z)$. So b_i is real for all i.

From $Q(z)$, we define $P(z)$ as

$$P(z) = b_0 + 2b_{-1}z^{-1} + \ldots + 2b_{-v}z^{-v} \tag{2.5}$$

From (2.3) through (2.5), it is evident that the interpolation requirement (2.1) is satisfied.

Notice that $P(z)$ is the solution of a conventional Hilbert transform problem: given $Q(e^{j\omega}) = \mathrm{Re}P(e^{j\omega})$, find $P(z)$ analytic in $|z|>\rho^{-1}$, for some $\rho>1$.

(b) **n even**, $n = 2v$.

We construct $Q(z)$ which interpolates $\mathrm{Re}[G(\exp j2\pi k/n)]$ and having the form

$$Q(z) = b_{v-1}z^{v-1} + b_{v-2}z^{v-2} + \ldots + b_0 + \ldots + b_{-v+1}z^{-v+1} + b_{-v}z^{-v} \tag{2.6}$$

This determines $Q(z)$ uniquely. Now $Q(\exp -j2\pi k/n) = \mathrm{Re}[G(\exp j2\pi k/n)]$ by evenness of $\mathrm{Re}[G(e^{j\omega})]$, and $b_{-v}z^{-v} = b_{-v}z^{v} = \frac{1}{2}(b_{-v}z^{-v}+b_{-v}z^{v})$ at every interpolating point. Hence the following function also interpolates the data:

$$\hat{Q}(z) = b_{v-1}z^{-(v-1)} + b_{v-2}z^{-(v-2)} + \ldots + b_0 + \ldots + b_{-v+1}z^{v-1} + b_{-v}z^{-v}$$

and by uniqueness, we must have $b_i = b_{-i}$, $i=0,\ldots,v-1$. Also since $Q(z)$ is real at $z = \exp j2\pi k/n$, we know that b_i is real, $i=0,\ldots v$.

From $Q(z)$, we define $P(z)$ as

$$P(z) = b_0 + 2b_{-1}z^{-1} + \ldots + 2b_{-v+1}z^{-v+1} + b_{-v}z^{-v} \tag{2.7}$$

It is easy to verify that (2.1) holds. Comparison of (2.7) with (2.5) shows that the coefficient of z^{-v} differs by a factor of two between n−even and n−odd cases.

Since $\mathrm{Re}P(e^{j\omega}) \neq Q(e^{j\omega})$, it is not true that $P(z)$ is the solution of a Hilbert transform problem with data $Q(e^{j\omega})$, in contrast to the case $n=2v+1$.

2.3 *Error in the Real Part*

In this subsection, we shall examine the error in $|\mathrm{Re}[P(e^{j\omega})] - \mathrm{Re}[G(e^{j\omega})]|$. To derive an error bound, we need some preliminary results. First we recall the following result, proved in [4]:

Lemma 2.1 Let $H(z)$ be a transfer function analytic in $\rho^{-1}<|z|<\rho$ for some $\rho>1$.

$$\text{If} \quad Q_{2v}(z) = \sum_{i=-v}^{v-1} b_i z^i$$

interpolates $H(z)$ in the (2v)th roots of unity, then for any R_1 with $1<R_1<\rho$

$$\|H-Q_{2v}\|_\infty \leq \frac{2R_1^v}{(R_1^{2v}-1)(R_1-1)}\left[\max_{\theta\in(-\pi,\pi]}|H(R_1^{-1}e^{j\theta})| + \max_{\theta\in(-\pi,\pi]}|H(R_1 e^{j\theta})|\right] \tag{2.8a}$$

$$\text{If} \quad Q_{2v+1}(z) = \sum_{i=-v}^{v} b_i z^i$$

interpolates $H(z)$ in the $(2v+1)$−th roots of unity, then for any R_1 with $1<R_1<\rho$,

$$\|H-Q_{2v+1}\|_\infty \leq \frac{2R_1^{v+1}}{(R_1^{2v+1}-1)(R_1-1)}\left[\max_{\theta\in(-\pi,\pi]}|H(R_1^{-1}e^{j\theta})| + \max_{\theta\in(-\pi,\pi]}|H(R_1 e^{j\theta})|\right] \tag{2.8b}$$

Actually, only the result for even n is proved in [4]. However, the result for odd n follows by virtually the same argument.

We shall also need a further result on the behaviour of the last coefficient b_{-v} in a (2v)−th roots of unity interpolation.

Lemma 2.2 Assume the same hypotheses as Lemma 2.1, with $n=2v$. Then

$$|b_{-v}| < \frac{R_1^v}{R_1^{2v}-1}\left[\max_{\theta\in(-\pi,\pi]} |H(R_1^{-1}e^{j\theta})| + \max_{\theta\in(-\pi,\pi]} |H(R_1 e^{j\theta})|\right] \tag{2.9}$$

Proof Omitted due to space limitations.

With these lemmas in hand, we can now bound the error between $\mathrm{Re}[P(e^{j\omega})]$ and $\mathrm{Re}[G(e^{j\omega})]$. This error is of course zero at the interpolating points, but in general only at these interpolating points. The determination of this error is a *first step for obtaining* a bound on $|P(e^{j\omega})-G(e^{j\omega})|$, as opposed to its real part. The error in the real part is bounded in a manner described in the following theorem:

Theorem 2.1 Assume $G(z)$ *satisfies the assumptions* listed in Section 2.1. Let $P(z)$ be defined as described in Section 2.2. If $n = 2v+1$, there holds for any R_1 with $1<R_1<\rho$

$$|\mathrm{Re}G(e^{j\omega})-\mathrm{Re}P(e^{j\omega})| < \frac{2R_1^{v+1}}{(R_1^{2v+1}-1)(R_1-1)}\left[\max_{\theta\in(-\pi,\pi]} |G(R_1 e^{j\theta})| + \max_{\theta\in(-\pi,\pi)} |G(R_1^{-1}e^{j\theta})|\right] \tag{2.10a}$$

while if $n = 2v$, there holds

$$|\mathrm{Re}G(e^{j\omega})-\mathrm{Re}P(e^{j\omega})| < \frac{R_1^v}{(R_1^{2v}-1)}\,\frac{R_1+1}{R_1-1}\left[\max_{\theta\in(-\pi,\pi]} |G(R_1 e^{j\theta})| + \max_{\theta\in(-\pi,\pi)} |G(R_1^{-1}e^{j\theta})|\right] \tag{2.10b}$$

Remark: The error is effectively of the form $KR_1^{-n/2}$.

Proof: *Omitted due to space limitations.*

The fact that we have a bound on $\mathrm{Re}[G(j\omega)-P(j\omega)]$ does not yield immediately a bound on $\mathrm{Im}[G(j\omega)-P(j\omega)]$, in view of the discontinuity argument presented in Section 1. However, by obtaining an additional bound on the derivative of $\mathrm{Re}[G(j\omega)-P(j\omega)]$, we will then be able to obtain the bound on $\mathrm{Im}[G(j\omega)-P(j\omega)]$.

2.4 *Error in the Derivative of the Real Part*

In this subsection, we shall derive an eror bound for $|d/d\omega\{\mathrm{Re}G(e^{j\omega}) - \mathrm{Re}P(e^{j\omega})\}|$. This is probably not of intrinsic interest. However, we use this bound to construct a bound for $|G(e^{j\omega}) - P(e^{j\omega})|$ *in the next subsection.*

The result is as follows.

Theorem 2.2 Assume $G(z)$ satisfies the assumptions listed in Section 2.1. Let $P(z)$ be defined as described in Section 2.2. Then for $n = 2v$, for any R_1 with $1<R_1<\rho$, there holds

$$\left\|\frac{d}{d\omega}(\mathrm{Re}G(e^{j\omega})-\mathrm{Re}P(e^{j\omega}))\right\|_\infty < \frac{R_1^v}{R_1^{2v}-1}\,\frac{2(R_1^2+1)v+R_1+1}{(R_1^2-1)}\left[\max_{\theta\in(-\pi,\pi]} |G(R_1 e^{j\theta})| + \max_{\theta\in(-\pi,\pi)} |G(R_1^{-1}e^{j\theta})|\right] \tag{2.11a}$$

and for $n = 2v+1$, there holds

$$\left\|\frac{d}{d\omega}(\mathrm{Re}G(e^{j\omega})-\mathrm{Re}P(e^{j\omega}))\right\| < \frac{2R_1^{v+1}}{R_1^{2v+1}-1}\,\frac{(R_1+1)(v+1)}{(R_1-1)^2}\left[\max_{\theta\in(-\pi,\pi]} |G(R_1 e^{j\theta})| + \max_{\theta\in(-\pi,\pi)} |G(R_1^{-1}e^{j\theta})|\right] \tag{2.11b}$$

Remark: The error is effectively of the form $K(n/2 + 1)R_1^{-(n/2)}$.

Proof: Omitted due to space limitations.

We remark *that in* [3], *one-sided interpolation* schemes are shown to yield errors with derivatives which behave like vR_1^{-v} for some $R_1>0$. It is therefore not surprising that, since the one-sided interpolation schemes can be extended to two-sided schemes as illustrated in [4], the error formulas for the derivatives should be similarly extendable.

2.5 *Errors in the approximation of the function* $G(e^{j\omega})$

Our goal here is to show that $P(e^{j\omega})$ is a good approximation of $G(e^{j\omega})$, and becomes better as n increases. The information at our disposal is that $\mathrm{Re}P(e^{j\omega})$ and its derivative are good approximations of $\mathrm{Re}G(e^{j\omega})$ and its derivative. It is the closeness of these derivatives as well as the real parts themselves which means that the Hilbert transform yields answers which are close, see e.g. [2].

The key *is the following Lemma.*

Lemma 2.3 Let $M(e^{j\omega}) = M_R(e^{j\omega})+jM_I(e^{j\omega})$, with $M_R(e^{j\omega})$ real and even in ω, $M_I(e^{j\omega})$ real and odd in ω, and $M(e^{j\omega})$ the Fourier transform of a real, causal impulse response, such that $\|M_R(e^{j\omega})\|_\infty$ and $\|d/d\omega\, M_R(e^{j\omega})\|_\infty$ are both finite. Then

$$M_I(e^{j\omega}) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{d}{d\theta}M_R(e^{j\theta})\right\}\ln\left|\sin\frac{\theta-\omega}{2}\right|d\theta \tag{2.14}$$

and

$$\|M_I(e^{j\omega})\|_\infty < 2\left\|\frac{d}{d\omega}M_R(e^{j\omega})\right\|_\infty \tag{2.15}$$

Proof Omitted due to space limitations.

To return to the interpolation problem of interest to us, we simply identify $M(z)$ in the Lemma with $G(z)-P(z)$. Putting together Theorem 2.1, Theorem 2.2 and the Lemma, we have

Main Theorem 2.3 *Suppose* $G(z)$ is analytic in $|z|>\rho^{-1}$ for some $\rho>1$, that values of $\mathrm{Re}[G(\exp\, j2\pi k/n)]$, $k=0,1,\ldots,n-1$ are prescribed, and that $G(z)$ corresponds to a real impulse response. Then we can *find* $P(z)$ a polynomial of degree $v=n/2$ or $v=(n-1)/2$ in z^{-1} such that,

$$\mathrm{Re}P(\exp j2\pi k/n) = \mathrm{Re}G(\exp j2\pi k/n) \quad k=0,1,\ldots,n-1$$

and such that for any R_1, with $1<R_1<\rho$,

$$\|G(e^{j\omega})-P(e^{j\omega})\|_\infty < [K_1(R_1)v+K_2(R_1)]\left[\max_{\theta\in(-\pi,\pi]} |G(R_1 e^{j\theta})| + \max_{\theta\in(-\pi,\pi]} |G(R_1^{-1}e^{j\theta})|\right]$$

with $K_1(R_1)$, $K_2(R_1)$ going to zero as R_1^{-v}.

Remark A very similar theorem naturally holds for the problem of passing from values of $\mathrm{Im}[G(\exp j2\pi k/n)]$, $k=0,1,\ldots,n-1$ plus the value of ReG at one particular point to an estimate of $G(e^{j\omega})$ using a polynomial of degree n.

2.6 *Rapprochement with the Discrete Hilbert Transform*

The discrete Hilbert transform (DHT) [1, p.353], [5] arises in one of two ways. Let $h(i)$, $i=0,\ldots,n/2$ be a finite impulse response of length $n/2+1$, with n assumed even. Let $H(k)$, $k=0,\ldots,n-1$ denote the evaluation of $\Sigma h(i)z^{-i}$ at $z = \exp(j2\pi k/n)$; then the discrete Hilbert transform constructs $\mathrm{Im}H(k)$, $k=0,1,\ldots,n-1$, from $\mathrm{Re}H(k)$, $k=0,1,\ldots,n-1$, or vice versa. *Second,* let $h(i)$ be a complex periodic sequence of period n, equal to zero for $n/2<i<n$. Let $H(k)$ denote the coefficients of a Fourier series for $h(i)$. Then the Hilbert transform links $\mathrm{Re}H(k)$ to $\mathrm{Im}H(k)$.

It is commonly suggested that if one has $\mathrm{Re}G(\exp j2\pi k/n)$ *for a stable* G, then $\mathrm{Im}G(\exp j2\pi k/n)$ should be estimated as if G were associated with a finite impulse response of length $n/2+1$, i.e. via the discrete Hilbert transform. Left unexplained is how the result of *this computation might* relate to the true value of $\mathrm{Im}G(\exp j2\pi k/n)$ and how the z-transform of the finite impulse response might relate to $G(z)$. As it turns out, we now have the wherewithal to clarify these questions.

In [1], the DHT construction is performed for n even. Comparison of Section 2.2 with [1, p.354] shows that the quantity $P(z)$ of (2.7) is precisely the z-domain transfer function computed by the DHT, *corresponding to an* FIR with support over $[0, n/2]$. Hence our main result provides a theoretical underpinning to the use of the DFT when the underlying transfer functions may not come from a finite impulse response: the underpinning rests on the error formulas of the Main Theorem 2.3.

3. INTERPOLATION OF GAIN–PHASE DATA

In this section, we shall concentrate on the problem where the given data is magnitude data. Of course, very similar results apply when the data is phase data.

There are two broad approaches, summed up in Figures 3.1 and 3.2, and we will discuss them in turn.

3.1 *Problem Statement*

For an unknown transfer function $G(z)$, we are given the following information:

(i) $G(z)$ and $G^{-1}(z)$ are analytic in $|z|>\rho^{-1}$ for some $\rho>1$

(ii) Values of $|G(\exp j2\pi k/n)|$ for $k=0,1,\dots,n-1$

(iii) $G(z)$ corresponds to a real impulse response, so that $\ln|G(e^{j\omega})|$ is even and $\arg G(e^{j\omega})$ is odd in ω.

From this data, we seek to find an interpolant of $G(z)$ of the form

$$P(z) = \sum_{z=0}^{v} b_i z^{-i} \text{ or } P(z) = \exp\Big(\sum_{i=0}^{v} b_i z^{-i}\Big) \text{ for}$$

$n=2v$ or $n=2v+1$, such that

$$|P(\exp j2\pi k/n)| = |G(\exp j2\pi k/n)| \quad k=0,1,\dots,n-1 \tag{3.1}$$

and we seek a bound of the form

$$\|1 - G^{-1}P\|_\infty = \max_{\omega} |1-G^{-1}(e^{j\omega})P(e^{j\omega})| \leq K \tag{3.2}$$

such that $K\to 0$ as $n\to\infty$ in a predictable fashion.

3.2 *First Approach: Interpolation of* $|G|^2$

Figure 3.1 outlines this approach to the construction of P. Suppose first that $n=2v+1$. We find

$$Q(z) = \sum_{i=-v}^{v} b_i z^i \tag{3.3}$$

such that $Q(\exp j2\pi k/n) = |G(\exp j2\pi k/n)|^2$. Arguing as in Section 2, it follows that $b_i = b_{-i}$ and is real. We shall argue later that for large enough v, $Q(e^{j\omega}) > 0$ for all ω. It follows that we can factor $Q(z)$ as

$$Q(z) = P(z)P(z^{-1}) \tag{3.4}$$

with

$$P(z) = c_0 + c_{-1}z^{-1} + \dots + c_{-v}z^{-v} \qquad c_i \text{ real} \tag{3.5}$$

and all zeros of $P(z)$ inside $|z|<1$. $P(z)$ is then the solution of the interpolation problem.

In case $n=2v$, define

$$\hat{Q}(z) = \sum_{i=-v}^{v-1} b_i z^i \tag{3.6}$$

such that $\hat{Q}(\exp j2\pi k/n) = |G(\exp j2\pi k/n)|^2$, and then set

$$Q(z) = \tfrac{1}{2}b_{-v}z^{v} + \sum_{i=-(v-1)}^{v-1} b_i z^i + \tfrac{1}{2}b_{-v}z^{-v} \tag{3.7}$$

At the points $\exp j2\pi k/n$, $\hat{Q}$ takes the same values as Q. As before, $b_i = b_{-i}$ and is real. Again, for large enough v, $Q(e^{j\omega})>0$ for all ω. We define $P(z)$ as in (3.4), (3.5).

In both cases $n=2v$ and $n=2v+1$, we have

$$|G(\exp j2\pi k/n)| = |P(\exp j2\pi k/n)| \quad k=0,1,\dots,n-1 \tag{3.8}$$

The determination of $P(z)$ from $Q(z)$ in (3.4) is not necessarily an easy task. It should be noted however that procedures exist which do not rely on determination of all the zeros of $Q(z)$, see e.g. [6,7] for a linearly convergent algorithm and [8] for a quadratically convergent algorithm.

3.3 *Error Bounds*

We shall set out the arguments justifying the error bounds in a much briefer way than in the last section.

Lemma 3.1 Assume $G(z)$ satisfies the assumptions listed in Section 3.1 and $H(z) = G(z)G(z^{-1})$. Let $Q(z)$ be formed as described in Section 3.2. Then for any R_1 with $1<R_1<\rho$

$$\|H(e^{j\omega})-Q(e^{j\omega})\|_\infty \leq R_1^{v}\alpha(R_1) \max_{\theta\in(-\pi,\pi]} |H(R_1e^{j\theta})| \tag{3.9}$$

$$\left\| \frac{d}{d\omega}\left[H(e^{j\omega})-Q(e^{j\omega})\right]\right\|_\infty \leq R_1^{v}[\beta(R_1)v+\gamma(R_1)] \max_{\theta\in(-\pi,\pi]} |H(R_1e^{j\theta})| \tag{3.10}$$

Here $\alpha(R_1)$, $\beta(R_1)$ and $\gamma(R_1)$ are functions of R_1 which are independent of v, but will vary according as n is even or odd.

Proof Recognize that Q interpolates H at $\exp(j2\pi k/n)$ in case n is odd, and $Q+\frac{1}{2}b_{-v}z^{-v}-\frac{1}{2}b_{-v}z^{v}$ interpolates H in case n is even. Then the same arguments as yielded Lemma 2.1, Lemma 2.2, Theorem 2.1 and Theorem 2.2 yield (3.9) and (3.10).

Remark At this stage, we can justify our earlier claim that $Q(e^{j\omega}) > 0$ for suitably large n. Because G has all its zeros inside $|z|<\rho^{-1}$, $|G|$ is nonzero on $|z|=1$. Hence $|H|$ is nonzero on $|z|=1$. By (3.9), for suitably large n, we can ensure that

$$\|H-Q\|_\infty < \min_{|z|=1} |H(e^{j\omega})|.$$

Then $Q(e^{j\omega}) > 0$ for all ω.

We need to deduce several consequences of this Lemma. The first two relate to the behaviour of $|PG^{-1}|$ and its derivative.

Corollary 3.1 Assume the same hypotheses as for Lemma 3.1, and assume that $P(z)$ is defined as described in Section 3.2. Suppose that

$$|G(e^{j\omega})| \geq c \quad \forall\omega \tag{3.11}$$

Set

$$M = \max_{\theta\in(-\pi,\pi]} |H(R_1e^{j\theta})| \tag{3.12}$$

Then

$$\left\|1 - \left|\frac{P}{G}\right|\right\|_\infty \leq c^{-2}R_1^{v}\alpha(R_1)M \tag{3.13}$$

Proof Omitted due to space limitations.

Corollary 3.2 Assume the same hypotheses as for Lemma 3.1, and assume that $P(z)$ is defined as in Section 3.2. With c, M as in Corollary 3.1, and

$$d = \left\| \frac{d|G(e^{j\omega})|}{d\omega} \right\|_\infty \tag{3.14}$$

there holds, for sufficiently large n, say $n\geq N$,

$$\left\|\frac{d}{d\omega}\left|\frac{P}{G}\right|\right\|_\infty \leq \frac{\frac{1}{2}c^{-2}\beta(R_1)M}{1-c^{-2}R_1^{v}\alpha(R_1)M}\, vR_1^{v} + \left[\frac{\frac{1}{2}c^{-2}\gamma(R_1)M}{1-c^{-2}R_1^{v}\alpha(R_1)M} + \alpha(R_1)c^{-3}d\right]R_1^{v} \tag{3.15}$$

Corollaries 3.1 and 3.2 allow us to construct a bound on $\arg(P/G)$. The tool is to appeal to much the same argument as used in Section 2, where we had bounds on the derivative of the real part of a function, and obtained a bound on its imaginary part.

Lemma 3.2 Assume the same hypotheses as Lemma 3.1 and suppose $P(z)$ is determined as described in Section 3.2. Then for all n exceeding some N, there exist constants K, L depending on

$$\inf_{\omega}|G(e^{j\omega})|,\ \left\|\frac{dG}{d\omega}\right\|_\infty,\ R_1 \text{ and } \max_{\theta\in(-\pi,\pi]} |H(R_1e^{j\theta})|$$

such that

$$\left\|\arg \frac{P}{G}\right\|_\infty < (Kv+L)R_1^{v} \tag{3.16}$$

Proof Omitted due to space limitations.

Finally this brings us to

Main Theorem 3.1 Let $G(z)$ be an unknown transfer function with $G(z)$, $G^{-1}(z)$ analytic in $|z|>\rho^{-1}$ for some $\rho>1$, and with values of $|G(\exp j2\pi k/n)|$ prescribed for $k=0,1,\dots,n-1$. Suppose $G(z)$ corresponds to a real impulse response, so that $\ln|G(e^{j\omega})|$ and $\arg G(e^{j\omega})$ are even and odd in ω. Let $P(z)$ be a polynomial in z^{-1} of degree v, where $n=2v$ or $2v+1$, with all zeros (regarded as a function of z) in $|z|<1$, such that $|P|$ interpolates $|G|$ at $\exp j2\pi k/n$, with $P(z)$ determined according to the procedures of Section 3.2. Then

$$\| 1-G^{-1}P \|_\infty < (K_1 v+L_1)R_1^v \qquad (3.17)$$

for constants K_1, L_1 depending on

$$\inf_\omega |G(e^{j\omega})|, \ \left\|\frac{dG}{d\omega}\right\|_\infty, \ R_1 \text{ and } \max_\omega |H(R_1 e^{j\omega})|,$$

where $H(z) = G(z)G(z^{-1})$.

Proof

Equations (3.13) and (3.16) yield the result.

3.4 *Second Approach: Interpolation of* ln(G)

In the first approach of Section 3, we found a rational interpolation of $|G|^2$, and from it a rational (actually FIR) interpolation of $|G|$ which approximated G. In the second approach, see Figure 3.2, we obtain a rational approximation of $\log|G|$ and from it a rational (FIR) approximation of logG. Exponentiation gives an approximation of G. Of course, the object approximating G is no longer rational, let alone FIR.

In more detail, let

$$Q(z) = \sum_{i=-v}^{v} b_i z^i \qquad (n=2v+1) \qquad (3.18a)$$

$$= \sum_{i=-v}^{v-1} b_i z^i \qquad (n=2v) \qquad (3.18b)$$

interpolate $\log|G(\exp j2\pi k/n)|$ for $k=0,1,\dots,n-1$.

Let

$$T(z) = b_0 + 2\sum_{i=-v}^{-1} b_i z^i \qquad (n=2v+1)\,(3.19a)$$

$$= \frac{b_0}{2} + 2\sum_{i=-(v-1)}^{-1} b_i z^i + b_{-v}z^{-v} \quad (n=2v)\,(3.19b)$$

Then $T(z)$ is an approximant of $\ln G(z)$. The line of argument of Section 2 carries through to yield the following conclusions: For any R_1 with $1<R_1<\rho$, and with

$$M = \max_{\theta\in(-\pi,\pi]} [\,|\ln G(R_1 e^{j\theta})|\,] + \max_{\theta\in(-\pi,\pi]} [\,|\ln G(R_1^{-1}e^{j\theta})|\,] \qquad (3.20)$$

there holds

$$\|\mathrm{Re}T(e^{j\omega})| - \ln|G(e^{j\omega})|\,\|_\infty < K(R_1)R_1^v M \qquad (3.21a)$$

$$\left\| \frac{d}{d\omega}\left[\mathrm{Re}T(e^{j\omega}) - \ln|G(e^{j\omega})|\right]\right\|_\infty < [K_1(R_1)+K_2(R_1)v]R_1^v M \qquad (3.21b)$$

and, using Lemma 2.3 to bound $\|\mathrm{Im}T(e^{j\omega}) - \arg G(e^{j\omega})\|$,

$$\|T(e^{j\omega}) - \log G(e^{j\omega})\|_\infty < \|\mathrm{Re}T(e^{j\omega}) - \ln|G(e^{j\omega})|\,\|_\infty + \|\mathrm{Im}T(e^{j\omega}) - \arg G(e^{j\omega})\|_\infty$$

$$< K(R_1)R_1^v M + 2[K_1(R_1)+vK_2(R_1)]R_1^v M = [K_1'(R_1) + vK_2'(R_1)]R_1^v M \qquad (3.22)$$

We can obtain an equivalent inequality involving $G^{-1}P$ as follows. By (3.21b),

$$|\ln(|P|/|G|)| < [K_1(R_1)+vK_2(R_1)]R_1^v M$$

so that

$$|P|/|G| < \exp[K_1(R_1)+vK_2(R_1)]R_1^v M$$
$$> \exp-[K_1(R_1)+vK_2(R_1)]R_1^v M \qquad (3.23a)$$

Also, as argued above,

$$|\arg P - \arg G| < 2[K_1(R_1)+vK_2(R_1)]R_1^v M \qquad (3.23b)$$

Hence, as in the proof of Main Theorem 3.1,

$$\left\| 1-\frac{P}{G}\right\|_\infty < \left\| 1-\left|\frac{P}{G}\right|\right\|_\infty + \left\|\arg\left[\frac{P}{G}\right]\right\|_\infty$$

and from (3.23) we see that decay at a rate R_1^v is evident for $\|I-G^{-1}P\|_\infty$.

The above procedure generates a rational approximation of logG. There is a minor variation that will generate a rational approximation of G: we find a

$$\bar{P}(z) = \sum_{i=0}^{n-1} \bar{p}_i z^{-i}$$

such that $\bar{P}(z)$ interpolates $\exp T(z)$ at $z = \exp(j2\pi k/n)$. Note that this means that $|\bar{P}(z)|$ interpolates $|G(z)|$ at the same points. The theory of [3] ensures that

$$\|\bar{P}(e^{j\omega}) - P(e^{j\omega})\|_\infty = \|\bar{P}(e^{j\omega}) - \exp T(e^{j\omega})\|_\infty < KR_1^{-n} \qquad (3.24)$$

where K depends on

$$\max_{(-\pi,\pi]} |\exp T(R_1^{-1}e^{j\theta})|.$$

This bound can be combined with that for $\|1-PG^{-1}\|_\infty$ to obtain a bound for $\|1-\bar{P}G^{-1}\|_\infty$.

We remark that in case n is even, the construction of $T(z)$ is just that of [1, p.357], where the discrete Hilbert transform is suggested as a procedure for recovering phase data. The inequality (3.22) allows us to pin down the error that may result in its use, without any prior assumption that $G(z)$ or $\log G(z)$ corresponds to a finite impulse response.

4. ERRORS IN THE DATA

To this point, we have considered the construction of a transfer function from data (i.e. real part, or magnitude at discrete points on the unit circle) which has been assumed error free. In this section, we postulate the existence of errors in this data, and consider the resulting (extra) errors which now arise between the true function and that constructed from the error−contaminated data.

We shall be particularly interested in the behaviour of the error with increasing n, the number of interpolating points. Most error models will permit a discrete increase in value at one point, and discrete decrease at an adjacent point. This means that as $n\to\infty$, i.e. as the spacing between adjacent points decreases, the derivative (with respect to angle around the unit circle) can approach infinity. In the light of remarks earlier in this paper and in [2], we must expect that the error in the constructed function could in some way diverge as $n\to\infty$.

Such behaviour is indeed what we discover when we postulate arbitrary variation of (up to) $\pm\Delta$ in the values, say real part, from which an interpolation function is constructed. However, if we postulate random fluctuations which are gaussian, we do not get such unpleasant behaviour, at least when we examine the variance in the error of the constructed function. Roughly, the averaging process somehow smooths out the difficulties caused by the derivative of the error possibly becoming unbounded as $n\to\infty$.

We shall consider this statistical approach first.

4.1 *Statistical Approach to Errors*

To fix ideas, let us consider the construction of a function from discrete values of its real part (a similar discussion would apply for magnitude data). To pursue a statistical approach, we might suppose that the values of $\mathrm{Re}G(z)$ at $z = \exp j2\pi k/n$, $k = 0,\dots,n-1$ are perturbed by independent, zero mean gaussian random variables of variance σ^2. However, we must reflect the fact that ReG is necessarily even. So this suggests that we assume G(1) is known to within standard deviation σ, and $\mathrm{Re}G(\exp j2\pi k/n)$ for $k<(n/2)$ to within standard deviation $\sigma/\sqrt{2}$ (where we exploit the fact that $\mathrm{Re}G(\exp j2\pi k/n)$ and $\mathrm{Re}G(\exp j2\pi(n-k)/n)$ take the same value.)

Let P be the interpolating FIR function that would be associated with noiseless data, and let $\hat{P}$ be the interpolating FIR function that is associated with noisy data. Since

$$\|G-\hat{P}\|_\infty < \|G-P\|_\infty + \|P-\hat{P}\|_\infty \qquad (4.1)$$

the damage done by the noisy data is reflected in $\|P-\hat{P}\|_\infty$, which is actually a random variable.

Space limitations prevent us giving the derivation, but it is not difficult to show that

$$E[(P(e^{j\omega}) - \hat{P}(e^{j\omega}))^2] = \sigma^2 \quad \forall\omega\in(-\pi,\pi] \qquad (4.2)$$

We see then that at any one frequency $|G-\hat{P}|$ differs from

$|G-P|$ by a random variable of variance σ^2. In particular, at that frequency for which $|G-\hat{P}|$ is maximum, i.e. $\|G-P\|_\infty = |G-P|$, we have $|G-\hat{P}|$ differing from $\|G-P\|_\infty$ by a random variable of variance $\hat{\sigma}^2$. However, we cannot make a firm statement about $\|G-\hat{P}\|_\infty$.

4.2 *Deterministic Approach to Error*

We shall now show, with the aid of an example, that if errors of up to $\pm\Delta$ are postulated in the value of the real part of $Re[\tilde{P}(e^{j\theta})]$ at interpolating points, $\|P(e^{j\omega})\|_\infty$ can grow as log n. This shows that in the presence of deterministic errors, the benefit obtained from finer and finer sampling (convergence in the sense of Section 2 or 3) is eventually outweighed by the effect of the errors in the sampled values.

Take n as a multiple of 4, viz. $n=2v=4u$. Consider errors in $Re\tilde{P}(e^{j\theta})$ with the property that

$$
\begin{aligned}
Re\tilde{P}(\exp j2\pi k/n) &= \Delta \quad \text{when } Re(\exp j2\pi k/n) > 0\\
&= 0 \quad \text{when } Re(\exp j2\pi k/n) = 0\\
&= -\Delta \quad \text{when } Re(\exp j2\pi k/n) < 0
\end{aligned} \tag{4.3}
$$

Again space limitations prevent us presenting the algebraic details. However, with some manipulation one can show that

$$
\begin{aligned}
\|\tilde{P}\|_\infty &= |\tilde{P}(\exp j\pi/2)|\\
&= \frac{4\Delta}{n}\sum_{m=1}^{n/4}\cot(m-\tfrac{1}{2})\theta
\end{aligned} \tag{4.4}
$$

It is not hard then to prove that

$$
\Delta\left[\frac{2}{\pi}\ln n - \frac{2}{\pi}\ln 2 + 2\right] > \left|\tilde{P}(\exp j\tfrac{\pi}{2})\right| > \Delta\left[\frac{2}{\pi}\ln n - \frac{2}{\pi}\ln \pi\right] \tag{4.5}
$$

Hence it is clear that $|\tilde{P}(\exp j\pi/2)|$ grows as ln n. In absolute terms, the values stay modest for large values of n, e.g. $n = 4096$ yields $|\tilde{P}(\exp j\pi/2)| = 5.3752\Delta$.

As foreshadowed, this example shows that the error can certainly diverge as fast as ln n. Whether or not there exist other constructions [i.e. choice of errors of which (4.3) is one example] such that divergence occurs faster than ln n is an open problem. However, it is true that divergence occurs no faster than $n^{\frac{1}{2}}\ln n$. We know that $|Re\tilde{P}(e^{j2\pi k/n})| < \Delta$ gives, from [9, Corollary 2.1],

$$
\|Re\tilde{P}(e^{j\omega})\|_\infty \leq \Delta\,(1 + 2/\pi\ln n) \tag{4.6}
$$

Now for n odd

$$
\begin{aligned}
Re\tilde{P}(e^{j\omega}) &= b_0 + \sum_{k=1}^{v} b_k(e^{jk\omega}) + b_k(e^{-jk\omega})\\
&= b_0 + \sum_{k=1}^{v} 2b_k\cos k\omega
\end{aligned} \tag{4.7}
$$

Using [10, 3.5.42, p.265], if $|Re\tilde{P}(e^{j\omega})| < M$, then

$$
\begin{aligned}
|\tilde{P}(e^{j\omega})| &= |b_0 + \sum_{k=1}^{v} 2b_k e^{jk\omega}|\\
&\leq |b_0| + \sum_{k=1}^{v} 2|b_k|\\
&\leq AMn^{\frac{1}{2}}
\end{aligned} \tag{4.8}
$$

where A is a constant. Identifying M with the right hand side of (4.6), then combining with (4.8) gives an $n^{\frac{1}{2}}\log n$ increase. A similar calculation applies for n even.

5. EXAMPLES

In this section, we present two examples of generating an approximate frequency response from interpolation data: the first example uses real part data; the second example operates on magnitude data. With frequency response plots we illustrate the salient features of each method, including the reduction in approximation error as the number of interpolation points increases.

The first example uses the stable transfer function

$$
G(z) = \frac{1}{z^2+0.64} + \frac{1}{z-0.5}
$$

(poles are at $z = \pm j0.8,\ 0.5$) and we evaluate for (a) $n=8$ and (b) $n=16$

$$
Re\ G(e^{j2\pi k/n}) \qquad k=0,1,\ldots,n-1
$$

From this real part data, we construct P(z), causal and FIR according to equations (2.3)−(2.5). Figure 5.1 charts the frequency responses $ReG(e^{j\omega})$ and $ReP(e^{j\omega})$; $ImG(e^{j\omega})$ and $ImP(e^{j\omega})$; $|G(e^{j\omega}) - P(e^{j\omega})|$. Comparison of Figure 5.1 (a) $n=8$ and (b) $n=16$ shows the effect of increasing the number of interpolation points. For $n=8$ the error $\|G-P\|_\infty = 1.6$; for $n=16$, $\|G-P\|_\infty = 0.6$.

These plots illustrate other points of interest. Note that the real parts of $G(e^{j\omega})$ and $P(e^{j\omega})$ coincide at the equispaced interpolation points, but, as expected, the imaginary parts do not act likewise. Observe carefully the frequencies where the error $|G(e^{j\omega}) - P(e^{j\omega})|$ is large. These are not coincident with frequencies where $ReG(e^{j\omega})$ is large, but rather at those frequencies where $|d/d\omega\ ReG(e^{j\omega})|$ is large, consistent with the importance we placed on the derivative of $ReG(e^{j\omega})$ in the error bounds of Section 3.

The second example illustrates the gain phase relationships. For this we need a stable, minimum phase transfer function G(z). Note that any strictly proper, rational transfer function G(z) has one or more zeros at infinity; therefore, it is non−minimum phase. For this example we use the stable mimimum phase

$$
G(z) = \frac{(z+0.6)\ (z-0.8)}{(z-0.4-j0.4)(z-0.4+j0.4)}
$$

and evaluate

$$
|G(e^{j2\pi k/n})| \qquad k=0,1,\ldots,n-1
$$

From this magnitude data we construct P(z) according to either "method 1" or "method 2" outlined in Figures 3.1 and 3.2 respectively. Figure 5.2 plots the results for $n=8$; Figure 5.3 for $n=10$.

Similar observations to the real−imaginary case apply here: the regularly spaced interpolation of magnitude plots; decreasing error $\|G-P\|_\infty$ as n increases from 8 to 10; the error $|G(e^{j\omega})-P(e^{j\omega})|$ being large at frequencies where the magnitude has large derivative. Contrasting method 1, which relies on spectral factorization, and method 2, based on the logarithm, shows that the former gives a smaller error. Whether this is true for general G(z) is not apparent from the error bound calculations of Sections 3.3 and 3.4. Moreover, Figures 5.2 and 5.3 show the additive error $|G-P|$; for multiplicative error $|1-P/G|$, the two methods may rank in the reverse order.

The two examples of this section are in qualitative agreement with the theoretical calculations of preceding sections. (Very lengthy calculations are required to check quantitative agreement and hardly seem justified). The examples illustrate that a good approximation to a frequency response given only real part interpolation data or magnitude interpolation data is indeed possible, provided the underlying transfer function is stable, or stable and miminum phase, and sufficient interpolation points are taken. The error phenomena associated with imprecise data and described in the previous section generally only become of concern for larger values of n, far larger in fact than the values required in these examples to secure good approximation.

6. CONCLUSIONS

Despite the discontinuity questions associated with use of the Hilbert transform, we have been able to clarify the circumstances under which Hilbert transform type calculations can be used to recover a transfer function from discrete frequency data for its real part, imaginary part, magnitude, or phase.

With a finite number of points, it is never possible to recover the transfer function exactly, unless the class of considered transfer functions is heavily restricted *a priori* e.g. to being FIR. Instead, we can hope to recover an FIR function which is close to the desired transfer function, with the error going to zero as the number of data points becomes infinite.

There is however a practical limit to this result imposed by data errors. For the effect of data errors on the determined FIR transfer function becomes greater as the number of data points increases, at least for certain types of error model.

There are a number of further questions which future work could examine. One could seek to extend the results of the paper to matrix transfer functions. The connection between real and imaginary part has nothing new in it; the connection between spectrum matrix and minimum phase stable spectral factors (the matrix analog of the magnitude to transfer function calculation) is somewhat different, since one lacks the ability to work with Hilbert transforms of the real and imaginary part of the logarithm of the spectrum.

Another direction of generalization would involve changing the postulates about the spacing of discrete frequencies. One could be interested in working with information over only a limited part of the unit circle, or with frequencies which were the bilinear transforms of (continuous time) frequencies uniformly spaced in a logarithmic sense, or with frequencies which are spaced more densely in the neighbourhood of high derivatives of the data.

A third direction of generalization would involve using discrete data to define a transfer function which was not FIR, for example IIR with prescribed poles, or all-pole. Besides explaining how to construct such a function from the data, one would have to establish a bound between the constructed transfer function and the class of all transfer functions consistent with the data and establish that the bound went to zero as the number of data points became infinitely dense.

Finally, we note that we have not established the worst possible behaviour of the error in a *Hilbert transform induced* by (deterministic) bounded errors in the data.

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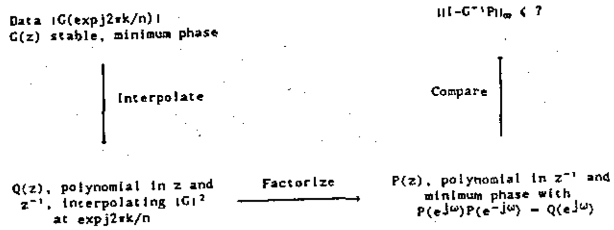

Figure 3.1: Outline approach to gain interpolation problem.

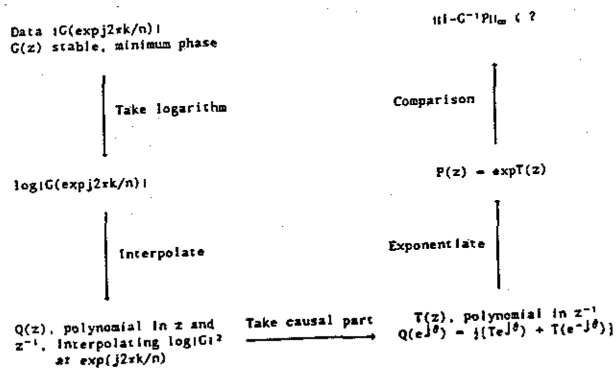

Figure 3.2: Alternative approach to gain interpolation problem.

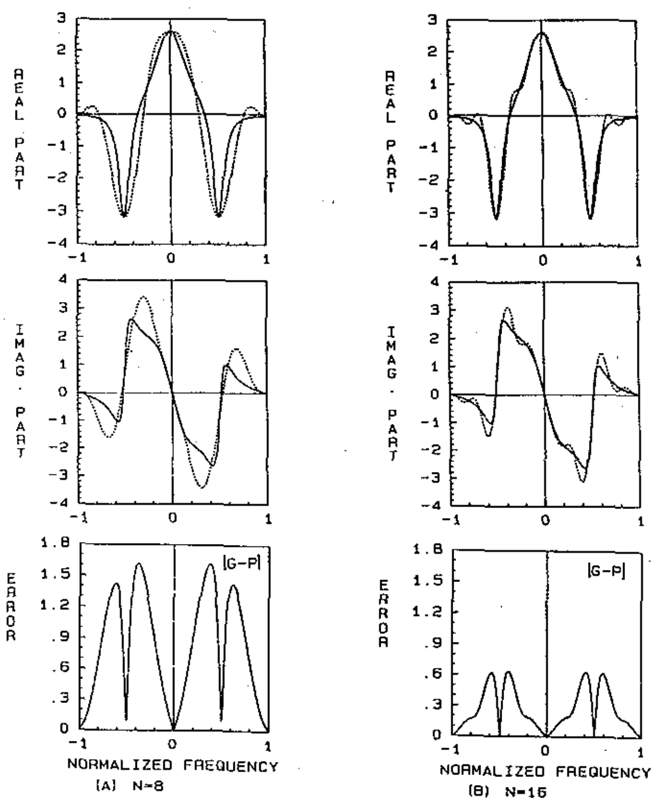

Figure 5.1 Frequency responses for the real part data example. For the real and imaginary part plots, solid line is G, broken line is P.

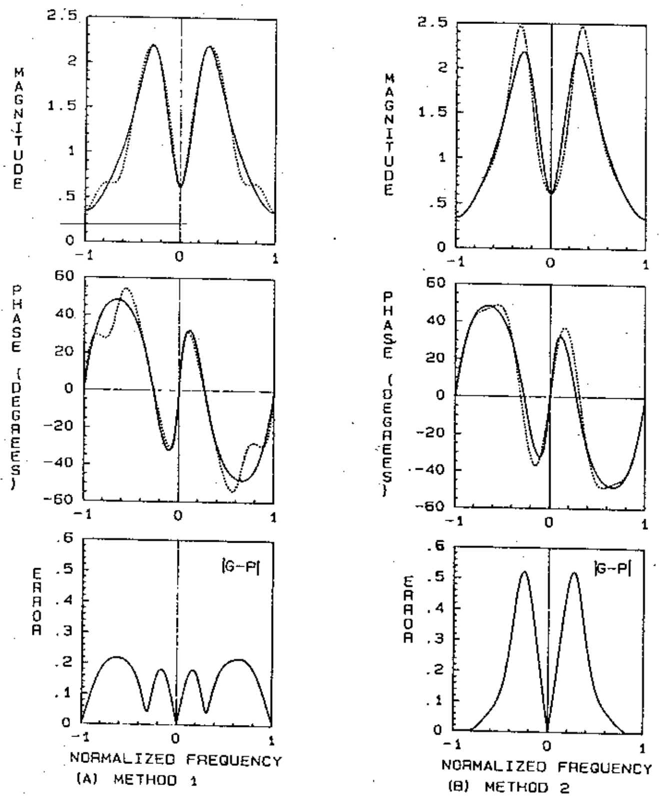

Figure 5.2 Frequency responses for the magnitude data example, n=8. Solid line G, broken line P, in the magnitude and phase plots.

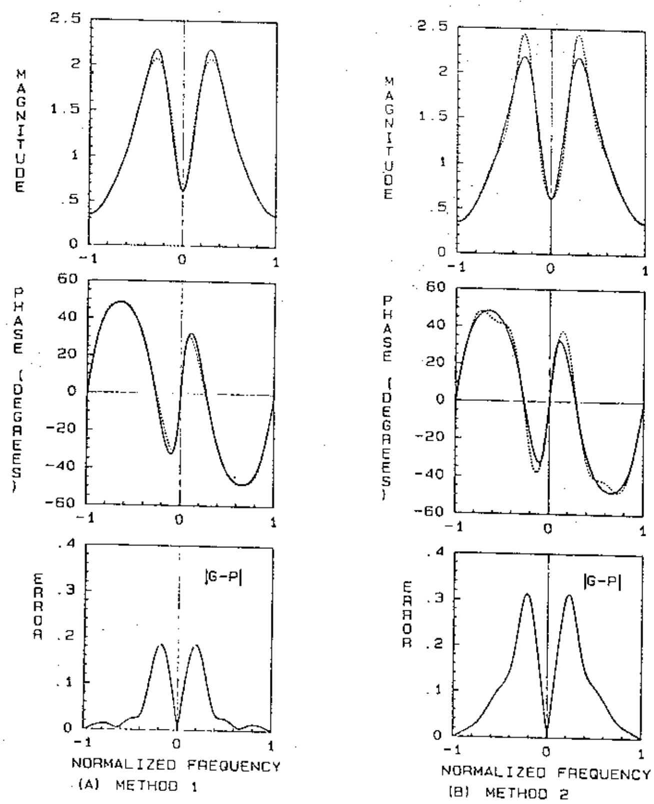

Figure 5.3 Frequency responses for the magnitude data example, n=10. Solid line G, broken line P in the magnitude and phase plots.