ERRORS-IN-VARIABLES IDENTIFICATION: LINEAR-GAUSSIAN STATIONARY RESULTS

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Abstract. This paper reviews the main results on errors-in-variables identification for linear systems (memoryless and dynamic) when second order statistics are available. Most errors-in-variables problems have nonunique solutions. The requirement of causality of a solution in dynamic problems, which are treated in a nonparametric manner, raises mathematical questions which are new to the identification literature. Several open questions are noted.

Keywords. Errors-in-variables identification; System identification; Nonparametric methods

1. INTRODUCTION

Errors-in-variables identification represents some departure from the conventions usually governing the statement of identification problems. We can illustrate the key issue simply, by contemplating a physics experiment in which the object is to determine the value of the gravitational acceleration constant by measuring the square root of the length $\sqrt{L_i}$ and the period T_i of a number of pendula. The ideal relationship is of course

$$T_i = 2\pi\sqrt{\left[\frac{L_i}{g}\right]} \qquad (1.1)$$

Consider the scatter diagram of Fig.1. The problem of determining g is then, in effect, the problem of determining the slope of the line of best fit passing through the origin. What do we mean by the line of best fit? We could assume a model

$$T_i = \frac{2\pi}{\sqrt{g}}\sqrt{L_i} + n_i \qquad (1.2)$$

regarding n_i as noise perturbing measurements of T_i, while measurements of $\sqrt{L_i}$ are assumed noiseless. The best fit line, call it l_1, is chosen to minimize Σn_i^2, i.e. the sum squared of vertical errors. But equally, one could assume

$$\sqrt{L_i} = \frac{\sqrt{g}}{2\pi} T_i + m_i \qquad (1.3)$$

with T_i known perfectly, and noise m_i contaminating the measurements of $\sqrt{L_i}$. The best fit line now minimizes Σm_i^2, the sum squared of the horizontal errors. A <u>different</u> best fit line, call it l_2, is obtained. Generically, it has a greater slope than l_1. Yet another possibility is to postulate that <u>both</u> $\sqrt{L_i}$ and T_i measurements contain noise, and then to somehow determine a best fit line.

This last possibility is representative of the errors-in-variables viewpoint - that no measurements are noise free. Those individuals accustomed to least squares identification will recognise this as an atypical assumption in the engineering literature dealing with system identification, where it is common to assume inputs are noise free.

In this paper, length constraints do not permit an exhaustive survey of errors-in-variables identification. We limit discussion to problems involving linear systems, excited by zero mean signals and noises, and from which second order statistics (only) are measured. A survey of results which cover the use of higher order statistics can be found in Deistler (1986).

A general characteristic of the problems we consider is that the identification problem generally has a nonunique solution. In fact, in the example above, an argument can be advanced (without extra assumptions on the noise other than independence of the noises which perturb measurements $\sqrt{L_i}$ and T_i) that any line of slope between either of the two best fit lines l_1 and l_2 can serve as a solution of the identification problem.

2. NONDYNAMIC PROBLEMS IN MANY VARIABLES

The set up we consider in this section is the same as that considered in Kalman (1982). There is an underlying noiseless stationary n-vector process $\hat{x}_t$, with $E[\hat{x}_t\hat{x}_s'] = 0$ for $t \neq s$ and $E[\hat{x}_t] = 0$. Measurements x_t of $\hat{x}_t$ are available, being given by

$$x_t = \hat{x}_t + n_t \qquad (2.1)$$

Here, $\{n_t\}$ is a zero mean process independent of $\{\hat{x}_t\}$, with $E[n_t n_s'] = 0$, $t \neq s$ and $E[n_t n_t'] = \tilde{\Sigma}$, where $\tilde{\Sigma}$ is some diagonal matrix (the entries of which are unknown). The matrices

$$\Sigma = E[x_t x_t'] \qquad \hat{\Sigma} = E(\hat{x}_t\hat{x}_t') \qquad (2.2)$$

are respectively known and unknown. Notice that

$$\Sigma = \hat{\Sigma} + \tilde{\Sigma} \qquad (2.3)$$

The problem considered in Kalman (1982) has several aspects:

(1) To infer from Σ what the maximum nullity is for $\hat{\Sigma}$ in any decomposition of the type (2.3), where $\hat{\Sigma}$ is nonnegative definite symmetric, and $\tilde{\Sigma}$ is diagonal. This is equivalent to defining the maximum number of linear relations that can exist among the entries of $\hat{x}_t$; any vector α in the nullspace of $\hat{\Sigma}$ defines a linear relationship $\alpha'\hat{x}_t = 0$ for all t.
(2) To characterize the linear relationships that can exist among the entries of $\hat{x}_t$.

In the case $n = 2$, the solution to these problems is long standing, see Gini (1921), Frisch (1934), Madansky (1959), Moran (1971), Aigner et al (1984), T.W. Anderson (1984). Suppose

$$\Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} \qquad (2.4)$$

Assume $\sigma_{12} \neq 0$ (else we have an uninteresting problem). Then the maximum nullity for $\hat{\Sigma}$ is 1. The desired linear relation is $\alpha\hat{x}_1 = \hat{x}_2$, where

$$\frac{|\sigma_{12}|}{\sigma_{11}} \leq |\alpha| \leq \frac{\sigma_{22}}{|\sigma_{12}|} \qquad \text{sgn}\alpha = \text{sgn}\sigma_{12} \qquad (2.5)$$

i.e. for <u>every</u> such choice of α, one can find $\hat{\Sigma} = \hat{\Sigma}' \geq 0$ of nullity 1 and $\tilde{\Sigma} \geq 0$, diagonal, such that $\Sigma = \hat{\Sigma} + \tilde{\Sigma}$. In fact, we have

$$\Sigma = \begin{bmatrix} \left|\frac{\sigma_{12}}{\alpha}\right| & \sigma_{12} \\ \sigma_{12} & |\alpha\sigma_{12}| \end{bmatrix} + \begin{bmatrix} \sigma_{11} - \left|\frac{\sigma_{12}}{\alpha}\right| & 0 \\ 0 & \sigma_{22} - |\alpha\sigma_{12}| \end{bmatrix}$$

[In relation to the experiment described in the introduction, the lines l_1 and l_2 have slopes defined respectively by the lower and upper values $\sigma_{12}\sigma_{11}^{-1}$ and $\sigma_{22}\sigma_{12}^{-1}$.]

The most important result due to Kalman describes the situation where $\hat{\Sigma}$ can have nullity no greater than 1, for arbitrary size vectors:

> Theorem. Suppose Σ is nonsingular, and suppose that (by transforming the sign of variables if necessary) the first row and column of Σ^{-1} are all positive[1]. If some entries of Σ^{-1} are not positive, then there exist decompositions (2.3) (with $\hat{\Sigma},\tilde{\Sigma}$ nonnegative, $\tilde{\Sigma}$ diagonal) in which $\hat{\Sigma}$ has nullity at least 2. If all entries of Σ^{-1} are positive, then there exist decompositions (2.3) where $\hat{\Sigma}$ is singular and in all of these $\hat{\Sigma}$ has nullity 1. In this latter case, the set of vectors α defining linear relations $\alpha'\hat{x}_t = 0$ is given by
>
> $$\alpha = \sum_{j=1}^{n} \mu_j s^{(j)} \quad \mu_j \geq 0 \quad \Sigma\mu_j = \mu > 0$$
>
> $s^{(j)}$ = j-th column of Σ^{-1}
> so that $\mu^{-1}\alpha$ is a convex combination of the elementary least squares regressions.

Complete results are not yet available which characterize in a precise manner those Σ for which $\hat{\Sigma}$ of maximum nullity 2, 3, ... exist, together with the associated linear relations. However, a substantial advance has been provided by Dr Moor and Vandewalle (1986a, 1986b). In contrast perhaps to Kalman (1982), who essentially focussed on obtaining conditions on $\tilde{\Sigma}$ to yield a solution, these authors focus on obtaining conditions on the nullvectors of Σ-$\tilde{\Sigma}$, viz. the defining linear relations. They conjecture that the solution set of such nullvectors is a collection of convex polyhedral sets in orthants, with the vertices of these sets defined by least squares solutions, and certain other vectors (termed allowed orthant null-invariant vectors) which contain as many zeros as possible. An algorithm is available for defining these vertex vectors.

The above results apply to real-valued processes, and Σ, $\hat{\Sigma}$, $\tilde{\Sigma}$ are all real matrices. Similar questions as those considered above can be posed when Σ is positive definite Hermitian. Again, complete results are not yet available.

3. DYNAMIC PROBLEMS IN TWO SCALAR VARIABLES

We turn now to consider the arrangement depicted in Figure 2. The standing assumptions are:

- $\hat{x}_k$, u_k, v_k are mutually independent, zero mean scalar stationary processes with fixed but unknown power spectra.
- the process $\hat{y}_k$ is related to $\hat{x}_k$ through a convolution operation

$$\hat{y}_k = \sum_{\ell} w_{k-\ell}\hat{x}_\ell \qquad (3.1)$$

(For the moment, no assumption concerning the causality of the convolution operation will be made.)

Once again, the task is to use second order information - now the spectral matrix of $[x_k\ y_k]'$ - to deduce as much as can be deduced about the convolution operator.

[1] The nongeneric situation where certain entries can be zero will be avoided for simplicity.

Dynamic problems involving two scalar variables are discussed in Maravall (1979), Aigner et al (1982), Hsiao (1982), Wegge (1983), Söderström (1980), Anderson and Deistler (1984) and Anderson (1985). The first four references are concerned with local identifiability. The next two deal with global identifiability, and identify a number of collections of special assumptions (e.g. that the input process or a noise process be autoregressive, or moving average) that ensure unique identifiability of the transfer function. The treatment of the last reference is followed here. It is by far the closest in spirit to the nondynamic work.

A little analysis of the arrangement of Figure 2 yields (in obvious notation)

$$\Sigma(e^{j\omega}) = \begin{bmatrix} \sigma_{xx}(e^{j\omega}) & \sigma_{xy}(e^{j\omega}) \\ \sigma_{yx}(e^{j\omega}) & \sigma_{yy}(e^{j\omega}) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\hat{x}\hat{x}}(e^{j\omega}) + \sigma_{uu}(e^{j\omega}) & W^*(e^{j\omega})\sigma_{\hat{x}\hat{x}}(e^{j\omega}) \\ W(e^{j\omega})\sigma_{\hat{x}\hat{x}}(e^{j\omega}) & |W(e^{j\omega})|^2\sigma_{\hat{x}\hat{x}}(e^{j\omega})+\sigma_{vv}(e^{j\omega}) \end{bmatrix} \qquad (3.2)$$

For convenience, let us assume that $\sigma_{xy}(e^{j\omega}) \neq 0$ for all ω. (It is known how to relax this assumption). From (3.1), one can deduce [compare with (2.5)]

$$\left|\frac{\sigma_{xy}(e^{j\omega})}{\sigma_{xx}(e^{j\omega})}\right| \leq |W(e^{j\omega})| \leq \left|\frac{\sigma_{yy}(e^{j\omega})}{\sigma_{xy}(e^{j\omega})}\right| \qquad (3.3a)$$

$$\arg W(e^{j\omega}) = \arg \sigma_{yx}(e^{j\omega}) \qquad (3.3b)$$

Equations (3.3) represent necessary conditions on the (transform of) the convolution operation. The equations are also sufficient in the following sense: pick any transfer function $W(e^{j\omega})$ defined for $\omega\in[0,2\pi]$ and satisfying (3.3). Then there exist nonnegative real functions $\sigma_{\hat{x}\hat{x}}(e^{j\omega})$, $\sigma_{uu}(e^{j\omega})$ and $\sigma_{vv}(e^{j\omega})$ such that (3.2) holds, i.e. the set-up of Fig.2 is consistent with the problem data, here the matrix $\Sigma(e^{j\omega})$.

This exercise requires nothing more than expanding the two-variable nondynamic result for real random variables stated in the previous section to cope with complex random variables, and applying the expanded result on a pointwise basis at each point around the unit circle.

The situation changes dramatically when we agree to restrict attention to a search for causal stable $W(e^{j\omega})$. Thus suppose in (3.1) we have

$$\hat{y}_k = \sum_{\ell=-\infty}^{k} w_{k-\ell}\hat{x}_\ell \qquad (3.4)$$

and

$$W(z) = \sum_{0}^{\infty} w_k z^k \qquad (3.5)$$

$$\sum_{k=0}^{\infty} |w_k|\rho^k < \infty \quad \text{for some } \rho > 1 \qquad (3.6)$$

The question here is one of characterizing the causal stable W(z) whose amplitude lies within certain bounds, see (3.3a), and whose phase is known exactly, see (3.3b).

Knowledge of the phase of $W(e^{j\omega})$ and its stability allows determination (through the argument principle) of the number of zeros in $|z|<1$ of W(z) - all W(z) consistent with the data have the same number of zeros in $|z|<1$, viz. $(2\pi)^{-1}$ times the change in $\arg W(e^{j\omega})$ as ω moves from 0 to 2π. If there are no zeros in $|z|<1$ then $W(z)$ is minimum phase, knowledge of the phase of $W(e^{j\omega})$ from (3.3b) determines $W(e^{j\omega})$ to within a scaling constant (see Anderson (1985) for a statement of this well known result) and the scaling constant can be bounded by (3.3a). Thus if $\bar{W}(e^{j\omega})$ is a minimum phase transfer function with phase identical with $\sigma_{xy}(e^{j\omega})$, the family of solutions $W(e^{j\omega})$ is

$$W(e^{j\omega}) = \mu\,\bar{W}(e^{j\omega}) \tag{3.7a}$$

$$\max_{\omega}\left|\frac{\sigma_{xy}(e^{j\omega})}{\sigma_{xx}(e^{j\omega})\bar{W}(e^{j\omega})}\right| \le \mu \le \min_{\omega}\left|\frac{\sigma_{yy}(e^{j\omega})}{\sigma_{xy}(e^{j\omega})\bar{W}(e^{j\omega})}\right|$$

A particular application of this idea to an econometric modelling problem can be found in Hinich and Weber (1986).

If $W(z)$ is determined as having a positive number of zeros in $|z|<1$, N say, then the family of causal solutions becomes parametrized by $N+1$ parameters - the zero positions and a scaling constant. Let $\alpha_1,\ldots,\alpha_N$ be candidate zeros of $W(z)$ in $|z|<1$, (complex zeros occurring in complex conjugate pairs). Set

$$U_A(z) = \prod_{i=1}^{N} (z-\alpha_i)(\alpha_i z-1)^{-1} \tag{3.8a}$$

Find a minimum phase $W_A(z)$ such that

$$\arg W_A(e^{j\omega}) = \arg \sigma_{yx}(e^{j\omega}) - \arg U_A(e^{j\omega}) \tag{3.9}$$

Then

$$W(e^{j\omega}) = \mu_A W_A(e^{j\omega}) U_A(e^{j\omega}) \tag{3.10}$$

with

$$\max_{\omega}\left|\frac{\sigma_{xy}(e^{j\omega})}{\sigma_{xx}(e^{j\omega})W_A(e^{j\omega})}\right| \le \mu_A \le \min_{\omega}\left|\frac{\sigma_{yy}(e^{j\omega})}{\sigma_{xy}(e^{j\omega})W_A(e^{j\omega})}\right| \tag{3.11}$$

In case (3.11) is vacuous, this means that there is no causal solution $W(z)$ with zeros at the particular set of values $\alpha_1,\ldots,\alpha_N$.

Several other points should be noted. First, the principle of the argument may lead to the conclusion that there are no causal stable $W(e^{j\omega})$ consistent with the data (if the change in argument of $\sigma_{yx}(e^{j\omega})$ around the unit circle is a negative multiple of 2π, this will be so). In this case, one should rather look for a model in which in Fig.2 $\{\hat{y}_k\}$ is an input sequence and $\{\hat{x}_k\}$ an output sequence of a causal object. (Thus in the two dynamic variable case, the sign of the change in argument gives the direction of causality.) Second, suppose that the true $\hat{x}_k$, u_k, v_k sequences are white and the true convolution operator is a scaling constant. Then application of the dynamic theory of this section will correctly identify the convolution operator as a scaling constant; the bounds will be the same as those predicted by the theory of Section 2. Third, in case $\sigma_{yx}(e^{j\omega})$ is rational, $W(e^{j\omega})$ will be rational. Fourth, errors in the knowledge of $\sigma_{yx}(e^{j\omega})$ can introduce errors in the solution class of $W(e^{j\omega})$ in a strange way, since the map constructing amplitude from phase of a minimum phase transfer function is continuous as a map on L_p for $1 \le p < \infty$, but not as a map on L_∞.

4. DYNAMIC PROBLEMS IN TWO VECTOR VARIABLES

We consider now an extension to the scheme of the last section, following Green and Anderson (1985). Figure 2 still applies. However, $\hat{x}_k$, $\hat{y}_k$, u_k, v_k are all <u>vector</u> processes, of the <u>same</u> dimensions. (In the rational case, we only need $\hat{x}_k$ and u_k to have the same dimension, and $\hat{y}_k$ and v_k to have the same dimension). The standing assumptions are:

- $\{\hat{x}_k\}$, $\{u_k\}$, $\{v_k\}$ are mutually independent, zero mean vector stationary processes with fixed but unknown power spectral matrices;
- The process $\hat{y}_k$ is related to $\hat{x}_k$ through a convolution operator

$$\hat{y}_k = \sum_{\ell} w_{k-\ell}\hat{x}_\ell \tag{4.1}$$

In order to eliminate consideration of awkward nongeneric cases, we assume further that

$$W(e^{j\omega})\Sigma_{\hat{x}\hat{x}}(e^{j\omega}) \text{ is nonsingular } \forall\omega\in[0,2\pi] \tag{4.2}$$

The effective identification task is to unravel the joint spectrum matrix of $\{x_k', y_k'\}'$:

$$\Sigma(e^{j\omega}) = \begin{bmatrix}\Sigma_{xx}(e^{j\omega}) & \Sigma_{xy}(e^{j\omega})\\ \Sigma_{yx}(e^{j\omega}) & \Sigma_{yy}(e^{j\omega})\end{bmatrix}$$

$$= \begin{bmatrix}\Sigma_{\hat{x}\hat{x}}(e^{j\omega})+\Sigma_{uu}(e^{j\omega}) & \Sigma_{\hat{x}\hat{x}}(e^{j\omega})W^*(e^{j\omega})\\ W(e^{j\omega})\Sigma_{\hat{x}\hat{x}}(e^{j\omega}) & W(e^{j\omega})\Sigma_{\hat{x}\hat{x}}(e^{j\omega})W^*(e^{j\omega})+\Sigma_{vv}(e^{j\omega})\end{bmatrix} \tag{4.3}$$

Here, $\Sigma_{\hat{x}\hat{x}}(e^{j\omega})$, $\Sigma_{uu}(e^{j\omega})$ and $\Sigma_{vv}(e^{j\omega})$ are all power spectrum matrices. The real interest is in the case when $W(e^{j\omega})$ is the transform of a causal stable convolution operator. Thus in (4.1), the summation runs over $(-\infty,k)$ and

$$\sum_{k=0}^{\infty} ||w_k||\rho^k < \infty \quad \text{for some } \rho > 1 \tag{4.4}$$

What makes this problem immediately different from that considered in the previous section is that we can no longer simply identify the phase of $W(e^{j\omega})$ - whatever this might be for the matrix case. The key to a solution is to use a Gohberg Krein factorization, see Clancey and Gohberg (1981), of $\Sigma_{yx}(e^{j\omega})$. The key result is that any square matrix function $F(e^{j\omega})$ nonsingular for $\omega \in [0,2\pi]$ can be factored as follows:

$$F(e^{j\omega}) = F_+(e^{j\omega})\, \mathrm{diag}\,[(e^{j\omega})^{k_i}]\, F_-(e^{j\omega}) \tag{4.5}$$

where $F_+(e^{j\omega})$ is the value on $|z| = 1$ of a matrix function $F_+(z)$ which is analytic together with its inverse in $|z| < 1$, and $F_-(e^{j\omega})$ is the value on $|z| = 1$ of a matrix function $F_-(z)$ which is analytic together with its inverse in $|z| > 1$. The quantities k_i are integers with $k_i \ge k_{i+1}$, and they are uniquely determined by $F(\cdot)$. While F_+, F_- are <u>not</u> unique, the family of F_+, F_- which can appear in (4.5) for a given $F(e^{j\omega})$ is finitely parametrised. In case all k_i are zero, F_+ is determined to within right multiplication by a constant matrix, a fact we use in a moment.

Minimum phase solutions played a prominent role in the theory of Section 2. This is also the case here. Let $\Theta(e^{j\omega})$ be a causal minimum phase spectral factor of $\Sigma_{\hat{x}\hat{x}}(e^{j\omega})$. Then (4.3) implies

$$\Sigma_{yx}(e^{j\omega}) = W(e^{j\omega})\, \Theta(e^{j\omega})\Theta^*(e^{j\omega}) \tag{4.6}$$

If $W(e^{j\omega})$ is causal and minimum phase, <u>then</u> $\Sigma_{yx}(e^{j\omega})$ <u>necessarily has a Gohberg Krein factorization with all</u> $k_i = 0$. <u>It is in fact not hard</u> also <u>to establish the converse</u>: if we have a factorization

$$\Sigma_{yx}(e^{j\omega}) = F_+(e^{j\omega})\, F_-(e^{j\omega}) = [F_+(e^{j\omega})H][H^{-1}F_-(e^{j\omega})] \tag{4.7}$$

then $W(e^{j\omega})$ is necessarily given, for some constant H, by

$$W(e^{j\omega}) = [F_+(e^{j\omega})H][H^{-1}F_-(e^{j\omega})]^{-*}$$
$$= F_+(e^{j\omega})HH^* F_-^{-*}(e^{j\omega}) \qquad (4.8)$$

and $W(e^{j\omega})$ is minimum phase. Thus study of $\Sigma_{yx}(e^{j\omega})$ enables parametrization of all $W(e^{j\omega})$, with the parametrization involving a type of constant gain matrix, here HH^*. As in the previous section, $\Sigma_{xx}(e^{j\omega})$ and $\Sigma_{yy}(e^{j\omega})$ impose bounds on the gain matrix:

$$HH^* \le F_+^{-1}(e^{j\omega})\Sigma_{yy}(e^{j\omega})F_+^{-*}(e^{j\omega}) \qquad \omega \ (4.9a)$$

$$(HH^*)^{-1} \le F_-^{-*}(e^{j\omega})\Sigma_{xx}(e^{j\omega})F_-^{-1}(e^{j\omega}) \qquad \omega \ (4.9b)$$

These inequalities are necessary and sufficient for the existence of nonnegative definite Hermitian $\Sigma_{uu}(e^{j\omega})$, $\Sigma_{vv}(e^{j\omega})$ satisfying (4.3). The spectrum $\Sigma_{\hat{x}\hat{x}}(e^{j\omega})$ is determined as $W^{-1}(e^{j\omega})\Sigma_{yx}(e^{j\omega})$, see (4.3), and is $F_-^*(e^{j\omega})H^{-*}H^{-1}F_-(e^{j\omega})$.

The case of nonminimum phase solutions is more complicated, and we outline the result with somewhat less explanation. First, nonminimum phase solutions correspond to Gohberg Krein factorizations of $\Sigma_{yx}(e^{j\omega})$ in which all k_i are nonnegative, and $k_i > 0$. [Should there be a factorization with a negative index, it is simply impossible for a set-up like Figure 1 with causal stable $W(z)$ to have produced the spectral matrix $\Sigma(e^{j\omega})$.] Suppose that

$$\Sigma_{yx}(e^{j\omega}) = F_+(e^{j\omega})D(e^{j\omega})F_-(e^{j\omega}) \qquad (4.10)$$

with $D(e^{j\omega}) = \mathrm{diag}[e^{j\omega k} \ 1]$. Then $W(e^{j\omega})$ and $\Sigma_{\hat{x}\hat{x}}(e^{j\omega})$ will solve our problem if and only if

$$W(e^{j\omega}) = F_+(e^{j\omega})D(e^{j\omega})H_-(e^{j\omega})H_-^*(e^{j\omega})F_-^{-*}(e^{j\omega}) \quad (4.11b)$$

$$\Sigma_{\hat{x}\hat{x}}(e^{j\omega}) = F_-^*(e^{j\omega})[H_-(e^{j\omega})H_-^*(e^{j\omega})]^{-1}F_-(e^{j\omega}) \quad (4.11b)$$

where H_- satisfies for all ω

$$H_-H_-^* \le D^{-1}F_+^{-1}\Sigma_{yy}F_+^{-*}D^{-1} \qquad (4.12a)$$

$$(H_-H_-^*)^{-1} \le F_-^{-*}\Sigma_{xx}F_-^{-1} \qquad (4.12b)$$

Finally, H_- is parameterised in a finite dimensional manner: H_- must be analytic together with its inverse in $|z| \ge 1$ and $D(e^{j\omega})H_-(e^{j\omega})$ must be analytic in $|z| \le 1$. The number of parameters in H_- is $n(n+K)$ where n is the dimension of x_k, y_k and $K = \Sigma k_i$.

There is an obvious parallel with the results of the previous sections. Note however that the treatment in this section is not the same as that of the nondynamic multivariable case discussed in Section 2, since all variables are not treated symmetrically. In particular, the input noise and output noise power spectra are not required to be diagonal, and thus the problem considered is really a generalization of the results of Section 3 to two vector variables, rather than a dynamic extension of section 2. It should be noted that, as in Section 3, if $\Sigma(e^{j\omega})$ evaluates as a constant or rational matrix, all solutions $W(e^{j\omega})$ are constant or rational.

All the above calculations assumed dimx = dimy. In case dimx ≠ dimy, we can extend the results above when $\Sigma(e^{j\omega})$ is rational, but not otherwise. (It is not how to check that there is no Gohberg Krein factorization of $[\exp(e^{j\omega}) \ 1]$ with F_- now 1x2).

A number of the results of this section can also be found in Picci and Pinzoni (1986). These authors start with the concept of a dynamic factor analysis model, in which there is no a priori assumption that one noiseless variable, $\hat{y}_k$ say, is determined by passing the other, $\hat{x}_k$, through a causal stable system. Rather, both $\hat{x}_k$ and $\hat{y}_k$ depend on a third variable w_k for which no measurements, noisy or noiseless, are available. Thus

$$x_k = \hat{x}_k + u_k = A_1(z)w_k + u_k \qquad (4.13a)$$

$$y_k = \hat{y}_k + v_k = A_2(z)w_k + v_k \qquad (4.13b)$$

The three processes $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ are mutually independent and stationary zero mean, with unknown power spectrum matrices. Now the nonuniqueness is even greater. For example, given one model (4.13), there always exists another model with the same spectrum for $[x_k \ y_k]'$, with $A_1(z)$, $A_2(z)$ causal and stable, and w_k white noise. One can seek to parametrise all models, or at least the irreducible ones - where w_k has least possible dimension-, and in Picci and Pinzoni (1986), special attention is given to those where the $\{w_k\}$ process is $[x_k' \ y_k']'$ measurable. The results earlier in this section apply when $A_1(z)$, $A_2(z)$ can be found such that $A_2(z) \ A_1(z)^{-L}$ is causal and stable.

5. DYNAMIC PROBLEMS WITH TWO SCALAR INPUTS AND ONE SCALAR OUTPUT

The ideas in this section extend, in a limited way, concepts raised in each of the preceding sections. The ideas are based on unpublished work of Anderson and Deistler.

Consider the arrangement depicted in Figure 3. This is a two input one output system, and we assume the noise processes u_1, u_2, u_3 are independent. Thus we are considering a dynamic three variable problem based on the approach of Section 2, not a special case of the theory of Section 4, where u_2 and u_3 would not be required to be independent and w_1, w_2 and the input spectrum would be required to be rational (since there are unequal numbers of inputs and outputs.)

The blocks w_2, w_3 in general are convolution blocks which define one linear relation among the processes $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$. In frequency domain terms

$$\hat{x}_1(e^{j\omega}) = w_2(e^{j\omega})\hat{x}_2(e^{j\omega}) + w_3(e^{j\omega})\hat{x}_3(e^{j\omega}) \quad (5.1)$$

To begin with, focus attention on a single fixed frequency ω_0, assuming available the 3x3 power spectrum matrix $\Sigma(e^{j\omega})$. This matrix is of course non-negative definite Hermitian, indeed positive definite Hermitian in the generic case. We are interested in a decomposition

$$\Sigma(e^{j\omega_0}) = \hat{\Sigma}(e^{j\omega_0}) + \tilde{\Sigma}(e^{j\omega_0}) \qquad (5.2)$$

in which $\tilde{\Sigma}$ is non-negative definite and diagonal and $\hat{\Sigma}$ is non-negative Hermitian and not full rank.

If $\hat{\Sigma}(e^{j\omega_0})$ has rank 2 (i.e. nullity 1) in (5.2) we can determine $w_2(e^{j\omega_0})$ and $w_3(e^{j\omega_0})$ to satisfy (5.1) evaluated at ω_0 via the null vector of $\hat{\Sigma}(e^{j\omega_0})$, since

$$[1 \ \ -w_2(e^{j\omega_0}) \ \ -w_3(e^{j\omega_0})] \times \begin{bmatrix} \hat{\sigma}_{11}(e^{j\omega_0}) & \hat{\sigma}_{12}(e^{j\omega_0}) & \hat{\sigma}_{13}(e^{j\omega_0}) \\ \hat{\sigma}_{21}(e^{j\omega_0}) & \hat{\sigma}_{22}(e^{j\omega_0}) & \hat{\sigma}_{23}(e^{j\omega_0}) \\ \hat{\sigma}_{31}(e^{j\omega_0}) & \hat{\sigma}_{32}(e^{j\omega_0}) & \hat{\sigma}_{33}(e^{j\omega_0}) \end{bmatrix} = 0 \qquad (5.3)$$

Thus the dynamic relationship (5.1) holds if and only if, for each fixed frequency ω_0, a decomposition (5.2) of $\Sigma(e^{j\omega_0})$ with $\hat{\Sigma}(e^{j\omega_0})$ rank 2 is possible. This suggests we should try to use Kalman's nondynamic result described in Section 2 to determine when this is the case. Kalman's result however deals only with real matrices, whereas we now need to consider complex matrices as we deal frequency by frequency with a Hermitian power spectrum. Consequently, we need to generalize Kalman's result to complex matrices. The result is follows:

> Theorem. Suppose Σ is a 3x3 positive definite Hermitian matrix of complex numbers with all entries non-zero. Let $S = \Sigma^{-1}$ and s_{ij}, σ_{ij} be the elements of S and Σ. The following are equivalent:
> (i) there exists a decomposition (5.2) with $\hat{\Sigma}$ of rank 1;
> (ii) $\sigma_{12}\sigma_{23}\sigma_{31}$ is real and positive and
> $$\sigma_{ii} \geq |\sigma_{ij}||\sigma_{ki}||\sigma_{jk}|^{-1} \text{ for } i \neq j \neq k;$$
> (iii) $s_{12}s_{23}s_{31}$ is real and negative.

The tie with Kalman's result in Section 2 is as follows. In the real Σ case, disregarding nongeneric situations, there are two posibilities: $s_{12}s_{23}s_{31} < 0$ (in which case $\hat{\Sigma}$ can have nullity 2 or rank 1) and $s_{12}s_{23}s_{31}>0$ (in which case $\hat{\Sigma}$ can never have nullity 2, but can have nullity 1). In the complex Σ case, $s_{12}s_{23}s_{31}$ may take a complex value. The theorem resolves whether or not under this circumstance, $\hat{\Sigma}$ can have nullity 2, by establishing this is in fact impossible.

The theorem can now be directly applied at all frequencies to determine whether or not there is one linear relation among the three processes (i.e. (5.1) holds). If there is just one linear relation possible we can consider whether or not causal w_2 and w_3 can be found consistent with an allowed linear relation. As in Section 3, the decisive quantity is the sign of the change in $\arg(w_i(e^{j\omega}))$, $i = 2,3$ as ω moves from 0 to 2π. It turns out that when $s_{12}(e^{j\omega})s_{23}(e^{j\omega})s_{31}(e^{j\omega})$ is not real and negative for any ω it is possible to determine the change around the unit circle in arg w_1, arg w_2 without actually computing the decomposition (5.2). That is, $\Delta\arg(w_i(e^{j\omega})$ $i = 2,3$ are independent of $\hat{\sigma}_{ii}(e^{j\omega})$. For there to be causal w_2, w_3 it is necessary for these changes in argument to be non-negative, as we saw in Section 3. Accordingly it is possible to check for the possibility of causal w_2, w_3 without actually calculating them first, as was also the case in Section 3.

The actual determination of causal w_2, w_3 however remains an open problem. If one does not insist on causal w_2, w_3 it is comparatively easy to characterize at each frequency the w_2, w_3 pairs which satisfy (5.3) for some $\hat{\Sigma} \geq 0$ with $\Sigma - \hat{\Sigma} \geq 0$ and diagonal, and hence to characterize not necessarily causal solutions to (5.1).

6. OPEN PROBLEMS

To conclude, we highlight some of the open problems. Outstanding seem:

(a) the real static problem with nullity greater than 1 in $\hat{\Sigma}$
(b) the complex static problem for more than three variables, even with nullity 1 for $\hat{\Sigma}$;
(c) the dynamic problem with causality assumption, in the first case with three variables.

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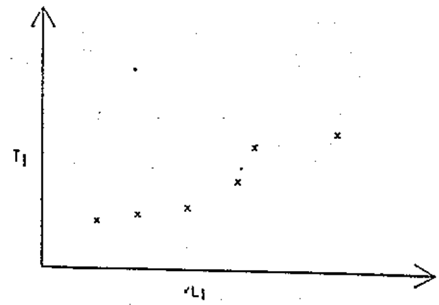

Figure 1: Period and square root of length of various pendula

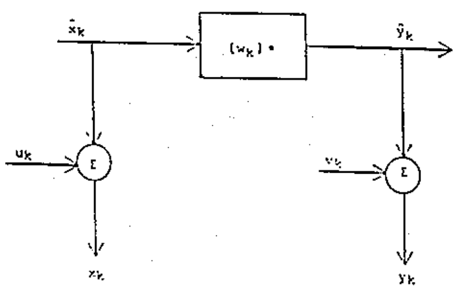

Figure 2: Set up for dynamic errors-in-variables problem with unknown scalar transfer function

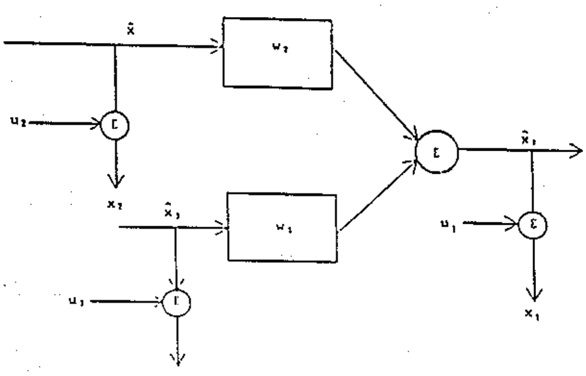

Figure 3: Set up for dynamic errors-in-variables problem with two unknown scalar transfer function