

# HILBERT TRANSFORMS FROM INTERPOLATION DATA

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**ABSTRACT:** This paper studies the generation of stable transfer functions for which the real or imaginary part takes prescribed values at discrete uniformly spaced points on the unit circle. Formulas bounding the error between a particular interpolating function and any function consistent with the data are presented, which have the desirable property that the error goes to zero exponentially fast with the number of interpolating points. The paper also examines generation of stable minimum phase transfer functions for which the magnitude takes prescribed values at uniformly spaced points on the unit circle, and presents error bounds for this problem. Connection with the discrete Hilbert transform is made. The effect of uncertainty in the original data is also examined.

## 1. INTRODUCTION

Suppose we have the real part of a (discrete-time) transfer function at  $n$  discrete points on the unit circle; moreover, suppose that the function is stable. How can one reconstruct the function, both real and imaginary parts, on the whole of the unit circle? A related problem is how to reconstruct, from values of the transfer function magnitude given at only  $n$  discrete points on the unit circle, the complete transfer function - magnitude and phase - on the entire unit circle. In addressing this problem we combine two ideas: interpolation and the Hilbert transform.

Apart from theoretical interest this problem has also practical motivation. Measurement data on a transfer function, either of the real part or of the magnitude, is probably only available at discrete frequencies: one must then recover the complete transfer function at all frequencies.

Given real part data of a stable transfer function on the entire unit circle, recovery of the imaginary part (and vice-versa) is a more-or-less standard problem, tackled via the Hilbert transform, see e.g.[1]. Recovery from phase or magnitude data given on the entire unit circle is almost as simple. Under the presumption that a transfer function is stable and minimum phase (all singularities of the function and its inverse lie in  $|z| < 1$ ), so that its logarithm is stable, then the logarithm of the gain and the phase, being the real and imaginary part of the logarithm of the function, are related by the Hilbert transform, and so can be determined from each other to within an arbitrary constant. Alternatively, given the magnitude squared, spectral factorization yields the whole function.

Error propagation is an important issue. Consider for example constructing the imaginary part from the real part of a stable transfer function. Then an arbitrarily small perturbation in the real part can induce an arbitrarily large change in the imaginary part. For suppose  $G(e^{j\omega}) = R(e^{j\omega}) + jI(e^{j\omega})$ . Then we have, see [1,p344],

$$I(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\theta}) \cot \frac{\theta - \omega}{2} d\theta \quad (1.1)$$

(The Cauchy principal value of the integral is to be calculated). Suppose there is a perturbation in  $R(e^{j\theta})$  by a constant amount  $\epsilon$  over  $(-\omega_0, \omega_0)$ . Then the corresponding perturbation in  $I(e^{j\omega})$  is

$$\Delta I(e^{j\omega}) = \frac{1}{2\pi} \epsilon \int_{-\omega_0}^{\omega_0} \cot \frac{\theta - \omega}{2} d\theta$$

Take  $\omega = -\omega_0$ ; then the integral diverges.

The same sort of discontinuity can arise in going between magnitude and phase of a minimum phase stable transfer function, see e.g. [2].

This discontinuity causes the comparative difficulty of doing Hilbert transforms with data at only discrete points. Suppose that one is given the values of the real part of a stable function at frequencies  $\omega_i = \exp(2\pi i/n)$   $i=0,1,\dots,n-1$ . A naive approach to reconstruction of the function, or an approximation to it, might involve fitting a curve through the interpolation data, and then Hilbert transforming as in (1.1). Because of the lack of continuity of the Hilbert transform, one could not expect, at least *a priori* that the reconstructed object is close to the true object.

Our approach exploits the "Hilbert transform for the discrete Fourier transform" [1, p.353]. Given  $\text{Re}\{G(e^{j2\pi k/n})\}$ ,  $k=0,1,\dots,n-1$ , there is a unique finite impulse response  $g(t)$ ,  $t=0,1,\dots,n-1$  with  $g(t)=0$  for  $[n/2] < t < n-1$  such that its discrete Fourier transform  $G(e^{j2\pi k/n})$  has a real part that coincides with the given data. Uniquely determined, therefore, is the transfer function

$$P(z) = \sum_{i=0}^{n-1} g(i)z^{-i}$$

While this guarantees interpolation in that

$$\text{Re}\{P(e^{j2\pi k/n})\} = \text{Re}\{G(e^{j2\pi k/n})\} \quad k = 0,1,\dots,n-1$$

it is not part of our problem specification that the given data corresponds to an FIR transfer function: in fact the "true" transfer function  $G(z)$  may be any member of the infinite class of stable functions which take the prescribed real values at certain discrete points on the unit circle. We hope that  $P(z)$  is "close" to the true function  $G(z)$ ; it is important to quantify the approximation's accuracy. Section 2 of this paper gives the construction  $P(z)$  and bounds the error between  $P(z)$  and  $G(z)$ .

The situation is less clear again when gain/phase data is involved. In [1, p.357], the following idea is advanced. Suppose  $G(z)$  is the  $z$ -transform of a finite length sequence of  $g(t)$ , and let  $\hat{G}(z) = \log G(z)$ . Choose  $n$  at least equal to twice the length of  $g(t)$ , and suppose that  $\hat{G}(\exp 2\pi jk/n)$  is known for  $k=0,1,\dots,n-1$ . Using the Hilbert transform of the discrete Fourier transform, a collection of values of argument is obtained. These are not exactly the values of  $\arg G(\exp 2\pi jk/n)$ , basically because  $\hat{G}(z)$  does not correspond to a finite impulse response. Further, when  $G(z)$  does not correspond to a finite length sequence, the validity (even in an approximate sense) is yet more difficult to pin down. Error formulas presented in Section 3, do however, display the validity. Also in Section 3 is an alternative approach, where we find an FIR, non-causal system which interpolates  $|G(e^{j2\pi k/n})|^2$ ,  $k=0,1,\dots,n-1$ ; then spectral factorization yields an FIR approximation to  $G(e^{j\omega})$ . For each of these problems, the error bounds become vanishingly small as the number of interpolating points becomes large.

Never is one given perfectly accurate data. In Section 4, we analyse the effect of uncertainty in the original (i.e. discrete frequency) data.

Illustrative examples follow in Section 5; the paper provides concluding remarks in Section 6.

## 2. FUNCTION CONSTRUCTION FROM REAL PART

### 2.1 Problem Statement

For an unknown transfer function  $G(z)$ , we are given the following information:

- (i)  $G(z)$  is analytic in  $|z| > \rho^{-1}$  for some  $\rho > 1$
- (ii) Values of  $\text{Re}[G(\exp j2\pi k/n)]$ ,  $k=0,1,\dots,n-1$
- (iii)  $G(z)$  corresponds to a real impulse response, so that  $\text{Re}[G(e^{j\omega})]$  and  $\text{Im}[G(e^{j\omega})]$  are respectively even and odd in  $\omega$ .

From this data, we seek

$$P(z) = \sum_{i=0}^v b_i z^{-i},$$

with  $n=2v$  or  $n=2v+1$  such that

$$\text{ReP}(\exp j2\pi k/n) = \text{ReG}(\exp j2\pi k/n) \quad k=0,1,\dots,n-1 \quad (2.1)$$

and we seek a value of  $K$  such that

$$\|G-P\|_{\infty} = \max_{\omega} \|G(e^{j\omega}) - P(e^{j\omega})\| < K \quad (2.2)$$

with the property that  $K \rightarrow 0$  as  $n \rightarrow \infty$  in a predictable fashion.

### 2.2 Construction of Interpolating Function $P(z)$

We proceed in two steps. We first construct a two-sided (non-causal) FIR  $Q(z)$  with values interpolating  $\text{ReG}(\exp j2\pi k/n)$ . Then from  $Q(z)$  we find a (causal) FIR  $P(z)$  satisfying (2.1). We treat the cases of  $n$  odd and  $n$  even separately.

(a)  $n$  odd,  $n = 2v+1$ .

By DFT methods, it is straightforward to construct  $Q(z)$ , of the form

$$Q(z) = b_v z^v + b_{v-1} z^{v-1} + \dots + b_0 + \dots + b_{-v+1} z^{-(v-1)} + b_{-v} z^{-v} \quad (2.3)$$

such that

$$Q(\exp j2\pi k/n) = \text{Re}[G(\exp j2\pi k/n)] \quad k=0,1,\dots,n-1 \quad (2.4)$$

Of course, with the FIR constraint of (2.3), the function  $Q(z)$  is unique. Because  $\text{ReG}(e^{j\omega})$  is even in  $\omega$ ,

$$Q(\exp -j2\pi k/n) = \text{Re}[G(\exp j2\pi k/n)]$$

so that  $Q(z^{-1})$  also interpolates the data. Hence, by uniqueness,  $Q(z) = Q(z^{-1})$  and  $b_i = b_{-i}$ . Also, if  $\bar{Q}(z)$  denotes  $Q(z)$  with complex conjugations of every coefficient, a similar argument gives  $\bar{Q}(z) = Q(z)$ . So  $b_i$  is real for all  $i$ .

From  $Q(z)$ , we define  $P(z)$  as

$$P(z) = b_0 + 2b_{-1} z^{-1} + \dots + 2b_{-v} z^{-v} \quad (2.5)$$

From (2.3) through (2.5), it is evident that the interpolation requirement (2.1) is satisfied.

Notice that  $P(z)$  is the solution of a conventional Hilbert transform problem: given  $Q(e^{j\omega}) = \text{ReP}(e^{j\omega})$ , find  $P(z)$  analytic in  $|z| > \rho^{-1}$ , for some  $\rho > 1$ .

(b)  $n$  even,  $n = 2v$ .

We construct  $Q(z)$  which interpolates  $\text{Re}[G(\exp j2\pi k/n)]$  and has the form

$$Q(z) = b_{v-1} z^{v-1} + b_{v-2} z^{v-2} + \dots + b_0 + \dots + b_{-v+1} z^{-v+1} + b_{-v} z^{-v} \quad (2.6)$$

This determines  $Q(z)$  uniquely.

By a similar argument to above,  $b_i = b_{-i}$ , and  $b_i$  is real,  $i=0,\dots,v$ .

From  $Q(z)$ , we define  $P(z)$  as

$$P(z) = b_0 + 2b_{-1} z^{-1} + \dots + 2b_{-v+1} z^{-v+1} + b_{-v} z^{-v} \quad (2.7)$$

It is easy to verify that (2.1) holds. Comparison of (2.7) with (2.5) shows that the coefficient of  $z^{-v}$  differs by a factor of two between  $n$ -even and  $n$ -odd cases.

Since  $\text{ReP}(e^{j\omega}) \neq Q(e^{j\omega})$ , it is not true that  $P(z)$  is the solution of a Hilbert transform problem with data  $Q(e^{j\omega})$ , in contrast to the case  $n=2v+1$ .

### 2.3 Error in the Real Part

In this subsection, we shall examine the error in  $|\text{Re}[P(e^{j\omega})] - \text{Re}[G(e^{j\omega})]|$ . To derive an error bound, we need some preliminary results. First we recall from [4]:

**Lemma 2.1:** Let  $H(z)$  be a transfer function analytic in  $\rho^{-1} < |z| < \rho$  for some  $\rho > 1$ . If

$$Q_{2v}(z) = \sum_{i=-v}^{v-1} b_i z^i$$

interpolates  $H(z)$  in the  $(2v)$ th roots of unity, then for any  $R_1$ , with  $1 < R_1 < \rho$

$$\|H - Q_{2v}\|_{\infty} < \frac{2R_1^v}{(R_1^{2v}-1)(R_1-1)} \left[ \max_{\theta \in (-\pi, \pi)} |H(R_1^{-1} e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |H(R_1 e^{j\theta})| \right] \quad (2.8a)$$

Also, if

$$Q_{2v+1}(z) = \sum_{i=-v}^v b_i z^i$$

interpolates  $H(z)$  in the  $(2v+1)$ -th roots of unity, then for any  $R_1$ , with  $1 < R_1 < \rho$ ,

$$\|H - Q_{2v+1}\|_{\infty} < \frac{2R_1^{v+1}}{(R_1^{2v+1}-1)(R_1-1)} \left[ \max_{\theta \in (-\pi, \pi)} |H(R_1^{-1} e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |H(R_1 e^{j\theta})| \right] \quad (2.8b)$$

Actually, only the result for even  $n$  is proved in [4]. However, the result for odd  $n$  follows by virtually the same argument.

We shall also need a further result on the behaviour of the last coefficient  $b_{-v}$  in a  $(2v)$ -th roots of unity interpolation.

**Lemma 2.2:** Assume the same hypotheses as Lemma 2.1, with  $n=2v$ . Then

$$|b_{-v}| < \frac{R_1^v}{R_1^{2v}-1} \left[ \max_{\theta \in (-\pi, \pi)} |H(R_1^{-1} e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |H(R_1 e^{j\theta})| \right] \quad (2.9)$$

**Proof:** Omitted due to space limitations, as are all subsequent proofs.

With these lemmas in hand, we can now bound the error between  $\text{Re}[P(e^{j\omega})]$  and  $\text{Re}[G(e^{j\omega})]$ .

**Theorem 2.1:** Assume  $G(z)$  satisfies the assumptions listed in Section 2.1. Let  $P(z)$  be defined as described in Section 2.2. If  $n = 2v+1$ , there holds for any  $R_1$ , with  $1 < R_1 < \rho$

$$|\text{ReG}(e^{j\omega}) - \text{ReP}(e^{j\omega})| < \frac{2R_1^{v+1}}{(R_1^{2v+1}-1)(R_1-1)} \left[ \max_{\theta \in (-\pi, \pi)} |G(R_1 e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |G(R_1^{-1} e^{j\theta})| \right] \quad (2.10a)$$

while if  $n = 2v$ , there holds

$$|\text{ReG}(e^{j\omega}) - \text{ReP}(e^{j\omega})| < \frac{R_1^v}{(R_1^{2v}-1)} \frac{R_1+1}{R_1-1} \left[ \max_{\theta \in (-\pi, \pi)} |G(R_1 e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |G(R_1^{-1} e^{j\theta})| \right] \quad (2.10b)$$

**Remark:** The error is effectively of the form  $K R_1^{-n/2}$ .

The fact that we have a bound on  $\text{Re}[G(j\omega) - P(j\omega)]$  does not yield immediately a bound on  $\text{Im}[G(j\omega) - P(j\omega)]$ , in view of the discontinuity argument presented in Section 1. However, by obtaining an additional bound on the derivative of  $\text{Re}[G(j\omega) - P(j\omega)]$ , we will then be able to obtain the bound on  $\text{Im}[G(j\omega) - P(j\omega)]$ .

### 2.4 Error in the Derivative of the Real Part

In this subsection, we shall derive an error bound for  $|d/d\omega[\text{ReG}(e^{j\omega}) - \text{ReP}(e^{j\omega})]|$ . This is probably not of intrinsic

interest. However, we use this bound to construct a bound for  $|G(e^{j\omega}) - P(e^{j\omega})|$  in the next subsection.

**Theorem 2.2:** Assume  $G(z)$  satisfies the assumptions listed in Section 2.1. Let  $P(z)$  be defined as described in Section 2.2. Then for  $n = 2v$ , for any  $R_1$ , with  $1 < R_1 < \rho$ , there holds

$$\left\| \frac{d}{d\omega} \{ \text{Re}G(e^{j\omega}) - \text{Re}P(e^{j\omega}) \} \right\|_{\infty} < \frac{R_1^v}{R_1^{2v-1}} \frac{2(R_1^2+1)^{v+R_1+1}}{(R_1^2-1)} \left[ \max_{\theta \in (-\pi, \pi)} |G(R_1 e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |G(R_1^{-1} e^{j\theta})| \right] \quad (2.11a)$$

and for  $n = 2v+1$ , there holds

$$\left\| \frac{d}{d\omega} \{ \text{Re}G(e^{j\omega}) - \text{Re}P(e^{j\omega}) \} \right\|_{\infty} < \frac{2R_1^{v+1}}{R_1^{2v+1-1}} \frac{(R_1+1)(v+1)}{(R_1-1)^2} \left[ \max_{\theta \in (-\pi, \pi)} |G(R_1 e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |G(R_1^{-1} e^{j\theta})| \right] \quad (2.11b)$$

**Remark:** The error behaves as  $K(n/2 + 1)R_1^{-(n/2)}$ .

### 2.5 Errors in the approximation of the function $G(e^{j\omega})$

Our goal here is to show that  $P(e^{j\omega})$  is a good approximation of  $G(e^{j\omega})$ , and becomes better as  $n$  increases. The information at our disposal is that  $\text{Re}P(e^{j\omega})$  and its derivative are good approximations of  $\text{Re}G(e^{j\omega})$  and its derivative. It is the closeness of these derivatives as well as the real parts themselves which means that the Hilbert transform yields answers which are close, see e.g. [2].

**Lemma 2.3:** Let  $M(e^{j\omega}) = M_R(e^{j\omega}) + jM_I(e^{j\omega})$ , with  $M_R(e^{j\omega})$  real and even in  $\omega$ ,  $M_I(e^{j\omega})$  real and odd in  $\omega$ , and  $M(e^{j\omega})$  the Fourier transform of a real, causal impulse response, such that  $\|M_R(e^{j\omega})\|_{\infty}$  and  $\|d/d\omega M_R(e^{j\omega})\|_{\infty}$  are both finite. Then

$$M_I(e^{j\omega}) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \left\{ \frac{d}{d\theta} M_R(e^{j\theta}) \right\} \ln \left| \sin \frac{\theta-\omega}{2} \right| d\theta \quad (2.14)$$

$$\|M_I(e^{j\omega})\|_{\infty} < 2 \left\| \frac{d}{d\omega} M_R(e^{j\omega}) \right\|_{\infty} \quad (2.15)$$

To return to the interpolation problem of interest to us, we simply identify  $M(z)$  in the Lemma with  $G(z) - P(z)$ . Putting together Theorem 2.1, Theorem 2.2 and the Lemma, we have

**Main Theorem 2.3:** Suppose  $G(z)$  is analytic in  $|z| > \rho^{-1}$  for some  $\rho > 1$ , that values of  $\text{Re}\{G(\exp j2\pi k/n)\}$ ,  $k=0,1,\dots,n-1$  are prescribed, and that  $G(z)$  corresponds to a real impulse response. Then the unique polynomial  $P(z)$  of degree  $v=n/2$  or  $v=(n-1)/2$  in  $z^{-1}$  such that,

$$\text{Re}P(\exp j2\pi k/n) = \text{Re}G(\exp j2\pi k/n) \quad k=0,1,\dots,n-1$$

has the property that for any  $R_1$ , with  $1 < R_1 < \rho$ ,

$$\|G(e^{j\omega}) - P(e^{j\omega})\|_{\infty} < [K_1(R_1)^v + K_2(R_1)] \left[ \max_{\theta \in (-\pi, \pi)} |G(R_1 e^{j\theta})| + \max_{\theta \in (-\pi, \pi)} |G(R_1^{-1} e^{j\theta})| \right]$$

with  $K_1(R_1)$ ,  $K_2(R_1)$  going to zero as  $R_1^{-v}$ .

### 3. INTERPOLATION OF GAIN-PHASE DATA

In this section, we shall concentrate on the problem where the given data is magnitude data. Of course, very similar results apply when the data is phase data. There are two broad approaches, summed up in Figures 3.1 and 3.2.

#### 3.1 Problem Statement

For an unknown transfer function  $G(z)$ , we are given the following information:

- (i)  $G(z)$  and  $G^{-1}(z)$  are analytic in  $|z| > \rho^{-1}$  for some  $\rho > 1$
- (ii) Values of  $|G(\exp j2\pi k/n)|$  for  $k=0,1,\dots,n-1$
- (iii)  $G(z)$  corresponds to a real impulse response, so that  $\ln|G(e^{j\omega})|$  is even and  $\arg G(e^{j\omega})$  is odd in  $\omega$ .

From this data, we seek to find an interpolant of  $G(z)$  of the form

$$P(z) = \sum_{i=0}^v b_i z^{-i} \text{ or } P(z) = \exp\left(\sum_{i=0}^v b_i z^{-i}\right)$$

for  $n=2v$  or  $n=2v+1$ , such that

$$|P(\exp j2\pi k/n)| = |G(\exp j2\pi k/n)| \quad k=0,1,\dots,n-1 \quad (3.1)$$

and we seek a bound of the form

$$\|1 - G^{-1}P\|_{\infty} = \max_{\omega} \|1 - G^{-1}(e^{j\omega})P(e^{j\omega})\| < K \quad (3.2)$$

such that  $K \rightarrow 0$  as  $n \rightarrow \infty$  in a predictable fashion.

#### 3.2 First Approach: Interpolation of $|G|$

Figure 3.1 outlines this approach to the construction of  $P$ . Suppose first that  $n=2v+1$ . We find

$$Q(z) = \sum_{i=-v}^v b_i z^i \quad (3.3)$$

such that  $Q(\exp j2\pi k/n) = |G(\exp j2\pi k/n)|^2$ . Arguing as in Section 2, it follows that  $b_i = b_{-i}$  and is real. We shall argue later that for large enough  $v$ ,  $Q(e^{j\omega}) > 0$  for all  $\omega$ . It follows that we can factor  $Q(z)$  as

$$Q(z) = P(z)P(z^{-1}) \quad (3.4)$$

with

$$P(z) = c_0 + c_{-1}z^{-1} + \dots + c_{-v}z^{-v} \quad c_i \text{ real} \quad (3.5)$$

and all zeros of  $P(z)$  inside  $|z| < 1$ .  $P(z)$  is then the solution of the interpolation problem.

In case  $n=2v$ , define

$$\hat{Q}(z) = \sum_{i=-v}^{v-1} b_i z^i \quad (3.6)$$

such that  $\hat{Q}(\exp j2\pi k/n) = |G(\exp j2\pi k/n)|^2$ , and then set

$$Q(z) = \frac{1}{2}b_{-v}z^v + \sum_{i=-(v-1)}^{v-1} b_i z^i + \frac{1}{2}b_{-v}z^{-v} \quad (3.7)$$

At the points  $\exp j2\pi k/n$ ,  $\hat{Q}$  takes the same values as  $Q$ . As before,  $b_i = b_{-i}$  and is real. Again, for large enough  $v$ ,  $Q(e^{j\omega}) > 0$  for all  $\omega$ . We define  $P(z)$  as in (3.4), (3.5).

In both cases  $n=2v$  and  $n=2v+1$ , we have

$$|G(\exp j2\pi k/n)| = |P(\exp j2\pi k/n)| \quad k=0,1,\dots,n-1 \quad (3.8)$$

The determination of  $P(z)$  from  $Q(z)$  in (3.4) is not necessarily an easy task. It should be noted however that procedures exist which do not rely on determination of all the zeros of  $Q(z)$ , see e.g. [6,7] for a linearly convergent algorithm and [8] for a quadratically convergent algorithm.

#### 3.3 Error Bounds

We shall set out the arguments justifying the error bounds in a much briefer way than in the last section.

**Lemma 3.1:** Assume  $G(z)$  satisfies the assumptions listed in Section 3.1 and  $H(z) = G(z)G(z^{-1})$ . Let  $Q(z)$  be formed as described in Section 3.2. Then for any  $R_1$ , with  $1 < R_1 < \rho$

$$\|H(e^{j\omega}) - Q(e^{j\omega})\|_{\infty} < R_1^v \alpha(R_1) \max_{\theta \in (-\pi, \pi)} |H(R_1 e^{j\theta})| \quad (3.9)$$

$$\left\| \frac{d}{d\omega} [H(e^{j\omega}) - Q(e^{j\omega})] \right\|_{\infty} < R_1^v [\beta(R_1)^{v+\gamma(R_1)}] \max_{\theta \in (-\pi, \pi)} |H(R_1 e^{j\theta})| \quad (3.10)$$

Here  $\alpha(R_1)$ ,  $\beta(R_1)$  and  $\gamma(R_1)$  are functions of  $R_1$ , which are independent of  $v$ , but will vary according as  $n$  is even or odd.

**Proof:** Recognize that  $Q$  interpolates  $H$  at  $\exp(j2\pi k/n)$  in case  $n$  is odd, and  $Q + \frac{1}{2}b_{-v}z^{-v} - \frac{1}{2}b_{-v}z^v$  interpolates  $H$  in case  $n$  is even. Then the same arguments as yielded Lemma 2.1, Lemma 2.2, Theorem 2.1 and Theorem 2.2 yield (3.9) and (3.10).

**Remark:** At this stage, we can justify our earlier claim that  $Q(e^{j\omega}) > 0$  for suitably large  $n$ . Because  $G$  has all its zeros inside  $|z| < \rho^{-1}$ ,  $|G|$  is nonzero on  $|z|=1$ . Hence  $|H|$  is nonzero on  $|z|=1$ . By (3.9), for suitably large  $n$ , we can ensure that

$$\|H-Q\|_\infty < \min |H(e^{j\omega})|, \quad |z|=1$$

Then  $Q(e^{j\omega}) > 0$  for all  $\omega$ .

We need to deduce several consequences of this Lemma. The first two relate to the behaviour of  $|PG^{-1}|$  and its derivative.

**Corollary 3.1:** Assume the same hypotheses as for Lemma 3.1, and assume that  $P(z)$  is defined as described in Section 3.2. Suppose that

$$|G(e^{j\omega})| > c \quad \forall \omega \quad (3.11)$$

Set

$$M = \max_{\theta \in (-\pi, \pi]} |H(R_1 e^{j\theta})| \quad (3.12)$$

Then

$$\left\| 1 - \frac{P}{G} \right\|_\infty < c^{-2} R_1^\gamma \alpha(R_1) M \quad (3.13)$$

**Corollary 3.2:** Assume the same hypotheses as for Lemma 3.1, and assume that  $P(z)$  is defined as in Section 3.2. With  $c, M$  as in Corollary 3.1, and

$$d = \left\| \frac{d|G(e^{j\omega})|}{d\omega} \right\|_\infty \quad (3.14)$$

there holds, for sufficiently large  $n$ , say  $n > N$ ,

$$\left\| \frac{d}{d\omega} \frac{P}{G} \right\|_\infty < \frac{\frac{1}{2} c^{-2} \beta(R_1) M}{1 - c^{-2} R_1^\gamma \alpha(R_1) M} v R_1^\gamma + \left[ \frac{\frac{1}{2} c^{-2} \gamma(R_1) M + \alpha(R_1) c^{-3} d}{1 - c^{-2} R_1^\gamma \alpha(R_1) M} \right] R_1^\gamma \quad (3.15)$$

Corollaries 3.1 and 3.2 allow us to construct a bound on  $\arg(P/G)$ . The tool is to appeal to much the same argument as used in Section 2, where we had bounds on the derivative of the real part of a function, and obtained a bound on its imaginary part.

**Lemma 3.2:** Assume the same hypotheses as Lemma 3.1 and suppose  $P(z)$  is determined as described in Section 3.2. Then for all  $n$  exceeding some  $N$ , there exist constants  $K, L$  depending on

$$\inf_{\omega} |G(e^{j\omega})|, \left\| \frac{dG}{d\omega} \right\|_\infty, R_1, \text{ and } \max_{\theta \in (-\pi, \pi]} |H(R_1 e^{j\theta})|$$

such that

$$\left\| \arg \frac{P}{G} \right\|_\infty < (Kv + L) R_1^\gamma \quad (3.16)$$

Finally this brings us to

**Main Theorem 3.1:** Let  $G(z)$  be an unknown transfer function with  $G(z), G^{-1}(z)$  analytic in  $|z| > \rho^{-1}$  for some  $\rho > 1$ , and with values of  $|G(\exp j2\pi k/n)|$  prescribed for  $k=0,1,\dots,n-1$ . Suppose  $G(z)$  corresponds to a real impulse response, so that  $\ln|G(e^{j\omega})|$  and  $\arg G(e^{j\omega})$  are even and odd in  $\omega$ . Let  $P(z)$  be a polynomial in  $z^{-1}$  of degree  $v$ , where  $n=2v$  or  $2v+1$ , with all zeros (regarded as a function of  $z$ ) in  $|z| < 1$ , such that  $P$  interpolates  $G$  at  $\exp j2\pi k/n$ , with  $P(z)$  determined according to the procedures of Section 3.2. Then

$$\|1 - G^{-1}P\|_\infty < (K_1 v + L_1) R_1^\gamma \quad (3.17)$$

for constants  $K_1, L_1$  depending on

$$\inf_{\omega} |G(e^{j\omega})|, \left\| \frac{dG}{d\omega} \right\|_\infty, R_1, \text{ and } \max_{\omega} |H(R_1 e^{j\omega})|,$$

where  $H(z) = G(z)G(z^{-1})$ .

### 3.4 Second Approach: Interpolation of $\ln(G)$

In the second approach, see Figure 3.2, we obtain a rational approximation of  $\log|G|$  and from it a rational (FIR) approximation of  $\log G$ . Exponentiation gives an approximation of  $G$ . Of course, the object approximating  $G$  is no longer rational, let alone FIR.

In more detail, let

$$Q(z) = \sum_{i=-v}^v b_i z^i \quad (n=2v+1) \quad (3.18a)$$

$$= \sum_{i=-v}^{v-1} b_i z^i \quad (n=2v) \quad (3.18b)$$

interpolate  $\log|G(\exp j2\pi k/n)|$  for  $k=0,1,\dots,n-1$ . Let

$$T(z) = b_0 + 2 \sum_{i=-v}^{-1} b_i z^i \quad (n=2v+1) \quad (3.19a)$$

$$= \frac{b_0}{2} + 2 \sum_{i=-(v-1)}^{-1} b_i z^i + b_{-v} z^{-v} \quad (n=2v) \quad (3.19b)$$

Then  $T(z)$  is an approximant of  $\ln G(z)$ . The line of argument of Section 2 carries through to yield the following conclusions: For any  $R_1$  with  $1 < R_1 < \rho$ , and with

$$M = \max_{\theta \in (-\pi, \pi]} [|\ln G(R_1 e^{j\theta})|] + \max_{\theta \in (-\pi, \pi]} [|\ln G(R_1^{-1} e^{j\theta})|] \quad (3.20)$$

there holds

$$\| \operatorname{Re} T(e^{j\omega}) - \ln |G(e^{j\omega})| \|_\infty < K(R_1) R_1^\gamma M \quad (3.21a)$$

$$\left\| \frac{d}{d\omega} [\operatorname{Re} T(e^{j\omega}) - \ln |G(e^{j\omega})|] \right\|_\infty < [K_1(R_1) + K_2(R_1) v] R_1^\gamma M \quad (3.21b)$$

and, using Lemma 2.3 to bound  $\| \operatorname{Im} T(e^{j\omega}) - \arg G(e^{j\omega}) \|_\infty$ ,

$$\begin{aligned} \|T(e^{j\omega}) - \log G(e^{j\omega})\|_\infty &< \| \operatorname{Re} T(e^{j\omega}) - \ln |G(e^{j\omega})| \|_\infty + \| \operatorname{Im} T(e^{j\omega}) - \arg G(e^{j\omega}) \|_\infty \\ &< K(R_1) R_1^\gamma M + 2[K_1(R_1) + vK_2(R_1)] R_1^\gamma M \\ &= [K_1(R_1) + vK_2(R_1)] R_1^\gamma M \end{aligned} \quad (3.22)$$

With  $P(z) = \exp T(z)$ , we can obtain an equivalent inequality involving  $G^{-1}P$  as follows. By (3.21b),

$$\| \ln(|P|/|G|) \|_\infty < [K_1(R_1) + vK_2(R_1)] R_1^\gamma M$$

so that

$$\begin{aligned} |P|/|G| &< \exp [K_1(R_1) + vK_2(R_1)] R_1^\gamma M \\ |P|/|G| &> \exp -[K_1(R_1) + vK_2(R_1)] R_1^\gamma M \end{aligned} \quad (3.23a)$$

Also, as argued above,

$$\| \arg P - \arg G \|_\infty < 2[K_1(R_1) + vK_2(R_1)] R_1^\gamma M \quad (3.23b)$$

Hence, as in the proof of Main Theorem 3.1,

$$\left\| 1 - \frac{P}{G} \right\|_\infty < \left\| 1 - \frac{P}{G} \right\|_\infty + \left\| \arg \frac{P}{G} \right\|_\infty$$

and from (3.23) we see that decay at a rate  $R_1^\gamma$  is evident for  $\|1 - G^{-1}P\|_\infty$ .

The above procedure generates a rational approximation of  $\log G$ . There is a minor variation that will generate a rational approximation of  $G$ : we find a

$$\bar{P}(z) = \sum_{i=0}^{n-1} \bar{p}_i z^{-i}$$

such that  $\bar{P}(z)$  interpolates  $\exp T(z)$  at  $z = \exp(j2\pi k/n)$ . Note that this means that  $\bar{P}(z)$  interpolates  $|G(z)|$  at the same points. The theory of [3] ensures that

$$\| \bar{P}(e^{j\omega}) - P(e^{j\omega}) \|_\infty = \| \bar{P}(e^{j\omega}) - \exp T(e^{j\omega}) \|_\infty < K R_1^{-n} \quad (3.24)$$

where  $K$  depends on

$$\max_{(-\pi, \pi]} |\exp T(R_1^{-1} e^{j\theta})|.$$

This bound can be combined with that for  $\|1 - PG^{-1}\|_\infty$  to obtain a bound for  $\|1 - PG^{-1}\|_\infty$ .

We remark that in case  $n$  is even, the construction of  $T(z)$  is just that of [1, p.357], where the discrete Hilbert transform is suggested as a procedure for recovering phase data. The inequality (3.22) allows us to pin down the error that may result in its use, without any prior assumption that  $G(z)$  or  $\log G(z)$  corresponds to a finite impulse response.

#### 4. ERRORS IN THE DATA

To this point, we have considered the construction of a transfer function from data (i.e. real part, or magnitude at discrete points on the unit circle) which has been assumed error free. In this section, we postulate the existence of errors in this data, and consider the resulting (extra) errors which now arise between the true function and that constructed from the error-contaminated data.

We shall be particularly interested in the behaviour of the error with increasing  $n$ , the number of interpolating points. Most error models will permit a discrete increase in value at one point, and discrete decrease at an adjacent point. This means that as  $n \rightarrow \infty$ , i.e. as the spacing between adjacent points decreases, the derivative (with respect to angle around the unit circle) can approach infinity. In the light of remarks earlier in this paper and in [2], we must expect that the error in the constructed function could in some way diverge as  $n \rightarrow \infty$ . This happens in a deterministic, but not stochastic, error model.

##### 4.1 Stochastic Approach to Errors

To fix ideas, let us consider the construction of a function from discrete values of its real part (a similar discussion would apply for magnitude data). To pursue a statistical approach, we might suppose that the values of  $\text{Re}G(z)$  at  $z = \exp j2\pi k/n$ ,  $k = 0, \dots, n-1$  are perturbed by independent, zero mean gaussian random variables of variance  $\sigma^2$ . However, we must reflect the fact that  $\text{Re}G$  is necessarily even. So this suggests that we assume  $G(1)$  is known to within standard deviation  $\sigma$ , and  $\text{Re}G(\exp j2\pi k/n)$  for  $k < (n/2)$  to within standard deviation  $\sigma/2$  (where we exploit the fact that  $\text{Re}G(\exp j2\pi k/n)$  and  $\text{Re}G(\exp j2\pi(n-k)/n)$  take the same value.)

Let  $P$  be the interpolating FIR function that would be associated with noiseless data, and let  $\hat{P}$  be the interpolating FIR function that is associated with noisy data. Since

$$\|G - \hat{P}\|_{\infty} \leq \|G - P\|_{\infty} + \|P - \hat{P}\|_{\infty} \quad (4.1)$$

the damage done by the noisy data is reflected in  $\|P - \hat{P}\|_{\infty}$ , which is actually a random variable.

Space limitations prevent us giving the derivation, but it is not difficult to show that

$$E[|P(e^{j\omega}) - \hat{P}(e^{j\omega})|^2] = \sigma^2 \quad \forall \omega \in (-\pi, \pi) \quad (4.2)$$

We see then that at any one frequency  $|G - \hat{P}|$  differs from  $|G - P|$  by a random variable of variance  $\sigma^2$ . In particular, at that frequency for which  $|G - P|$  is maximum, i.e.  $\|G - P\|_{\infty} = |G - P|$ , we have  $|G - \hat{P}|$  differing from  $\|G - P\|_{\infty}$  by a random variable of variance  $\sigma^2$ . However, we cannot make a firm statement about  $\|G - \hat{P}\|_{\infty}$ .

##### 4.2 Deterministic Approach to Error

We shall now show, with the aid of an example, that if errors of up to  $\pm\Delta$  are postulated in the value of the real part of  $\text{Re}\{\hat{P}(e^{j\theta})\}$  at interpolating points,  $\|P - \hat{P}\|_{\infty}$  can grow as  $\ln n$ . This shows that in the presence of deterministic errors, the benefit obtained from finer and finer sampling (convergence in the sense of Section 2 or 3) is eventually outweighed by the effect of the errors in the sampled values.

Take  $n$  as a multiple of 4, viz  $n = 2v = 4u$ . Consider errors  $\text{Re}\{\hat{P}(e^{j\theta})\}$  with the property that

$$\begin{aligned} \text{Re}\{\hat{P}(\exp j2\pi k/n)\} &= \Delta \text{ when } \text{Re}(\exp j2\pi k/n) > 0 \\ &= 0 \text{ when } \text{Re}(\exp j2\pi k/n) = 0 \\ &= -\Delta \text{ when } \text{Re}(\exp j2\pi k/n) < 0 \end{aligned} \quad (4.3)$$

With some manipulation one can show that

$$\|P - \hat{P}\|_{\infty} = |P(\exp j\pi/2)| = \frac{4\Delta}{\pi} \sum_{m=1}^{n/4} \cot(m-\frac{1}{2})\theta \quad (4.4)$$

It is not hard then to prove that

$$\Delta \left[ \frac{2}{\pi} \ln n - \frac{2}{\pi} \ln 2 + 2 \right] > |P(\exp j\pi/2)| > \Delta \left[ \frac{2}{\pi} \ln n - \frac{2}{\pi} \ln \pi \right] \quad (4.5)$$

Hence it is clear that  $|P(\exp j\pi/2)|$  grows as  $\ln n$ . In absolute terms, the values stay modest for large values of  $n$ , e.g.  $n = 4096$  yields  $|P(\exp j\pi/2)| = 5.37524$ .

As foreshadowed, this example shows that the error can certainly diverge as fast as  $\ln n$ . Whether or not there exist

other constructions [i.e. choice of errors of which (4.3) is one example] such that divergence occurs faster than  $\ln n$  is an open problem. However, a simple calculation shows that divergence occurs no faster than  $n^{1/2}$ .

#### 5. EXAMPLES

We present an example of generating an approximate frequency response from real part interpolation data. With frequency response plots we illustrate the salient features of the method, including the reduction in approximation error as the number of interpolation points increases.

The example uses the stable transfer function

$$G(z) = \frac{1}{z^2 + 0.64} + \frac{1}{z - 0.5}$$

(poles are at  $z = \pm j0.8, 0.5$ ) and we evaluate, for (a)  $n=8$  and (b)  $n=16$ ,  $\text{Re} G(e^{j2\pi k/n})$   $k=0, 1, \dots, n-1$ . From this real part data, we construct  $P(z)$ , causal and FIR according to equations (2.3)-(2.5). Figure 5.1 charts the frequency responses  $\text{Re}G(e^{j\omega})$  and  $\text{Re}P(e^{j\omega})$ ;  $\text{Im}G(e^{j\omega})$  and  $\text{Im}P(e^{j\omega})$ ;  $|G(e^{j\omega}) - P(e^{j\omega})|$ . Comparison of Figure 5.1 (a)  $n=8$  and (b)  $n=16$  shows the effect of increasing the number of interpolation points. For  $n=8$  the error  $\|G - P\|_{\infty} = 1.6$ ; for  $n=16$ ,  $\|G - P\|_{\infty} = 0.6$ .

These plots illustrate other points of interest. Note that the real parts of  $G(e^{j\omega})$  and  $P(e^{j\omega})$  coincide at the equispaced interpolation points, but, as expected, the imaginary parts do not act likewise. Observe carefully the frequencies where the error  $|G(e^{j\omega}) - P(e^{j\omega})|$  is large. These are not coincident with frequencies where  $\text{Re}G(e^{j\omega})$  is large, but rather at those frequencies where  $|d/d\omega \text{Re}G(e^{j\omega})|$  is large, consistent with the importance we placed on the derivative of  $\text{Re}G(e^{j\omega})$  in the error bounds of Section 3.

The example of this section is in qualitative agreement with the theoretical calculations of preceding sections. (Very lengthy calculations are required to check quantitative agreement and hardly seem justified). The example illustrates that a good approximation to a frequency response given only real part interpolation data is indeed possible, provided the underlying transfer function is stable and sufficient interpolation points are taken. The error phenomena associated with imprecise data and described in the previous section generally only become of concern for larger values of  $n$ , far larger in fact than the values required in these examples to secure good approximation.

#### 6. CONCLUSIONS

Despite the discontinuity questions associated with use of the Hilbert transform, we have been able to clarify the circumstances under which Hilbert transform calculations recover a transfer function from discrete frequency data of its real part, imaginary part, magnitude, or phase.

With a finite number of points, it is never possible to recover the transfer function exactly, unless the class of considered transfer functions is heavily restricted *a priori* e.g. to being FIR. Instead, we can hope to recover an FIR function which is close to the desired transfer function, with the error going to zero as the number of data points becomes infinite.

However, data errors impose a practical limit to this result.

There are a number of further questions which future work could examine. One could seek to extend the results of the paper to matrix transfer functions. The connection between real and imaginary part has nothing new in it; the connection between spectrum matrix and minimum phase stable spectral factors (the matrix analog of the magnitude to transfer function calculation) is somewhat different, since one lacks the ability to work with the logarithm of the spectrum.

Another direction of generalization would involve changing the postulates about the spacing of discrete frequencies. One could be interested in working with information over only a limited part of the unit circle, or with frequencies which were the bilinear transforms of (continuous time) frequencies uniformly spaced in a logarithmic sense, or with frequencies which are spaced more densely in the neighbourhood of high derivatives of the data.

A third direction of generalization would involve using discrete data to define a transfer function which was not FIR, for example IIR with prescribed poles, or all-pole.

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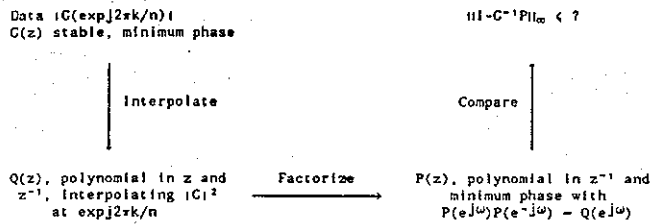


Figure 3.1: Outline approach to gain interpolation problem.

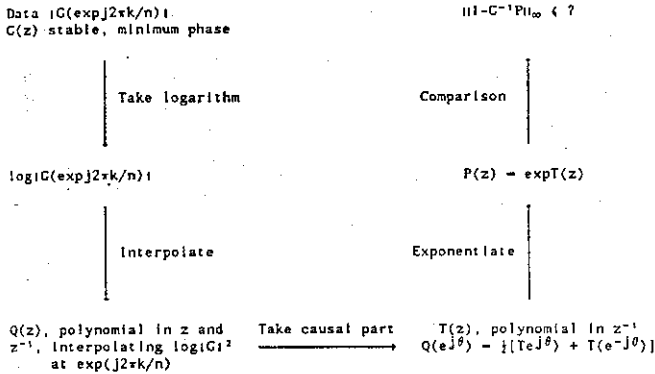


Figure 3.2: Alternative approach to gain interpolation problem.

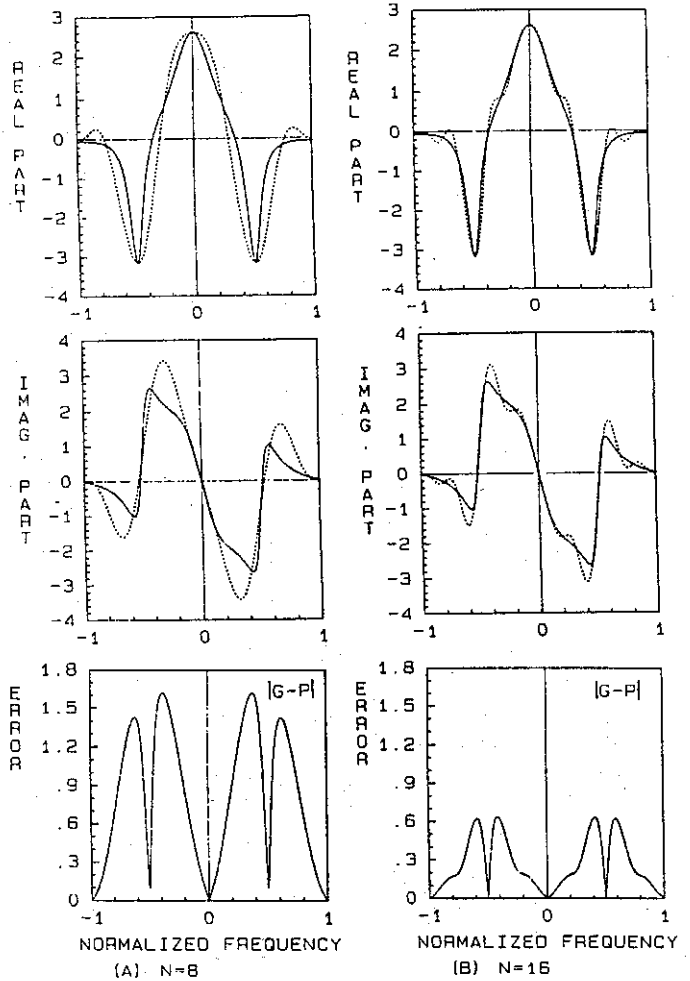


Figure 5.1 Frequency responses for the real part data example. For the real and imaginary part plots, solid line is G, broken line is P.