PROC. 1987 AMERICAN CONTROL CONFERENCE
MINNEAPOLIS

WAP - 8:30

CONTROLLER REDUCTION: CONCEPTS AND APPROACHES

Brian D.O. Anderson*
Yi Liu*

Department of Systems Engineering, Research School of Physical Sciences, Australian National University, G.P.O. Box 4, Canberra, A.C.T., Australia.

ABSTRACT

This paper considers the problem of passing from a linear time-invariant high order controller designed for a linear time-invariant plant (of presumably high order) to a low order approximation of the controller. The approximation problem is often best posed as a frequency weighted L_∞ approximation problem. Many different controller representations are possible, giving different performance of the various reduction algorithms.

1. Introduction

Simple linear controllers are normally to be preferred to complex linear controllers for linear time-invariant plants: there are fewer things to go wrong in the hardware or bugs to fix in the software, they are easier to understand, and the computational requirements are less. For this reason, there is a desire to have methods available for designing low order controllers for high order plants. Such methods can broadly be divided into two classes: *direct* in which the parameters defining a low order controller are computed by some optimization or other procedure; and *indirect*, in which a high order controller is first found, and then a procedure used to simplify it.

Examples of direct methods include the work of Ly [1] (see the third case study in [1]) which draws on [2], and of Bernstein and Hyland [3-5]. It is not our purpose to discuss this work here.

As for indirect methods, there are at least two sophisticated approaches to the design of high order controllers, LQG and H^∞, and at least for the former, a great deal of qualitative/conceptual knowlege exists which is vital in the applications of the design algorithms, to practical problems. Less well developed are the procedures for reducing high order controllers to low order controllers; such procedures are the subject of this paper.

A somewhat crude approach (that nevertheless can often be successful) to controller reduction is modal reduction, see e.g. [6]. It can be argued that, under certain circumstances, this is very like balanced truncation.

Yet another approach is to approximate the plant rather than the controller. Then a low order controller is designed using a low order of approximation of the plant, with the low order controller then used on the correct plant. It is argued in [7] that *satisfactory* approximation of the plant requires some knowledge in advance of the controller. So the designer is caught in an awkward logical loop. More generally, one can argue that in any design process, it is wise to postpone approximation, in case the effects of approximation propagate in an unhelpful way through the subsequent steps of the design process.

It is crucial to accept that the problem of controller reduction is distinct from the problem of (open-loop) model reduction, because of the presence of the plant. This is argued in Section 2, which makes the case that controller reduction can be regarded as a *frequency-weighted* L_∞ approximation problem, with no simple procedure the best for defining the weight. Unfortunately, there are no nice algorithms available for solving such problems. There are however algorithms which can come close. We describe such algorithms in Section 3. Such algorithms often offer less appealing performance when used on unstable systems — this motivates a study in Section 4 of controller reduction using what are termed "fractional representations" of the controller. Section 5 shows how to trade-off stability and performance considerations in some of the reduction procedures, and Section 6 contains concluding remarks.

2. Controller Reduction and Frequency Weighting

There is a fundamental difference between model reduction and controller reduction. Model reduction is, at least normally, based on open-loop considerations. On the other hand, any controller reduction procedure, if rationally based, ought to take into account the existence of the plant. Controller reduction should after all preserve *closed loop* stabilty, and (as far as possible) the *closed-loop* performance and *closed-loop* transfer function.

Making these arguments more precise turns out to generate frequency weighted approximation problems, as we shall now show. The choice of frequency weight is influenced by the choice of criterion thought most important in the approximation process, viz. stability, performance (the term being used in a loose sense), or closed-loop transfer function.

2.1 Stability Margin Considerations for Frequency Weighting: Let $G(s)$ be the transfer function matrix of a given linear time-invariant plant (with ℓ inputs and m outputs), and let $K(s)$ be a stabilizing high order compensator (obtained by some standard procedure). Let $K_r(s)$ be a reduced order compensator, which we are seeking. Regard the closed loop system with $K_r(s)$ replacing $K(s)$ as being equivalent to that of Figure 2.1. It can then be concluded using this redrawing [8] (and it is now well known) that if

(i) $K(s)$ and $K_r(s)$ have the same number of poles in $Re(s)>0$ and no poles on the imaginary axis†;

and

(ii) either

$$\|[K(s)-K_r(s)]G(s)[I+K(s)G(s)]^{-1}\|_\infty < 1 \quad (2.1)$$

or

$$\|[I+G(s)K(s)]^{-1}G(s)[K(s)-K_r(s)]\|_\infty < 1 \quad (2.2)$$

then $K_r(s)$ is a stabilizing compensator. (The notation $\|A(s)\|_\infty$ means

† This restriction to no $j\omega$-axis poles can be circumvented by requiring K and K_r to have identical $j\omega$-axis poles and residues.

$$\sup_{\omega} \max_{i} \lambda_i^{\frac{1}{2}}[A^*(j\omega)A(j\omega)]).$$

This suggest a minimization problem: find a $K_r(s)$ satisfying (i) which at the same time minimizes the left side of (2.1) or (2.2), and has prescribed degree. The matrix $G(I+KG)^{-1} = (I+GK)^{-1}G$ acts as a *weighting matrix* in this case.

Remarks

(2.1) It is easy to see that $\bar{\sigma}\{G(j\omega)[I+K(j\omega)G(j\omega)]^{-1}\}$ is small when either $\bar{\sigma}\{G(j\omega)\}$ is small or $\underline{\sigma}\{K(j\omega)\}$ is large and so often the frequency weighting obtained from the above stability margin argument will be greatest near the unity gain crossover frequencies of the loop gain $G(j\omega)K(j\omega)$. Here $\bar{\sigma}$ and $\underline{\sigma}$ stand for the largest and the smallest singular value of the matrix, respectively. This means that it is more important to have accurate approximation in this band, an idea familiar from classical control.

(2.2) If a K(s) of n^{th} degree is designed by an LQG optimal procedure, and we then find the lower order $K_r(s)$ which minimizes $\|[K(j\omega)-K_r(j\omega)]G(j\omega)[I+K(j\omega)G(j\omega)]^{-1}\|_\infty$ (or $\|[I+GK]^{-1}G(K-K_r)\|_\infty$, in the other case), there is no implication that $K_r(s)$ is in any sense LQG optimal.

2.2 Performance Considerations for Frequency Weighting: Consider the original closed−loop system in the presence of process and measurement noise, w(t) and v(t), as depicted in Figure 2.2. It is possible to compute the spectrum $\Phi_{qq}(j\omega)$ of the noise process of q(t). (We assume stationarity of the excitation noises and closed−loop stability, so that the spectrum exists).

In order that a low order approximation $K_r(s)$ to K(s) be a good approximation, it is important that it be most accurate in those frequency bands encountered in actual operation on performance. Thus if q(t) has little spectral energy in one band, $K(j\omega)$ need not be closely approximated there by $K_r(j\omega)$, while if the spectral energy in another band is high, approximation needs to be accurate. Let $V(j\omega)$ be a stable, minimum phase spectral factor of $\Phi_{qq}(j\omega)$. (Thus $VV^* = \Phi_{qq}$). Then the approximation problem becomes: find $K_r(j\omega)$ of nominated degree such that:

(i) K(s) and $K_r(s)$ have the same number of poles in Re(s)>0 and no poles on the imaginary axis

(ii) $\|[K(s) - K_r(s)]\, V(s)\|_\infty$ is minimized

2.3 Closed−Loop Transfer Function Considerations for Frequency Weighting: The closed−loop transfer function matrices with K(s), $K_r(s)$ are

$$W(s) = G(s)K(s)[I+G(s)K(s)]^{-1} = I - [I+G(s)K(s)]^{-1}$$
$$W_r(s) = G(s)K_r(s)[I+G(s)K_r(s)]^{-1} = I - [I+G(s)K_r(s)]^{-1}$$

Approximately, there holds

$$W_r(s)-W(s) = [I+G(s)K(s)]^{-1}G(s)[K_r(s)-K(s)][I+G(s)K(s)]^{-1}$$

and this suggests the following approximation problem: find $K_r(j\omega)$ of nominated degree so that

(i) K(s) and $K_r(s)$ have the same number of poles in Re(s)>0 and no $j\omega$−axis poles

(ii) $\|V_1(s)[K_r(s)-K(s)]V_2(s)\|_\infty$ is minimized, where $V_1 = (I+GK)^{-1}G$, $V_2 = (I+GK)^{-1}$

Comparing (ii) with (2.2) shows that there is reduced weighting placed on frequencies in the high loop−gain region in this third approach as compared with the first approach.

2.4 Further Issues:

- Other weights may be appropriate on occasion. For example, if the spectrum of external inputs were known, that could appear in a weight.
- As will be later seen, other representations of the controller lead to different frequency weighted problems, formulated however with the same conceptual basis (e.g. stability) as above.
- One would have to expect that concentration on stability could lead to poorly performing controllers, while concentration on other performance measures could lead to instability.
- The weighted approximation problem cannot in general be easily solved. Related problems can however be comparatively easily solved, as described in the sequel.
- The approximation problems posed are not fully appropriate for controllers with unstable or $j\omega$ axis−poles. Consider a controller containing a pure integrator. The approximation problem posed demands that any approximation also contain a pure integrator with *precisely* the same residue. This shows that the approximation problem is in some way unnecessarily restrictive.
- Quite apart from the suitability of the approximation problems posed, frequency weighted (or for that matter unweighted) approximation when unstable poles occur in the object being approximated can cause further headaches in the actual approximation process. One approach is to copy the unstable part (under additive decomposition) of K(s) into $K_r(s)$ and then just to approximate the stable part of K(s) with the (lower order) stable part of $K_r(s)$.

3. Frequency Weighted Model Reduction Techniques

There are now at least three important and also rather popular state−space based model reduction techniques, namely, truncation of the internally balanced realization [9,10], Hankel norm optimal approximation [11−14], and q−covariance equivalent realization (q−cover) [15−18]. In their usual form, they replace one stable high order model by a second stable low order model that usually is *not* an optimal L_∞ approximation; further, usually no frequency weighting is employed. There are available for the first two methods bounds on the L_∞ error of approximation. Frequency weighted versions of the first two methods are also available [7,19,20], but an L_∞ error bound is available only for frequency−weighted Hankel norm reduction [20]. There is not to this point a frequency weighted q−cover approximation procedure.

Through lack of a better alternative, these reduction techniques have been or can be used for controller reduction. In the remainder of this seciton we shall highlight some features of these techniques. It is important to remember that all these techniques are faute de mieux procedures that only come close to solving the L_∞ approximation problems defined in the last section.

3.1 Balancing Approximation: Given an n^{th} order, linear time−invariant and asymptotically stable system with transfer function matrix G(s), a minimal realization of $G(s) = C(sI-A)^{-1}B$ is internally balanced if {A,B,C} satisfy

$$A\Sigma + \Sigma A' + BB' = 0 \quad (3.1)$$
$$A'\Sigma + \Sigma A + C'C = 0 \quad (3.2)$$
$$\Sigma = \mathrm{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_n\},\ \sigma_i > \sigma_{i+1} > 0,\ i=1,\ldots,n-1 \quad (3.3)$$

The matrix Σ is both the controllability and observability gramian. Partition the system {A,B,C} as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\}r, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\}r$$

$$C = [C_1,\ C_2], \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\}r \quad (3.4)$$

Set $A_r = A_{11}$, $B_r = B_1$, $C_r = C_1$. Then the reduced order system $\{A_r,B_r,C_r\}$ is a good approximation of system $\{A,B,C\}$ if $\sigma_r \gg \sigma_{r+1}$. In fact, we have the following two properties:

(i) Subsystems $\{A_{ii},B_i,C_i\}$, $i = 1,2$, are asymptotically stable if $\sigma_r > \sigma_{r+1}$ [10].

(ii) There exists a frequency domain error bound for the balancing approximation [7,14]:

$$\|[C(j\omega I-A)^{-1}B - C_r(j\omega I-A_r)^{-1}B_r]\|_\infty \leq 2(\sigma_{r+1}+\ldots+\sigma_n) = 2tr(\Sigma_2) \quad (3.5)$$

It also turns out [14] that *any* approximation of $C(j\omega I-A)^{-1}B$ of degree r necessarily has an L_∞ error of at least σ_{r+1}. Sometimes the actual L_∞ error achieved on the left of (3.5) can be compared against this generally unachievable lower bound.

Enns [7] introduced frequency weighting into the balancing technique. Consider an asymptotically stable frequency weighting function $W_i(s) = E_i+C_i(sI-A_i)^{-1}B_i$ as an input weighting to the asymptotically stable system G(s) in Fig.3.1. The basic idea is to change the controllability gramian to reflect the introduction of the 'frequency weighting. Thus we find a *"frequency weighted"* controllability gramian which equals the observability gramian and then both are diagonalized. In outline, this is done as follows.

Let system matrices of the cascade system be defined as

$$\tilde{A} = \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} BE_i \\ B_i \end{bmatrix}$$

Assume

$$\tilde{U} = \begin{bmatrix} U & U_{21}' \\ U_{21} & U_{22} \end{bmatrix}$$

is the solution of the following Lyapunov equation:

$$\tilde{A}\tilde{U} + \tilde{U}\tilde{A}' + \tilde{B}\tilde{B}' = 0 \quad (3.6)$$

Now, U can be regarded as the frequency weighted controllability gramian for the original system. Let Y be the observability gramian, i.e. Y satisfies

$$YA + A'Y + C'C = 0 \quad (3.7)$$

Consider a co-ordinate basis change to (A,B,C) which makes

$$U_{new} = Y_{new} = diag\{\lambda_1,\lambda_2,\ldots,\lambda_n\},\ \lambda_i \geq \lambda_{i+1},\ i = 1,2,\ldots,n-1 \quad (3.8)$$

There is no change to (A_i,B_i,C_i). We call this new realization a frequency-weighted balanced realization.

Now, as before, the frequency weighted approximation is achieved by eliminating rows and columns of A,B,C corresponding to smallest $\{\lambda_n,\lambda_{n-1},\ldots,\lambda_{r+1}\}$.

Remarks:

3.1 The dual procedure for output frequency weighting is similar. In this case, one uses a *frequency weighted* observability gramian instead of a *frequency weighted* controllability gramian. One can also carry out a two-sided weighted balancing aprpoximation by diagonalizing and equalizing the frequency-weighted controllability gramian, and the frequency weighted observability gramian.

3.2 For input frequency-weighted or output frequency-weighted balancing approximation, the reduced order system is generically asymptotically stable. But a proof of the stability of the reduced order system is lacking for the two sided frequency-weighted balancing reduction procedure.

3.3 No error bound formula analogous to (3.5) is available.

3.4 We now can apply this technique to the controller reduction problem. As we discussed before, we set $V(s) = G(s)[I+K(s)G(s)]^{-1} = C_i(sI-A_i)^{-1}B_i$ to be the input frequency weighting or the output frequency weighting for the reduction of controller K(s).

3.5 If K(s) is determined by combining an estimator and state feedback law, and is stable, then (3.6) generically has order 2n. Otherwise, if K(s) is stable, of order n, generically (3.6) has order 3n. If K(s) has unstable poles, these do not contribute to the order of (3.6), as we approximate only the stable part of K(s).

3.6 An alternative method of introducing frequency weighting is suggested in the work of [19] on frequency-weighted Hankel norm reduction. First note that $\|(K-K_r)V\|_\infty = \|(K-K_r)\bar{V}\|$ where $\bar{V}(-s)$ and $\bar{V}^{-1}(-s)$ are stable and $VV^* = \bar{V}\bar{V}^*$ (given a nonsingular V with no $j\omega$-axis poles and zeros, such a $\bar{V}$ can always be found). Let $[\ ]_+$ denote the operation of taking the stable part. Let $L = [K\bar{V}]_+$. Let L_r approximate L (no frequency weighting) and set $K_r = [L_r\bar{V}^{-1}]_+$. Note that $degK = degL$, $degK_r = degL_r$, if K is stable.

3.7 There have been other approaches employing balancing to achieve controller reduction. Jonckheere and Silverman [21] suggested balancing of the two Riccati equations in the LQG design procedure, followed by truncation. But this scheme does not eliminate any uncontrollable or unobservable modes in the controller (which can arise in LQG design), and so appears unattractive. Also, no frequency domain error bound is available for it. The same idea was advanced also by Verriest, see [22]. Youssuf and Skelton [23] proposed use of an unweighted balancing approximation directly on the controller if it is stable. This scheme has a frequency domain error bound and has no restriction that the controller be obtained from an LQG design. However, as we have argued for controller reduction, it is better to have frequency weighting to improve stabilty or some aspect of the closed loop performance. If the controller is unstable, a modification of the scheme is available [23], but the underlying rationale for the modification is hard to see. A variant of the modification with more rationale was suggested by Davis and Skelton [24]. However, the same objection applies to this variant as applies to Jonckheere and Silverman's scheme.

3.2 Hankel Norm Approximation: Another very important model reduction approach is Hankel norm optimal approximation, which has been first considered in [11], then in [12] and [13]. Glover [14] characterized all stable approximations of a linear time-invariant stable system G(s) of McMillan degree n by $G_r(s)$ of Macmillan degree r ($r<n$) which minimize the "Hankel norm" error $\|G(s)-G_r(s)\|_H$. Note that an exact solution of an approximation problem is achieved, but the approximation problem solved is different from that for which a solution is desired, involving the Hankel norm rather than L_∞ norm. There is however a connection. If $G_r(s)$ is an optimal Hankel norm approximant of order r to G(s), then

$$\|G-G_r\|_\infty \leq (\sigma_{r+1} + \ldots + \sigma_n) \quad (3.9)$$

while, as earlier noted, no r^{th} order approximant can ever achieve $\|G-G_r\|_\infty < \sigma_{r+1}$. Here the σ_i are the singular values appearing in the balancing realization theory, ordered with $\sigma_i \geq \sigma_{i+1}$. (They are also known as the Hankel singular values.) Glover's calculations actually involve manipulation of a balanced realization of G to get G_r, the manipulations being somewhat more complicated than mere truncation.

Note that (3.9) gives an error bound half that for balanced truncation, but in (3.9), $G_r(s)$ is allowed to have $G_r(\infty)$ nonzero, while in (3.5), $G_r(\infty)$ must be zero.

Latham and Anderson [19] proposed a frequency-weighted version of the Hankel norm approximation to find a stable $G_r(s)$ of McMillan degree r, which minimized the Hankel norm error $\|[G(s)-G_r(s)]V(s)\|_H$ with $V(s)$ and $V^{-1}(s)$ completely unstable. (The approach of Remark 3.6 combined with Glover's method was used.) A formula bounding $\|[G(s)-G_r(s)]V(s)\|_\infty$ has been derived [20], though it is not as easy to evaluate the bound as in the unweighted case.

Reference [13] reports use of the unweighted procedure to simplify controllers.

3.3 q-Cover Approximation: The basic idea of q-covariance equivalent approximations applied to controller reduction is to approximate a high order controller by a low order one with two properties [15-18]:

(i) The first q Markov parameters of the high and low order controllers are the same;

(ii) The output covariances and their first q derivatives evaluated at time t = 0 of both high and low order controllers are equal. (This presupposes noise excitations to the closed loop.)

In the scalar case, q is the actual dimension of the reduced order controller. The mechanics of the reduction procedure involve the use of Hessenberg form representations. The basic idea behind q-cover approximation is to match transient behaviour *and* steady state behaviour; the transient behaviour is reflected in the Markov coefficients and the steady state behaviour in the covariance data. Of course, to the extent that frequency weighting cannot be accommodated, the technique is deficient; one must assume white noise excitation of the controllers. But note that equality of the first q Markov parameters of two controllers implies (nontrivially) the corresponding equality for the first q-Markov parameters of the two associated *closed-loop* systems.

So the failure to use frequency weighting is more of concern in securing the right steady state behaviour. Preservation of closed-loop stability is not directly sought or secured in q-cover approximation. Examples of reduced order controller design using q-covers can be found in [18].

3.4 Additional Remarks: There are important, partly concealed, limitations of the methods, even frequency-weighted methods, espoused above:

- no approximation is actually optimal from an L_∞ point of view, although this is what we want.
- the approximation procedures do not cope well with nonstable controllers. It is *not* the case that copying of the unstable part of the high order controller into the low order controller is likely to be optimal, see e.g. [25], and as noted in the last section, one can even query the appropriateness of the weighted L_∞ approximation problem when the high order controller is unstable.

Some of these issues are addressed in the succeeding section.

4. Controller Reduction Using Fractional Representations

In this section, for the most part we shall consider controllers formed from a combination of a stabilizing feedback law and an estimator. At one point, we shall even suppose an LQG-based design. Thus, for a given linear time-invariant system $G(s) = C(sI_n-A)^{-1}B$, and an LQ index

$$J = \int_0^\infty (x'Qx+u'Ru)dt, \quad Q' = Q \geq 0, \quad R' = R > 0 \tag{4.1}$$

we design the optimal feedback gain F which minimizes (4.1). Let L be the Kalman gain for the estimator. Then the compensator is

$$K(s) = F(sI_n-A+BF+LC)^{-1}L \tag{4.2}$$

(This formula applies also if F,L are designed by non-optimal pole-positioning methods.) Assume all conditions are satisfied which ensure A-BF and A-LC have all eigenvalues in $Re(s)<0$.

Define

$$\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} \triangleq \begin{bmatrix} I_\ell & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix} (sI_n-A+BF)^{-1}[B,L] \tag{4.3}$$

and

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \triangleq \begin{bmatrix} I_\ell & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} F \\ -C \end{bmatrix} (sI_n-A+LC)^{-1}[B,L] \tag{4.4}$$

Then it has been proved, see [26,27], that

$$G(s) = X(s)Y^{-1}(s) = \tilde{Y}^{-1}(s)\tilde{X}(s) \tag{4.5a}$$

$$K(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s) \tag{4.5b}$$

and the Bezout identity holds:

$$\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix}\begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} Y(s) & -N(s) \\ X(s) & D(s) \end{bmatrix}\begin{bmatrix} \tilde{D}(s) & \tilde{N}(s) \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \tag{4.6}$$

The transfer function matrices $N(s)$, $D(s)$, $\tilde{N}(s)$, $\tilde{D}(s)$, $X(s)$, $Y(s)$, $\tilde{X}(s)$ and $\tilde{Y}(s)$ are all stable. Hence, the Bezout identity (4.5) means that $N(s)D^{-1}(s)$ $(\tilde{D}^{-1}(s)\tilde{N}(s))$ is a stable right (left) coprime factorization of K(s). At the same time, $X(s)Y^{-1}(s)$ $(\tilde{Y}^{-1}(s)\tilde{X}(s))$ is a stable right (left) coprime factorization of G(s).

4.1 Coprime Factorization Reduction Without Frequency Weighting: With the controller defined as in (4.2), we can draw the closed loop as shown in Fig. 4.1.

Now think of the controller as being defined by *an interconnection rule together with a stable transfer function matrix*, viz

$$\begin{bmatrix} C \\ F \end{bmatrix}(sI_n-A+BF)^{-1}L = \begin{bmatrix} D(s)-I_m \\ N(s) \end{bmatrix} \tag{4.7}$$

(see Fig.4.2 — the minus sign attached to F is inessential). This suggests that we can approximate

$$\begin{bmatrix} D(s)-I_m \\ N(s) \end{bmatrix}$$

by some

$$\begin{bmatrix} D_1(s)-I_m \\ N_1(s) \end{bmatrix}$$

or, approximate

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \text{ by } \begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$$

with the second matrix having order r $(r<n)$, and then recover a reduced order controller by replicating the interconnection rule. Of course, this is equivalent to approximating the parts of a fractional representation of the controller [27], where such representations are over stable rational transfer functions.

How we reduce the controller, i.e. balancing truncation, Hankel norm or q−cover is a separate issue. Note that there are no difficulties with instabilities in the object being approximated, as D(s) and N(s) are guaranteed stable.

Should we introduce frequency weighting? If we take the spectral (performance oriented) viewpoint, as opposed to the stability or closed−loop transfer function viewpoint, the answer is no. For the process q(t) depicted in Fig.4.1 is the *innovations* process, and as such is white, with covariance $V\delta(t-\tau)$, the covariance of the measurement noise process v(t). The only weighting which we should then use is a *constant* matrix, $V^{\frac{1}{2}}$. Fortuitously, the redescription of the controller eliminates the frequency weighting; the idea of avoiding frequency weighting in this way comes indirectly from [18], and it allows application of q−cover approximation without the concern that the need for frequency weighting is being overlooked.

As shown by examples in [28], this method of controller reduction can be very efficacious.

Remark 4.1: The dual approach to the above is to start with the left stable coprime factorization of $K(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, then to approximate $[\tilde{D}(s)-I_\ell, \tilde{N}(s)] = F(sI_n-A+LC)^{-1}[B,L]$ by some low order $[\tilde{D}_1(s)-I_\ell, \tilde{N}_1(s)]$ and implement the reduced order controller as $K_r(s) = \tilde{D}_1^{-1}(s)\tilde{N}_1(s)$. (The significance of the transfer function matrix being approximated is evident from Fig.4.3.) In this case however, we are unable to justify constant frequency weighting (at the output now, by duality, see [28, Remark (C)]). As it turns out, for some examples this procedure works well and for some it works badly.

The fact that the efficacy of the procedure above and its dual cannot always be guaranteed (with primal and dual often having quite different quality of result in terms of stability) leads us to focus on the use of a stability−oriented method in conjunction with the above style of controller description. The key is to exploit the Bezout identity.

4.3 Coprime Factorization Reduction with Bezout Identity Induced Frequency Weighting: Refer again to Fig. 4.3, and regard the compensator as a two (vector) input, single (vector) output system with transfer function

$$\tilde{\Gamma}(s) = [\tilde{D}(s)-I_\ell \quad \tilde{N}(s)] \qquad (4.8)$$

Regard the "plant" as defined by

$$\tilde{H}(s) = \begin{bmatrix} I_\ell \\ G(s) \end{bmatrix} \qquad (4.9)$$

so that Fig.4.3 is equivalent to Fig.4.4. Let us seek to approximate $\tilde{\Gamma}(s)$ by a lower order stable $\tilde{\Gamma}_r(s)$. Taking the stability point of view, we should seek to minimize (see Section 2)

$$\tilde{\rho} = \|[\tilde{\Gamma}(s)-\tilde{\Gamma}_r(s)]\tilde{H}(s)[I+\tilde{\Gamma}(s)\tilde{H}(s)]^{-1}\|_\infty$$

Defining $\tilde{V}(s) = \tilde{H}(s)[I+\tilde{\Gamma}(s)\tilde{H}(s)]^{-1}$, it is easy, using the Bezout identity, to prove that

$$\tilde{V}(s) = \tilde{H}(s)[I+\tilde{\Gamma}(s)\tilde{H}(s)]^{-1} = \begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} - \begin{bmatrix} I_\ell \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ C \end{bmatrix}(sI_n-A+BF)^{-1}B \qquad (4.11)$$

where Y(s),X(s) were as defined in (4.3). Hence our goal becomes one of minimizing or finding a procedure that will approximately minimize

$$\tilde{\rho} = \|[\tilde{D}(s)-\tilde{D}_r(s) \quad \tilde{N}(s)-\tilde{N}_r(s)]\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix}\|_\infty \qquad (4.12)$$

over stable $[\tilde{D}_r \quad \tilde{N}_r]$ of prescribed degree.

Provided $[\tilde{D}_r \quad \tilde{N}_r]$ causes $\tilde{\rho}<1$, we know from Section 2.1 that the controller based on $[\tilde{D}_r \quad \tilde{N}_r]$ will necessarily be stabilizing. Obviously though the smaller $\tilde{\rho}$ is, the better off we are likely to be.

Remark 4.2: A dual result is available. We think of the controller as being defined by

$$\Gamma(s) = \begin{bmatrix} D(s)-I \\ N(s) \end{bmatrix} \qquad (4.13)$$

(See Fig.4.2 again). The actual controller transfer function matrix is $N(s)D^{-1}(s)$, and can be obtained by implementing Γ(s) and an interconnection rule. Carrying through reasoning analogous to the above leads to the conclusion that we should use $[\tilde{Y}(s) \quad \tilde{X}(s)]$ as the output frequency weighting, seeking D_r, N_r such that $[D_r' \quad N_r']'$ has a prescribed degree, and

$$\rho = \|[\tilde{Y}(s) \quad \tilde{X}(s)]\begin{bmatrix} D(s)-D_r(s) \\ N(s)-N_r(s) \end{bmatrix}\|_\infty \qquad (4.14)$$

is minimum. We remark that $\rho<1$ (which guarantees stability with the reduced order controller) is itself guaranteed if

$$\left\|\begin{matrix} D-D_r \\ N-N_r \end{matrix}\right\|_\infty < \frac{1}{\|\tilde{Y} \quad \tilde{X}\|_\infty} \qquad (4.15)$$

In [28], see Lemma 3.2, a more restrictive condition guaranteeing closed loop stability was given on the approximating controller.

Remarks:

4.3 The frequency weightings are all stable, as are the components in the fractional representations of the controller. So there are no problems associated with open−loop instability (or jω−axis poles) of K(s). Moreover, the methods do not require an underlying LQG assumption (in order, for example, to guarantee whiteness of some signal), but only the existence of F,L for which A−BF and A−LC have all left half plane eigenvalues.

4.4 q−cover approximation methods have not yet been extended to cope with weighting, and Hankel norm approximations require invertible weights. Hence without further development, we necessarily use balanced truncation. The dimension of the key Lyapunov equation is 2nx2n; however, simple algebraic transformations readily break this down, so that only one nxn equation has to be solved in obtaining the frequency−weighted controllability gramian. Compare this with the usual frequency−weighted balanced truncation procedure where the dimension is 3nx3n (for an arbitrary n^{th} order controller) and 2nx2n (for an n^{th} order controller obtained from feedback law and estimator).

Example 1: We use the example in [28] to compare the effects of different controller reduction procedures including Bezout identity type of frequency weighting. The plant to be controlled is a four−disk system, represented as a linear, time−invariant, single input and single output, unstable, non−minimum phase plant of eighth order. It was studied by Enns [7]. The plant $G(s)=C(sI-A)^{-1}B$ with transfer function described in [28] has minimal realization:

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -3.982 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.0 \\ 0.0 \\ 6.4432\times10^{-3} \\ 2.3196\times10^{-3} \\ 7.1252\times10^{-2} \\ 1.0002 \\ 0.10455 \\ 0.99551 \end{bmatrix} \tag{4.16}$$

(In [28] this is incorrectly termed an observable canonical form realization.) The weightings for an LQG design are given by $Q=q_1H'H$, $R=1$, with $H=[0,0,0,0,0.55,11,1.32,18]$, $q_1=10^{-6}$ and the filter covariance matrices are $W=q_2BB'$, $V=1$ where q_2 is a design parameter.

The following table extends Table 2 of [28]. It depicts the closed-loop stability of the system with reduced order controllers of different orders obtained by different reduction methods, starting with different LQG designs. These designs are obtained by adjusting the input noise intensity. For a minimum phase plant, choosing q_2 large would correspond to loop transfer recovery based designs. The unweighted and weighted right coprime factorization design referred to in the table correspond to the schemes set out in [28] and Section 4.1 (unweighted), and described in Remark 4.2 (weighted).

In terms of ensuring closed-loop stability with the reduced order controller, the right coprime factorization frequency-weighted method is the best.

Controller Order	Controller Reduction Method	$q_2=100$	$q_2=1000$	$q_2=2000$
7	Unweighted Balanced Truncation (Youseff and Skelton) [Y]	U	U	U
	(Davis and Skelton) [D]	S	S	S
	Unweighted Hankel-norm Reduction (Clover and Limebeer) [G]	S	U	S
	Weighted Balanced Truncation (Enns) [E]	S	S	S
	Unweighted Right Coprime Factorization (Liu and Anderson) [RCFU]	S	S	U
	Weighted Right Coprime Factorization [RCFFW]	S	S	S
6	Y	U	U	U
	D	S	S	S
	G	U	U	U
	E	S	S	S
	RCFU	S	S	U
	RCFFW	S	S	S
5	Y	U	U	U
	D	S	U	U
	G	U	U	U
	E	S	S	S
	RCFU	S	S	S
	RCFFW	S	S	S
4	Y	U	U	U
	D	S	U	U
	G	U	U	U
	E	S	S	U
	RCFU	S	S	S
	RCFFW	S	S	S
3	Y	U	U	U
	D	U	U	U
	G	U	U	U
	E	S	S	S
	RCFU	S	U	U
	RCFFW	S	S	S
2	Y	U	U	U
	D	U	U	U
	G	S	U	U
	E	U	U	U
	RCFU	S	S	S
	RCFFW	S	S	S

TABLE 1: Closed Loop Stability of the Reduced-Order Controllers (S= stable, U= unstable)

4.4 Coprime Factorization Reduction With More General Weighting: In many control system design problems, sometimes a single LQG designed controller is not enough. For instance, one may need an extra integral control loop to zero the steady state error, or a shaping filter at input or even output to achieve some performance objective. (See Fig.4.5 for an illustration that encompasses simultaneously several possibilities.) Controller reduction may be required with the constraint that shaping filters or integral feedback loops are maintained. We now seek to explain how this may easily be achieved, using the preceding ideas.

Let us consider a special case in detail. Assume $F_I(s)=0$, $SF_2(s)=1$, $SF_1(s)=D_F+C_F(sI-A_F)^{-1}B_F$, $G(s)=C(sI-A)^{-1}B$, and $\tilde{\Gamma}(s)=F(sI-A+LC)^{-1}[B,L]$ as the LQG controller. Then the closed-loop system is equivalent to that of Figure 4.4 with

$$\tilde{H}(s) = \begin{bmatrix} I_\ell \\ G(s) \end{bmatrix} SF_1(s) \tag{4.17}$$

With $\tilde{\Gamma}_r(s)$ the reduced order controller, we seek $\tilde{\Gamma}_r(s)$ to minimize

$$\| ([\tilde{\Gamma}(s)-\tilde{\Gamma}_r(s))V(s)] \| \tag{4.18}$$

where

$$V(s) = \tilde{H}(s)[I+\tilde{\Gamma}(s)\tilde{H}(s)]^{-1} \tag{4.19}$$

it is not too difficult to show that

$$V(s) = \begin{bmatrix} D_F \\ 0 \end{bmatrix} + \begin{bmatrix} C_F & -D_F F \\ 0 & C \end{bmatrix} [sI_{n+f} - \begin{bmatrix} A_F & -B_F F \\ BC_F & A-BD_F F \end{bmatrix}]^{-1} \begin{bmatrix} B_F \\ BD_F \end{bmatrix} \tag{4.20}$$

It is very interesting to note that now the frequency weighting has order $n+f$. Were we to use the non-factorization procedure of Enns' frequency weighting, the order of weighting would be of the order of $2n+f$ in this case; worse, the shaping filter structure is lost.

Note here that generally speaking if the filter $SF_1(s)$ is not a constant, we do not readily obtain a dual formulation of the same problem.

4.5 Other Approaches: We wish to record an approach currently under development by colleagues [29], see Fig.4.6. Given a stabilizing controller transfer function matrix K(s), it is possible to embed this in a 2x2 block matrix

$$K_a(s) = \begin{bmatrix} K(s) & L(s) \\ M(s) & N(s) \end{bmatrix} \tag{4.21}$$

sometimes of the same degree as K_a with the property that all stabilizing controllers are given by the set

$$K(s) + L(s)Q(s)[I-N(s)Q(s)]^{-1}M(s) \tag{4.22}$$

where Q ranges over all stable transfer functions. Controller reduction is achieved by reducing $K_a(s)$ as opposed to K(s) and then adding a constant Q chosen to optimize stability or performance.

The rationale for the above approach is to reduce the controller while maintaining robustness properties, or equivalently the ability to tolerate arbitrary stable Q(s) as in (4.22).

The approach is readily extended to maintain performance in the reduction process by augmenting $K_a(s)$ with additional outputs, the variance of which represents the performance of the controller.

5. Controller Reduction With Consideration of the Closed Loop Performance

It is obvious that the purpose of controller reduction is not only to maintain closed loop stability but also to maintain as much as possible the closed loop performance. In previous sections, we have concentrated on the stability problem and it is evident that the Bezout identity type of frequency weighting in controller reduction handles this issue well. Now, we shift to the performance problem.

We recall the LQG design procedure described in Section 4. For the given linear time-invariant system $G(s) = C(sI_n-A)^{-1}B$ and the LQ index

$$J = \int_0^\infty (x'Qx + u'Ru)dt \qquad Q' = Q \geq 0, \quad R' = R>0 \tag{5.1}$$

one can design the feedback gain F which minimizes (5.1) by full state feedback law $u=-Fx$. If not all states are measurable or there is process noise and measurement noise, one has to design an estimator to obtain the estimate of the states $\hat{x}$ to form the feedback law as $u= -F\hat{x}$.

Let $Q = H'H$. The index (5.1) then suggests that we should be concerned about getting a good approximation of Hx, or the next best thing, $H\hat{x}$. Continuing this heuristic argument, this suggests that we add an extra output y_a to the controller, see Fig.5.1, and reflect this into our statement of the approximation problem.

Thus we might seek to approximate

$$\begin{bmatrix} F \\ H \end{bmatrix} (sI_n-A+LC)^{-1}[B \quad L]$$

rather than just $F(sI_n-A+LC)^{-1}[B \quad L]$, as in the right coprime factorization methods. To develop the idea further, note that the performance index focuses our attention on $u = -F\hat{x}$, or more accurately $R^{\frac{1}{2}}u = -R^{\frac{1}{2}}F\hat{x}$. This suggests that it might be worthwhile to replace F by $R^{\frac{1}{2}}F$ (the negative sign is inessential). Recall that without the presence of H we are "stability-based" in our thinking, whereas introduction of H allows performance-based thinking. By introducing a scalar parameter $\alpha>0$, we can adjust our relative weighting of the two. This leads us then to focus on

$$\begin{bmatrix} R^{\frac{1}{2}}F \\ \alpha H \end{bmatrix} (sI_n-A+LC)^{-1}[B \quad L]$$

as the object to be reduced. Write the reduced order object as

$$\begin{bmatrix} R^{\frac{1}{2}}F_1 \\ \alpha H_1 \end{bmatrix} (sI_n-A_1)^{-1}[B_1 \quad L_1]$$

Then the reduced order controller is defined by $F_1(sI_n-A_1)^{-1}[B_1 \quad L_1]$. Both the unweighted left coprime factorization scheme, see Remark 4.1 [28, Remark C] and the frequency-weighted scheme, see Section 4.2, can be used for reduction, with the latter to be preferred, addressing as it does the stability problem directly.

Note also that the scaling $R^{\frac{1}{2}}$ and α play a very important role in improving the closed loop performance. Though $R^{\frac{1}{2}}$ is fixed, it is very important to use it as a scaling for multi-input systems. A properly chosen α can make much difference in performance; in many cases, if we choose large α, we can (not unexpectedly) run into instability problems. Note also that since H has *m* rows, we can in principle use a diagonal scaling matrix $\Lambda = \text{diag}\{\alpha_1,\alpha_2,...,\alpha_m\}$ instead of the scalar α in reduction.

Example 2: Now we use an example to illustrate how the artificial output involvement in reduction can effect the closed loop performances in the coprime factorization reduction scheme, with or without frequency weighting.

The system and LQG controller design set up are the same as for Example 1. The index is

$$J = \int_0^\infty (x'Qx+u'Ru)dt = \int_0^\infty (x'H'q_1Hx+u^2)dt$$

where q_1 and H are defined as in Example 1 and R=1. In the first case, with the design parameter $q_2=1.0$, we reduce the 8th order controller to a 7th order one without use of frequency-weighting, via the left coprime factorization scheme. As illustrated in Fig.5.2 if we do not involve the artificial output in the reduction (i.e.$\alpha=0$), the closed loop performance as indicated by the step response is a very poor approximation. But if we use $\alpha=9$ in balancing $[F' \ \ \alpha H']'(sI_n-A+LC)^{-1}[B \ L]$, then reducing the controller, the closed loop performance improves dramatically. The gain margins for the three cases are 14.38db (full order), -3.55db and 14.26db ($\alpha=0$) and 14.65db ($\alpha=9$). The phase margins are (in degrees) 48.86 (full order), 22.34 ($\alpha=0$) and 47.9 ($\alpha=9$). So there is apparently little damage to robustness in using $\alpha=9$. In the second case, with the design parameter $q_2=100$ and with $\alpha=0$, the unweighted left coprime factorization balancing reduction scheme yielded a stabilizing 6th order controller (with lower order controllers not stabilizing). If we use the Bezout identity type frequency weighting for reduction (frequency-weighted balancing), still with $\alpha=0$, we obtain all stabilizing controllers from 3rd to 7th order (another indication of the effectiveness of the Bezout identity type of frequency weighting in preserving stability). Then we introduce the artificial output to the reduction scheme. A properly chosen scaling factor α can make significant difference in the closed loop performance, as is illustrated in Fig.5.3, where we reduced the controller to 5th order with $\alpha=35$ (via frequency-weighted balancing). The gain margins are 16.95 (full order), 17.76 ($\alpha=0$) and -15.92 and 14.45 ($\alpha=35$). The phase margins are 53.21, 12.53 and 41.06. Again, there is little damage to robustness.

We have also used this reduction scheme on a complicated example, the pitch control system of an F-111 airplane. The plant is a multi-input and multi-output system with 23rd order LQG controller, 5th order filter and integral control loop, and the controller is reduced to fourth order. Performance appears very similar to that obtained with a pre-existing fourth order controller obtained via modal reduction. The results for this example will be reported elsewhere.

6. Conclusions

Throughout this paper, we have exhibited many choices facing those seeking to reduce the order of a controller. A number can be listed as follows:

- Frequency-weighted/unweighted approximation
- Choice of weight based on stability, spectrum or closed-loop transfer functions
- Reduction method based on balanced truncation, Hankel norm or q-cover
- Conventional representation or fractional representations
- Exclusion/inclusion of augmented output to improve performance

We have also thrown up a number of questions or open problems, for example:

- frequency-weighted q-cover reduction
- nonsquare frequency-weighted Hankel norm reduction
- preservation of stability in balanced truncation with input *and* output weighting
- reduction of unstable systems
- advance prediction that a particular reduction method will out-perform another

In connection with this last point, let us note that examples and some partially developed theory suggest that left fractional controller representations are to be preferred to right representations if the filter has been designed using loop recovery ideas, and right representations preferred to left representations if the LQ law has been designed using loop recovery ideas. Moreover, in using the preferred representation there will often be litle difference between the effect of frequency-weighted or unweighted reduction. Hitherto unexplored is the use of multiplicative rather that additive controller reduction, see [30] for results on (unweighted) model reduction.

Last, we remark that we have said very little on the issue of scaling. Proper scaling of inputs and outputs can cause great variations in the efficacy of a reduction procedure.

Acknowledgements

Many helpful discussions with U.L. Ly and J.B. Moore are gratefully acknowledged.

REFERENCES

[1] D. Gangsaas, K.R. Bruce, J.D. Blight and U.-L. Ly, "Application of modern synthesis to aircraft control: three case studies", IEEE Trans.Auto.Control, AC-31, No.11, 1986, pp.995-1104.

[2] U.-L. Ly, "A design algorithm for robust low order controller", Ph.D. Thesis, Department of Aeronautics and Astronautics, Stanford University, CA, U.S.A..

[3] D.S. Bernstein and D.C. Hyland, "The optimal projection equations for fixed-order dynamic compensation", IEEE Trans.Auto.Control, AC-29, No.11, 1985, pp.1034-1037.

[4] D.C. Hyland and D.S. Bernstein, "The optimal proejction equations for model reduction and the relationships among the methods of Wilson, Skelton and Moore", IEEE Trans. Auto. Control, AC-30, No.12, 1985, pp.1201-1211.

[5] D.S. Bernstein and D.C. Hyland, "Optimal projection/maximum entropy stochastic modelling and reduced-order design synthesis", Proc. IFAC Workshop on Model Errors and Compensation, 1985, Boston, MA, U.S.A.

[6] U.-L. Ly and D. Gangsaas, "Application of modified linear quadratic Gaussian design to active control of a transport airplane", Proc. AIAA Guidance and Control Conference, 1979, Boulder, Colorado, USA.

[7] D. Enns, "Model reduction for control systems design", Ph.D. Thesis, Department of Aeronautics and Astronautics, Stanford University, CA, USA, 1984.

[8] J.C. Doyle and G. Stein, "Multivariable feedback design: concepts for a classical/modern synthesis", IEEE Trans.Auto.Control, AC-26, No.1, 1981, pp.4-16.

[9] B.C. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction", IEEE Trans.Auto.Control, AC-26, No.1, 1982, pp.17-32.

[10] L. Pernebo and L.M. Silverman, "Model reduction via balanced state space representations", IEEE Trans.Auto.Control, AC-27, No.2, 1982, pp.382-387.

[11] V.M. Adamjan, D.Z. Arov and M.G. Krein, "Analytic properties of Schmidt pairs for Hankel operator and the generalized Schur-Takagi problem", Math.USSR Sbornik, Vol.15, No.1, 1971, pp.31-73.

[12] S.Y. Kung and D.W. Lin, "Optimal Hankel-norm model reductions: multivariable systems", IEEE Trans.Auto.Control, AC-26, No.4, 1981, pp.382-852.

[13] K. Glover and D.J.N. Limebeer, "Robust multivariable control system design using optimal reduced order plant methods", Proc. ACC, 1983, San Francisco, CA, USA, pp.644-

[14] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds", Int.J.Control, Vol.39, No.6, 1984, pp.1115-1193.

[15] A. Yousuff, D.A. Wagie and R.E. Skelton, "Linear system approximation via covariance equivalent realizations", J.Math.Analy.Appl., Vol.106, No.1, 1985, pp.91-115.

[16] A. Yousuff and R.E. Skelton, "Covariance equivalent realizations with application to model reduction of large-scale systems", Control and Dynamics Systems, Academic Press Inc., Vol.22, 1985, pp.273-348.

[17] B.D.O. Anderson and R.E. Skelton, "The generation of all q-Markov covers", submitted for publication.

[18] A. Youseff and R.E. Skelton, "Controller reduction by component cost analysis, IEEE Trans.Auto.Control, AC-29, No.6, 1984, pp.520-530.

[19] G.A. Latham and B.D.O. Anderson, "Frequency-weighted optimal Hankel norm approximation of stable transfer functions", Sys.& Control Letters, Vol.5, No.4, 1985, pp.229-236.

[20] B.D.O. Anderson, "Weighted Hankel norm approximation: calculation of bounds", Sys. & Control Letters, Vol.7, No.4, 1986,, pp.247-255.

[21] E.A. Jonckheere and L.M. Silverman, "A new set of invariants for linear systems - application and approximation", Int.Symp.Theory Networks & Systems, 1981, Santa Monica, CA, USA.

[22] E.J. Verriest, "Suboptimal LQG design via balanced realizations", Proc. 20th IEEE CDC, San Diego, CA, USA, 1981, pp.686-687.

[23] A. Yousuff and R.E. Skelton, "A note on balanced controller reduction", IEEE Trans.Auto.Control, AC-29, No.3, 1984, pp.254-257.

[24] J.A. Davis and R.E. Skelton, "Another balanced controller reduction algorithm", Sys.& Control Letters, Vol.4, No.2, pp.79-83, 1984.

[25] P.J. Parker and B.D.O. Anderson, "Unstable rational function approximation", Int.J. Control, to appear.

[26] C.N. Nett, C.A. Jacobson and M.J. Balas, "A connection between state-space and doubly coprime fractional representations", IEEE Trans.Auto.Control, AC-29, No.9, 1984, pp.831-832.

[27] M. Vidyasagar, Control System Synthesis: A Factorization Approach, MIT Press, Cambridge, MA, USA, 1985.

[28] Y. Liu and B.D.O. Anderson, "Controller reduction via stable factorization and balancing", Int.J.Control, Vol.44, No.2, 1986, pp.507-531.

[29] J.B. Moore and U.-L. Ly, "Controller reduction methods maintaining stability, performance and robustness", in preparation.

[30] K. Glover, "Multiplicative approximation of linear multivariable systems with L^{∞} error bounds", Proc. ACC, Seattle, 1986, pp.1705-1709.

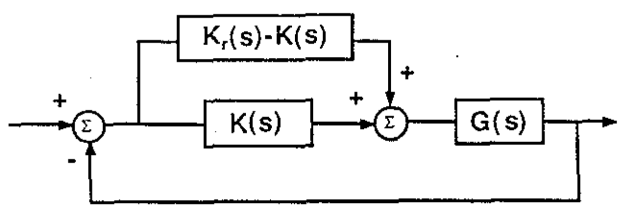

Figure 2.1

Rearrangement of feedback system with reduced order compensator

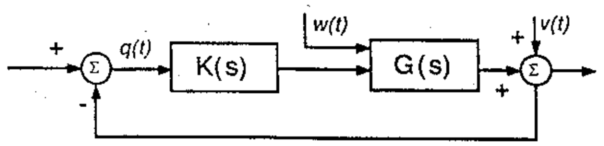

Figure 2.2

Original feedback scheme showing noise excitation

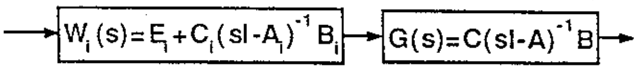

Figure 3.1

Introduction of input weighting

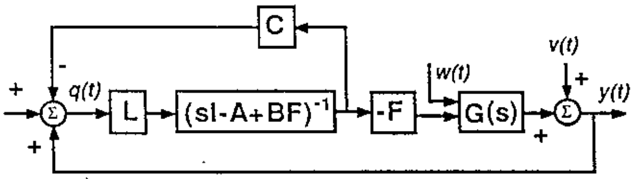

Figure 4.1

State feedback /estimation based controller with plant

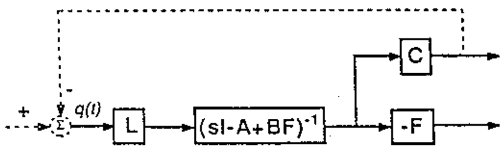

Figure 4.2

"Controller" (heavy line) and interconnection rule (dashed line)

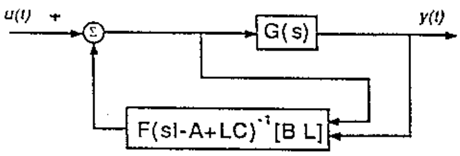

Figure 4.3

State feedback law / estimator design of controller

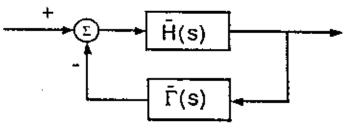

Figure 4.4

Redrawing of the scheme of Figure 4.3

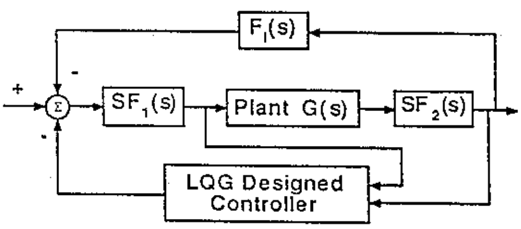

Figure 4.5

General setup allowing integral control and/or shaping filter

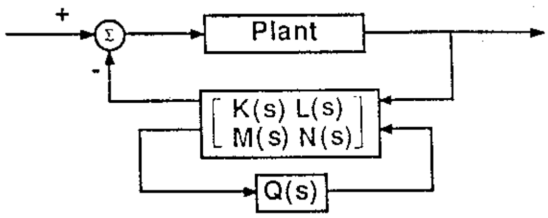

Figure 4.6

Closed-loop stable for all stable Q(s)

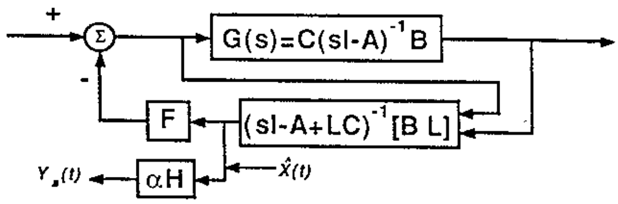

Figure 5.1

Introduction of artificial output

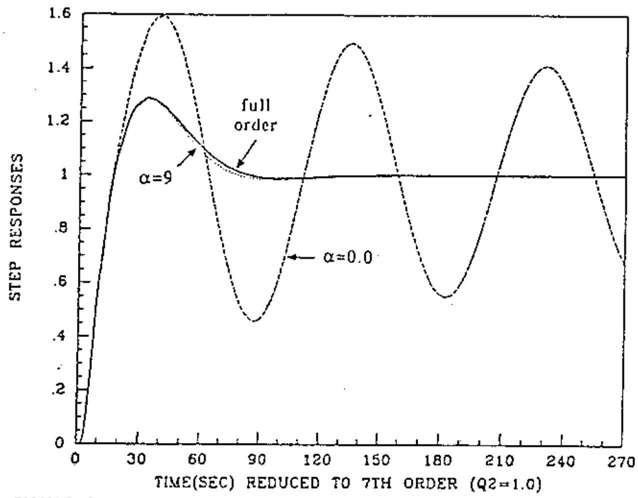

FIGURE 5.2 EFFECT OF CONSIDERING PERFORMANCE IN REDUCTION

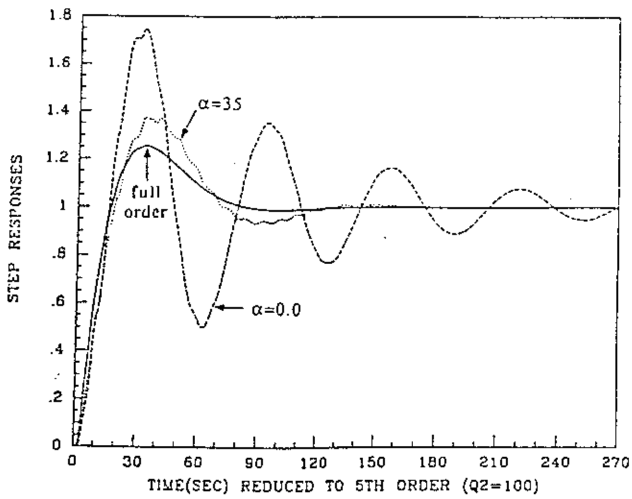

FIGURE 5.3 EFFECT OF CONSIDERING PERFORMANCE IN REDUCTION