Lecture Notes in Control and Information Sciences

Edited by M. Thoma and A. Wyner

INRIA 83

Analysis and Optimization of Systems

Proceedings of the Seventh International Conference on Analysis and Optimization of Systems
Antibes, June 25-27, 1986

Edited by
A. Bensoussan and J. L. Lions

Springer-Verlag
Berlin Heidelberg New York Tokyo

A SUMMARY OF RECENT RESULTS ON THE SCALAR RATIONAL INTERPOLATION PROBLEM

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Abstract. A summary of the results obtained in ANTOULAS [1986a] is presented.

Consider the pairs of points (x_i, y_i), $i \in \underline{N}$, where each entry belongs to some arbitrary but fixed field. The fundamental problem to be investigated is to find all rational functions

$$y(x) = n(x)/d(x);\ n,\ d\text{: coprime polynomials,} \tag{1}$$

in particular the ones having minimal complexity, which interpolate the above points. If these points are distinct, i.e. $x_i \neq x_j$, $i \neq j$, then we must have $y(x_i) = y_i$, $i \in \underline{N}$.

The straightforward approach to the problem is the following. Let $y(x)$, defined by (1), be an interpolating function of degree m (i.e. $deg\ y := \max\{deg\ n, deg\ d\} = m$). We define X to be the $N \times (m+1)$ Vandermonde matrix whose i-th row is $(1\ x_i\ \cdots\ x_i^{m+1})$, and $Y := diag\,(y_1, \ldots, y_N)$ (it is assumed for simplicity that all pairs (x_i, y_i) are finite). Let ν, δ be (m+1)-column vectors containing the coefficients of the polynomials $n(x)$, $d(x)$, starting with the constant term. Clearly (1) implies:

$$\begin{bmatrix} X & -YX \end{bmatrix} \begin{bmatrix} \nu \\ \delta \end{bmatrix} = 0. \tag{2}$$

The question arises as to whether every $(\nu'\ \delta')'$ in the kernel of $(X\ \ -YX)$, for some m, defines a function which interpolates the given pairs of points. The answer is no, because in case the resulting $n(x)$, $d(x)$ have common factors, the additional condition

$$\begin{bmatrix} X & -YX \end{bmatrix} \begin{bmatrix} \bar{\nu} \\ \bar{\delta} \end{bmatrix} = 0,$$

must be satisfied, where $\bar{\nu}, \bar{\delta}$ are the coefficient vectors of the coprime numerator and denominator polynomials of the rational function $n(x)/d(x)$. The set of all functions interpolating the given points is thus determined by the coefficient vectors satisfying (2), as well as

$$resultant\,(\nu, \delta) \neq 0; \tag{3}$$

this ensures the coprimeness of $n(x)$, $d(x)$. The problem thus reduces to the solution of the set of linear equations (2) subject to the constraint (3). The latter being rather difficult to handle, the problem needs to be reformulated so that (3) becomes easier to deal with.

For this purpose, we notice that one rational interpolating function $y(x)$ is given by:

$$\sum_{i \in \underline{N}} c_i \frac{y(x) - y_i}{x - x_i} = 0, \quad c_i \neq 0. \tag{4}$$

Clearly $y(x) = y_i$ if and only if $c_i \neq 0$. Depending on the particular choice of the $c_i's$, the degree of $y(x)$ is at most N - 1 (generically this upper bound is attained).

Our goal is to investigate the algebraic structure of the problem of parametrizing all interpolating functions , in particular those of minimal complexity (degree). One way for doing this is to try to determine those non-zero values of the coefficients c_i, $i \in \underline{N}$, in (4) for which we have the greatest number of pole-zero cancellations between the numerator and denominator polynomials of y. Another way for minimizing the degree of y, which is the one we have adopted, is the following. We consider a summation as in (4) containing only $q < N$ summands; for any set of non-zero $c_i's$, the rational function y, of generic degree q - 1, interpolates the first q points. Making use of the freedom in choosing the $c_i's$, we then try to achieve the interpolation of the remaining N - q points. Let $c := (c_1 \cdots c_q)'$; in order for the remaining N - q points to be interpolated, c must be in the kernel of the $(N-q) \times q$ matrix.

$$L := \left[\frac{y_i - y_j}{x_i - x_j}\right], \; j = 1, 2, ..., q, \; i = q+1, ..., N. \tag{5}$$

This is a *Löwner or divided-differences matrix* derived from the given (distinct) pairs of points. If we have multiple points the corresponding matrix is called *generalized Löwner matrix.* The (generalized) Löwner matrix turns out to be the fundamental tool for the investigation of the rational interpolation problem. It allows constraint (3) to be treated in a straightforward way.

The main result (ANTOULAS [1986a, Section 2]) asserts that the minimal degree of the interpolating function(s) is either rank L or N - rank L, according to whether certain explicitly stated conditions are satisfied or not. In the former case the minimal interpolating function is unique, while in the latter it is non-unique, having N - 2rank L + 1 degrees of freedom. A parametrization of all minimal and non-minimal interpolating functions follows. The problems of proper rational and polynomial interpolation are briefly examined. The third section deals with the problem of recursiveness. The main question is how to (minimally) update the interpolating function whenever additional points are provided, without having to start from scratch. It is first shown how to parametrize all minimal interpolating functions, given a single one; then, how to find one minimal updating of a given interpolating function. These two results combined provide a parametrization of all minimal updatings. The results on recursiveness derived in Section 3, are based on a linear fractional representation formula, much as in the partial realization case (see ANTOULAS [1986b]).

All the results described above have been derived in the case of multiple interpolation points.

Few accounts on the algebraic aspects of the interpolation problem have appeared in the literature. BELEVITCH [1970] discusses some of the connections between Löwner matrices and the interpolation problem in the case of distinct points. More recently FIEDLER [1984] discusses various properties of square Löwner matrices. Both of these papers deal essentially with the generic case, i.e. the case where $2m + 1$ pairs of points are interpolated by a *rational function of degree* m.

The (partial) realization problem of linear system theory, can be viewed as a special case of the rational interpolation problem described above, where all the $x_i's$ are the same (conventionally taken to be the point at infinity). The main tool for the study of the (partial) realization problem is the (partially defined) Hankel matrix (see e.g. KALMAN [1979] and BOSGRA [1983]). The question arises as to what the generalization of the Hankel matrix is in the case of the general interpolation problem. It turns out that the generalized Löwner matrix, defined for pairs of points with the same $x_i's$, has Hankel structure, and indeed is part of the Hankel matrix of the corresponding partial realization problem. Thus the theory of the (scalar) rational interpolation problem presented in ANTOULAS [1986a], constitutes the generalization of the (scalar partial) realization problem.

The interpolation problem has numerous applications in control theory. In ANDERSON and LINNEMANN [1985] a problem of compensator complexity in decentralized control is shown to reduce to an interpolation problem. Recently also, the close connection between H^∞– optimization in linear control systems and the interpolation problem (with stability requirements) has been demonstrated (see CHANG and PEARSON [1984]).

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