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A SUMMARY OF RECENT RESULTS ON THE SCALAR RATIONAL INTERPOLATION PROBLEM

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Abstract. A summary of the results on the scalar rational interpolation problem obtained in ANTOULAS and ANDERSON [1986] is presented.

The summary.

Consider the pairs of points $(x_i, y_i), i \in \underline{N}$, where each entry belongs to some arbitrary but fixed infinite field. The fundamental problem to be investigated is to parametrize all rational functions

$$y(x) = n(x)/d(x), \tag{1}$$

in particular the ones having minimal complexity, which interpolate the above points. If these points are distinct, i.e. $x_i \neq x_j, i \neq j$, then we must have $y(x_i) = y_i, i \in \underline{N}$.

The straightforward approach to the problem is the following. Let $y(x)$, defined by (1), be a rational function of degree m, i.e.

$$deg\, y := \max\,\{deg\, n, deg\, d\} = m.$$

We define X to be the $N \times (m+1)$ Vandermonde matrix whose i-th row is $X_i = (1 \;\; x_i \;\; \cdots \;\; x_i^m)$, and $Y := diag\,(y_1, ..., y_N)$ (it is assumed for simplicity that all pairs (x_i, y_i) are finite and distinct). Let ν, δ be (m+1)-column vectors containing the coefficients of the polynomials $n(x), d(x)$, starting with the constant term. It is readily checked that one parametrization of the set of all interpolating functions of degree at most m, is given as follows:

$$\begin{bmatrix} X & -YX \end{bmatrix} \begin{bmatrix} \nu \\ \delta \end{bmatrix} = 0, \tag{2}$$

subject to the constraints

$$X_i\delta \neq 0\,, i \in \underline{N}, \tag{3}$$

which ensure that $x_i, i \in \underline{N}$, is not a common factor of the polynomials $n(x), d(x)$. The problem thus reduces to finding those m for which equations (2), subject to (3), have a solution. The difficulty with this setting is that m is not properly encoded in X, Y and can be deduced only by trial and error. The need for a different approach, i.e. a repackaging of the data, becomes apparent.

For this purpose, we notice that one rational interpolating function $y(x)$ is determined by the equation:

$$\sum_{i \in \underline{N}} c_i \frac{y(x) - y_i}{x - x_i} = 0, \;\; c_i \neq 0. \tag{4}$$

Clearly, $y(x_i) = y_i$, if $c_i \neq 0$. Depending on the particular choice of the $c_i's$, the degree of $y(x)$ is at most N - 1 (generically this upper bound is attained).

The goal is to investigate the algebraic structure of the problem of parametrizing all interpolating functions , in particular those of minimal complexity (degree). One way for doing this is to try to determine those non-zero values of the coefficients $c_i, i \in \underline{N}$, in (4) for which we have the greatest number of pole-zero cancellations between the numerator and denominator polynomials of y. Another way for minimizing the degree of y, which is the one we have adopted, is the following. We consider a summation as in (4) containing only $q < N$ summands; for any set of non-zero $c_i's$, the rational function y, of generic degree q - 1, interpolates the first q points. Making use of the freedom in choosing the $c_i's$, we then try to achieve the interpolation of the remaining N - q points. Let $c := (c_1 \; \cdots \; c_q)'$; in order for the remaining N - q points to be interpolated, c must be in the kernel of the $(N-q) \times q$ matrix

$$L := \begin{bmatrix} \frac{y_i - y_j}{x_i - x_j} \end{bmatrix}, \;\; j = 1, 2, ..., q, \;\; i = q+1, ..., N. \tag{5}$$

This is a *Löwner or divided-differences matrix* derived from the given (distinct) pairs of points (see BELEVITCH [1970]). The corresponding matrix for multiple points is called *generalized Löwner matrix*. The (generalized) Löwner matrix turns out to be the fundamental tool for the investigation of the rational interpolation problem. The main property of this matrix is that its rank is related in a simple way, to the degree of the corresponding minimal-degree interpolating function(s).

The main result in ANTOULAS and ANDERSON [1986], is Theorem (2.25); it asserts that the minimal degree of the interpolating function(s) is either rank L or N - rank L, according to whether certain explicitly stated conditions are satisfied or not. In the former case the minimal interpolating function is unique, while in the latter it is non-unique, having N - 2rank L + 1 degrees of freedom. There follows a parametrization of all minimal and non-minimal interpolating functions in the form (4), for appropriate q and c. The next section of the above mentioned reference deals with the problem of recursiveness. The main question is how to (minimally) update the interpolating function whenever additional points are provided, without having to start from scratch. It is first shown how to parametrize all minimal interpolating functions, given a single one of them; the second step consists in showing how to determine one minimal updating of a given interpolating function. These two results combined provide a parametrization of all minimal updatings. The investigation of recursiveness is based on a linear fractional representation formula, much as in the partial realization case (see ANTOULAS [1986]). The results just described have been derived for the general case of multiple interpolation points.

The (partial) realization problem of linear system theory, can be viewed as a special case of the rational interpolation problem, where all the $x_i's$ are the same (conventionally taken to be the point at infinity). The main tool for the study of the (partial) realization problem is the (partially defined) Hankel matrix (see e.g. KALMAN [1979] and BOSGRA [1983]). The question arises as to what the generalization of the Hankel matrix is in the case of the general interpolation problem. An important consequence of our approach is the fact that the generalized Löwner matrix, defined for pairs of points with the same $x_i's$, has Hankel structure, and indeed is part of the Hankel matrix of the corresponding partial realization problem. This shows that in the context of interpolation problems, Hankel matrices are generalized to Löwner matrices. Thus the theory of the (scalar) rational interpolation problem presented in this paper constitutes the generalization of the (scalar partial) realization problem.

The interpolation problem has numerous applications in network, system and control theory. A classical paper on the use of interpolation in network and system theory is YOULA and SAITO [1967]. More recent references include CHANG and PEARSON [1984], ANDERSON and LINNEMANN [1985], to mention but two. In the first, the close connection between H^∞-optimization in linear control systems and the interpolation problem (with stability requirements) is demonstrated. In the second it is shown that a problem of compensator complexity in decentralized control reduces to an interpolation problem.

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