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THE APPROXIMATION OF POWER SPECTRA BY PHASE MATCHING

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ABSTRACT

We consider the problem of approximating high order multivariable power spectra by lower order power spectra using the phase matching approach of Jonckheere, Helton and Harshavadhana [1,2,3]. The motivation for the phase matching approach is the identification [1,2,3] of the Hankel operator associated with the "phase function" as Akaike's canonical correlation operator [4]. Thus the phase matching approach can also be viewed as an algorithm for the approximate realization of a stochastic process by canonical variables, which is desirable for the implementation of an approximate Kalman filter of the power spectrum.

1. INTRODUCTION

Given a (high order) pxp, real, proper, rational power spectrum P(s) we study the problem of approximating P(s) by a power spectrum $\hat{P}(s)$ of lower order.

The recent development of model reduction techniques for deterministic systems (eg. [5,6]) motivates an examination of how these techniques should be adapted for stochastic model reduction. One such technique is described by Desai and Pal in [7], and is a modified version of Moore's technique of balanced truncation [6]. The essential difference between [6] and [7] is that [7] introduces a new balanced structure for stochastic realizations, called a stochastically balanced realization, which differs from the conventional balanced realization of [6]. The criterion for discarding state vector components in [7] is based on the canonical correlation coefficient of each state vector component, a set of numbers introduced in [4].

The canonical correlation coefficient is a precise indicator of the correlation that the corresponding state vector component has with the future of the output stochastic process. Thus the canonical correlation coefficient "provides a rational basis for the decision of fitting a lower dimensional approximation" (Akaike [4], page 163).

Another stochastic model reduction technique based on canonical correlation analysis is the phase matching approach developed by Jonckheere, Helton and Harshavadhana in [1,2,3]. Reference [1] deals with the discrete time scalar case, whilst [2] and [3] consider the continuous time scalar case, although [3] envisions a multivariable generalization of the algorithm.

The phase matching algorithm, as the name suggests, is based on the approximation, by deterministic model reduction techniques (eg. those of [5,6]), of a function called the phase function of the stochastic process (or power spectrum). *The significant result of* [1,2,3] is that the Hankel operator of the process phase function is precisely Akaike's canonical correlation operator, implying that the Hankel singular values of the process phase function are Akaike's canonical correlation coefficients. Consequently model reduction techniques such as optimal Hankel norm approximation [5] or balanced truncation [6] applied to the process phase function select and discard state vector components on the basis of canonical correlation coefficients, and via the phase matching algorithm, produce approximations of the Markovian models developed by Akaike [4].

This paper develops the multivariable phase matching algorithm from the outline of [3], although as a by-product some significant insights relevant to the scalar case are obtained. In particular, the scalar phase matching algorithm described in [1,2,3] appears to have some problems when the power spectrum is non-singular (ie. non-zero) at infinity, whilst the algorithm of [7] applies only when the power spectrum is non-singular at infinity. We give a unified treatment for both singular-at-infinity and non-singular-at-infinity power spectra in the multivariable case. This shows that the algorithm of [7] is in fact just phase matching for non-singular-at-infinity power spectra, using balanced *truncation as the model reduction technique*. Also, as has already been observed in [3], a stochastically balanced realization is equivalent (in a way we will see in section 6) to a balanced realization of the process phase matrix.

The organization of the paper is as follows. In section 2 we review the scalar phase matching algorithm. In section 3 some of the more obvious difficulties involved in a multivariable extension are described. (Some of these were noted in [3], which outlined how multivariable phase matching might work.) Section 4 is a summary of the *factorization results of* [8] which we need to solve the problems described in section 3. Section 5 gives the multivariable phase matching algorithm and finally section 6 carries the entire algorithm through for non-singular-at-infinity power spectra using state space formulas.

2. OUTLINE OF THE SCALAR ALGORITHM

This section provides a brief summary of the important features of the scalar phase matching algorithm as developed in [1,2,3]. The algorithm is schematically described in figure 1, which we now explain.

As indicated, the algorithm can be viewed as two stages, the second of which is the inverse of the first. Stage 1 is the process of obtaining the stable part of the phase of a stochastic process. Stage 2, easily seen to be the more difficult, involves finding the power spectrum associated with a given stable part of a phase function.

Stage 1: *Given a strictly proper, rational, power spectrum p(s), with $p(j\omega) > 0$ for all real finite ω, the strictly proper, stable, minimum phase, spectral factor $v(s)$* is defined (uniquely to within ± 1 multiplier) by the equation

$$p(s) = v(s)v(-s) \qquad (2.1)$$

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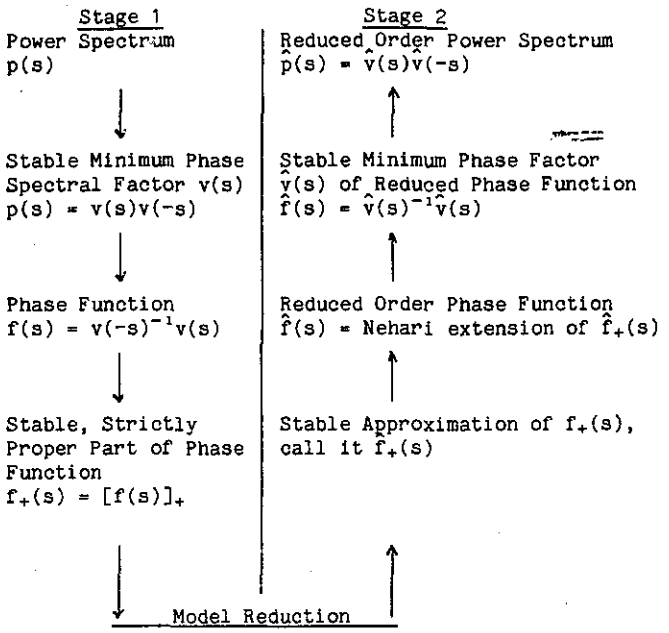

Figure 1: The Scalar Algorithm

The phase function f(s) is then defined as

$$f(s) = \frac{v(s)}{v(-s)} \qquad (2.2)$$

Note that f(s) is all-pass, and the name phase function is justified by the formula

$$f(j\omega) = \exp(2j \times \text{phase } v(j\omega)) \qquad (2.3)$$

Finally, we denote by $f_+(s)$ the stable, strictly proper part of f(s). That is

$$f(s) = f_+(s) + f_-(s) \qquad (2.4)$$

where $f_+(s)$ is stable and strictly proper and $f_-(s)$ is completely unstable (ie. $f_-(-s)$ is stable).

As mentioned in the introduction, the Hankel operator of $f_+(s)$ and the Hankel singular values of $f_+(s)$ are none other than Akaike's canonical correlation operator and canonical correlation coefficients [1,2,3].

Stage 2: *Having obtained a reduced order model of* $\hat{f}_+(s)$, called $\hat{f}_+(s)$, we interpret this function as the stable part of an all-pass function, our approximate phase function $\hat{f}(s)$. It is shown in [2] that f(s) is the (unique) Nehari extension of $f_+(s)$. Thus to obtain $\hat{f}(s)$ from $\hat{f}_+(s)$ we take $\hat{f}(s)$ to be the (unique) Nehari extension of $\hat{f}_+(s)$. The $\hat{f}(s)$ so obtained will be all-pass provided the largest Hankel singular value of $\hat{f}_+(s)$, called $\sigma_1(\hat{f}_+)$, is 1. It is shown in [2] that this is the case if the model reduction technique is balanced truncation or one-step-at-a-time Hankel norm approximation. It is also known [9] that Nehari extensions (with $\sigma_1 = 1$), at least in the scalar case, can be written

$$\hat{f}(s) = \pm\frac{\hat{v}(s)}{\hat{v}(-s)} \qquad (2.5)$$

with $\hat{v}(s)$ strictly proper, stable and minimum phase, and it is shown in [2] that in the phase matching situation the + sign holds. Note $\hat{v}(s)$ is unique up to a scaling constant, which we can choose so that $v(0) = \hat{v}(0)$, or by some other normalization.

Our final comment on the scalar case concerns the assumption that p(s), and consequently v(s), is strictly proper. When p(s) is proper but not strictly proper it is stage 2 which produces difficulties, described in [2]. The essence of the problem is that in this case stage 2 is no longer the inverse of stage 1. This is because $\sigma_1(f_+)$ and $\sigma_1(\hat{f}_+)$ are strictly less than one, implying f(s) is not the Nehari extension of $f_+(s)$, and the Nehari extension $\sigma_1(\hat{f}_+)\hat{f}(s)$ of $\hat{f}_+(s)$ is not all-pass, so it is not a candidate phase function. Moreover, $\hat{f}(s)$ will have a factorization like (2.2) with strictly proper, stable, minimum phase $\hat{v}(s)$, implying the candidate reduced order power spectrum is strictly proper, even though the original spectrum is not. In short, stage 2 is no longer the inverse of stage 1.

Some ad hoc rescue measures are suggested in [2] for this case, but they are hard to justify because they do not recover the essential structure of the phase matching approach, which is the inverse relationship between stages 1 and 2. Under the ad hoc fix-up, following through stage 1 and then the modified stage 2 with no model reduction at all would not return us to our original spectrum. Clearly this is unsatisfactory.

The correct modifications necessary to generalize to non-strictly proper power spectra will be seen when we provide a unified treatment of proper and strictly proper power spectra in the multivariable case. Before proceeding, however, we will point out some further difficulties in the generalization to multivariate power spectra, whether singular or non-singular at infinity.

3. DIFFICULTIES FOR A MULTIVARIABLE ALGORITHM

As might be guessed, it is stage 2 which produces all the problems. However since we intend to maintain the inverse relationship between stages 1 and 2, we will couch the dissusion in terms of the forward and reverse directions of stage 1. What we must do is, firstly, define all the functions of stage 1 (in particular the phase function) and, secondly, relate each function to the one preceding it and following it in a unique way.

The Spectral Factors:

A pxp proper, real, rational, power spectrum P(s), positive definite for all $s = j\omega$, ω real and finite, has left and right spectral factors V(s), W(s) satisfying

$$P(s) = V(s)V(-s)^t = W(-s)^tW(s) \qquad (3.1)$$

with V(s), W(s) proper, stable and minimum phase.
If V(s), W(s) are solutions of (3.1), all other solutions are given by

$$V_X(s) = V(s)X^t \qquad W_Y(s) = Y^tW(s) \qquad (3.2)$$

where X and Y are are arbitrary orthogonal matrices [10]. In the scalar case, X,Y = ±1. We will normalize at some frequency ω_o, which may be infinity, so that

$$W(-j\omega_o)^t = V(j\omega_o) \qquad (3.3)$$

Definition of the Phase Matrix:

What is the appropriate definition of the phase of a multivariable process for this problem?

In the scalar case we saw that f(s) was the transfer function whose Hankel operator is the canonical correlation operator. Thus we must find such a transfer function for the multivariable case. This has been done in [3], where it is shown that the appropriate transfer matrix is

$$F(s) = V(-s)^{-1}W(s)^t \qquad (3.4)$$

where V(s), W(s) are any solutions of (3.2). The normalization (3.3) is defined equivalently by the phase matrix normalization

$$F(j\omega_0) = I \qquad (3.5)$$

Using (3.4) and (3.1) we see that $F(s)F(-s)^t = I$, so F(s) is all-pass.

Nehari Extension:

Obtaining the strictly proper, stable part of F(s) is easy. The reverse operation is not. Any scalar, stable rational function $f_+(s)$ has a unique Nehari extension, so that provided $\sigma_1(f_+) < 1$, f(s) is uniquely determined from $f_+(s)$ by the Nehari extension. In general, a pxp stable matrix function has an infinite number of Nehari extensions [5]. Thus particular attention must be focused on the essential features of the phase matrix F(s). We only want to consider Nehari extensions which have the right properties to be candidate phase matrices. We will show how this defines F(s) from $F_+(s)$ up to a normalization parameter.

Essential Properties of the Phase Matrix:

The most important features of a phase matrix are that it is all-pass and that it has a factorization as in (3.4), with V(s), W(s) proper, stable and minimum phase. Thus we need only consider Nehari extensions which are all-pass and have a factorization as in (3.4). These factorizations are the subject of [8], which we will use to show how a phase function can be determined from its stable part. In some cases (namely when P(s) is non-singular at infinity) F(s) will not be a Nehari extension of $F_+(s)$, but a different extension.

4. FACTORIZATION OF ALL-PASS MATRICES

This section summarizes the results in [8] which are necessary for the multivariable phase matching algorithm.

Definition: Let G(s) be a pxp, proper, rational, matrix function and $G_+(s)$ its stable part with McMillan degree m. The i^{th} Hankel singular value of $G_+(s)$ will be denoted $\sigma_i(G_+)$ and

$$\sigma_i(G) = \sigma_i(G_+) \qquad i = 1,2,\ldots,m \qquad (4.1)$$

We assume the usual ordering

$$\sigma_i(G) \geq \sigma_{i+1}(G) \qquad i = 1,2,\ldots,m-1 \qquad (4.2)$$

(see [5] for details)

Lemma 4.1: Let E(s) be a pxp all-pass with m_1 stable and m_2 unstable poles (counting multiplicities). Then if $m_1 > m_2$

$$\sigma_i(E) = 1 \quad i = 1,\ldots,m_1-m_2 \qquad (4.3a)$$

$$\sigma_i(E) \leq 1 \quad i = m_1-m_2+1,\ldots,m_1 \qquad (4.3b)$$

and if $m_1 \leq m_2$ then

$$\sigma_i(E) \leq 1 \quad i = 1,\ldots,m_1 \qquad (4.3c)$$

This is a generalization of the result that stable all-pass matrices have all their Hankel singular values equal to 1 [5].

The multiplicity of 1 as a Hankel singular value of E(s) is crucial in deciding the factorization properties of E(s). Thus let r denote the multiplicity of 1 as a Hankel singular value of E(s). Of course if $\sigma_1(E) < 1$, we take r = 0. Thus Lemma 4.1 amounts to the equation

$$m_2 \geq m_1 - r \qquad (4.4)$$

Definition 4.2: A pxp all-pass matrix E(s) with m_1 stable and m_2 unstable poles and r as above will be called a minimal all-pass matrix if

$$m_2 = m_1 - r \qquad (4.5)$$

Remarks:

1. A large class of minimal all-pass matrices are those which result from the construction of a Nehari extension via Theorem 6.3 of [5].
2. A scalar all-pass function is minimal if and only if $m_1 \geq m_2$. That is, for p = 1 in Lemma 4.1, the inequalities (4.3b,c) are strict [8].

Theorem 4.1:

Let E(s) be a pxp all-pass matrix. Then E(s) has a factorization as

$$E(s) = V(-s)^{-1}W(s)^t \qquad (4.6)$$

with V(s), W(s) proper, stable and minimum phase if and only if E(s) is a minimal all-pass matrix. Furthermore,

i) V(∞), W(∞) are non-singular if and only if r = 0 (so $m_1 = m_2$)
ii) V(∞), W(∞) are singular if and only if E(s) is a Nehari extension of $E_+(s)$ (ie. r > 0)
iii) V(∞), W(∞) can be zero, ie. V(s), W(s) strictly proper, if and only if E(s) is the unique Nehari extension of $E_+(s)$.

Remarks:

1. Clearly by part (i), the Nehari extension is not the appropriate tool to relate the phase matrix to its stable part when V(s), W(s) are non-singular at infinity.
2. It also follows from Theorem 4.1 and the identification of the Hankel singular values of the phase function of a power spectrum with Akaike's canonical correlation coefficients that processes with P(∞) non-singular have canonical correlation coefficients strictly less than 1.

Lemma 4.2: Let E(s) be a minimal all-pass matrix and let $V_i(s)$, $W_i(s)$ i = 1,2 be two solutions of (4.6) satisfying the associated constraints. Then

$$W_2(s) = W_1(s)U(s) \qquad (4.7a)$$

$$V_2(s) = U(-s)^tV_1(s) \qquad (4.7b)$$

where U(s) is a (non-arbitrary - see [8] for details) unimodular polynomial matrix in s. If r = 0 (or if p = 1, ie scalar E(s)), then U(s) = U is a constant non-singular matrix.

Remark: This result means that normalization of the factors V(s), W(s) could be a problem when r > 0 and P(s) is not scalar.

Since all the phase matrices in the multivariable phase matching algorithm are formed via equation (3.4), we see from Theorem 4.1 that phase matrices are minimal all-pass matrices. Thus to execute the approximation proceedure, we need to know how to construct a minimal all-pass matrix from its stable part. This construction is provided by the minimal all-pass extension theorem, which is a slight modification of Theorem 6.3 of [5] to allow for the case r = 0, ie. $\sigma_1(E_+) < 1$.

Theorem 4.2: (Minimal All-Pass Extension Theorem)

Let G(s) be a pxp, stable, strictly proper, rational, matrix function of McMillan degree m with balanced realization (A,B,C) satisfying

$$G(s) = C(sI - A)^{-1}B \qquad (4.8a)$$

$$A\Sigma + \Sigma A^t + BB^t = 0 \qquad (4.8b)$$

$$A^t\Sigma + \Sigma A + C^tC = 0 \qquad (4.8c)$$

with

$$\Sigma = \mathrm{diag}(I_r, \sigma_{r+1}, \ldots \sigma_m)$$
$$0 < \sigma_m \leq \ldots \leq \sigma_{r+1} < 1 \qquad (4.9a)$$
$$= \mathrm{diag}(I_r, \Sigma_2) \qquad (4.9b)$$

where $r \geq 0$ and I_r denotes the rxr identity matrix. (note: if $r = 0$, omit I_r so $\Sigma_2 = \Sigma$).

Partition A,B,C conformally with Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad C = [C_1, C_2] \tag{4.10}$$

and let D be any orthogonal matrix satisfying

$$B_1 = -C_1{}^t D \tag{4.11}$$

(Note: if $r = 0$, B_1, $C_1{}^t$ have no rows, ie. $B_2 = B$, $C_2 = C$ and D can be an arbitrary orthogonal matrix). Define

$$\tilde{A} = \Gamma^{-1}(A_{22}{}^t + \Sigma_2 A_{22} \Sigma_2 - C_2{}^t D B_2{}^t) \tag{4.12a}$$

$$\tilde{B} = \Gamma^{-1}(\Sigma_2 B_2 + C_2{}^t D) \tag{4.12b}$$

$$\tilde{C} = (C_2 \Sigma_2 + D B_2{}^t) \tag{4.12c}$$

where

$$\Gamma = (\Sigma_2{}^2 - I) \tag{4.13}$$

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix} \quad B_e = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix} \quad C_e = [C, -\tilde{C}] \tag{4.14}$$

Then

$$E(s) = D + C_e(sI - A_e)^{-1} B_e \tag{4.15}$$

is the unique minimal all-pass matrix such that

$$E_+(s) = G(s) \qquad E(\infty) = D \tag{4.16}$$

Remarks:

1. The result is essentially distilled from [5]. In fact for $r > 0$, it is basically a specialized version of Theorem 6.3 of [5]. Note that the 1 and 2 blocks (of Σ, A, B, C) are reversed in their order compared with Theorem 6.3 of [5]. This is to make the notation more natural when we consider model reduction (of G(s)). When $r = 0$, the result is proved in the same way as Theorem 6.3 of [5]. Thus all that needs comment is the minimality of E(s) and the uniqueness.

It follow from parts 3b and 3d of Theorem 6.3 of [5] that A is completely unstable and that (A,B,C) is controllable and observable. Since A is (m-r)x(m-r) it follows that E(s) has m stable and m-r unstable poles, and so is a minimal all-pass matrix, since $E_+(s) = G(s)$ and G(s) has r Hankel singular values equal to 1 (see (4.9)).

The uniqueness result follows from the further characterization of all-pass matrices in section 8 of [5]. A summary of the relevant results is in [8]

2. The minimal all-pass extension theorem says that a minimal all-pass matrix is uniquely determined by its stable part and its value at one point (a particularly convenient choice being infinity, as in Theorem 4.2). Obviously the value at the particular point is limited by the stable part via (4.11). Indeed when rank B_1 (= rank C_1) equals p, D is uniquely determined by (4.11). This is the case if and only if E(s) is the unique Nehari extension of G(s) [5]. Thus there are two extreme cases

i) $r = 0$:
D is a completely arbitrary orthogonal matrix

ii) rank $B_1 = p$ (implies $r \geq p$):
D is uniquely determined by B_1, C_1 via (4.11).

3. The results reported in Theorems 4.1 and 4.2 suggest that a multivariable phase matching algorithm may be possible. We know what all-pass matrices are phase matrices (minimal ones), we know they have the required factorization and we know we can obtain them uniquely from their stable parts provided some normalization is observed. However Lemma 4.2 indicates that the problem of normalizing the factors of the phase matrix when $r > 0$ and $p > 1$ could be a problem.

The following theorem makes the easiest case ($r = 0$) completely solvable with the use of closed form state space formulas.

Theorem 4.3:
Let G(s) be as in Theorem 4.2, with (A,B,C) as in (4.8), (4.9), (4.10) with $r = 0$. Let E(s) be any minimal all-pass extension of G(s) and let $D = E(\infty)$. Then V(s), W(s) satisfy (4.6) and the associated conditions if and only if

$$W(s) = (I - B^t(sI - A^t)^{-1}\bar{B})T \tag{4.17a}$$

$$V(s) = T^t D^t (I - \tilde{C}\Gamma^{-1}(sI - A^t)^{-1}C) \tag{4.17b}$$

for some non-singular matrix T, where $\bar{B}$, $\tilde{C}$, Γ are given by (4.12b,c) and (4.13).

5. THE MULTIVARIABLE PHASE MATCHING ALGORITHM

Given the results of section 4, all that really remains is to write down the algorithm and to clear up some of the details concerning normalization. Figure 2 is a schematic representation of the algorithm

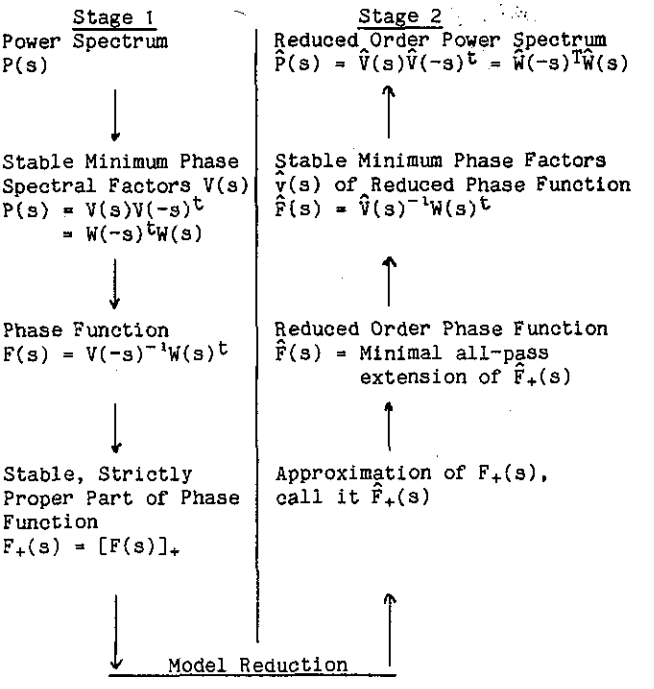

Figure 2: The Multivariable Algorithm

Normalization of the Phase Matrix:

Let us assume, for definiteness, that our normalization point is infinity. Thus we assume $F(\infty) = I$. This means, taking $G(s) = F_+(s)$ in Theorem 4.2 (the minimal all-pass extension theorem), that $D = I$ is a solution of (4.11).

To ensure $\hat{F}(\infty)$ is also normalized to I, we must ensure that I is still a solution of (4.11) when we take $G(s) = \hat{F}_+(s)$ in Theorem 4.2. This is certainly the case when the model reduction algorithm is balanced truncation (see section 6 for complete details), but is not always the case when we use one-step-at-a-time or optimal Hankel norm approximation. Interestingly, when $p = 1$ (the scalar case), 1 as a solution of (4.11) is preserved by one-step-at-a-time Hankel approximation. This comes about because of the very simple structure of orthogonal scalars, which are ± 1. A detailed exposition of phase matching using Hankel norm

approximation is therefore delayed pending further investigation. The details for balanced truncation will be dealt with in section 6.

Normalization of the Factors:

Assuming that we have $F(\infty) = \hat{F}(\infty) = I$, we still have the problem of determining which factors of $\hat{F}(s)$ ra should take. Lemma 4.2 indicates this is rather difficult except for $p = 1$ (the scalar case) or $r = 0$ (the non-singular at infinity case). When $r = 0$, we just choose a non-singular matrix U (in Lemma 4.2) so that $V(\infty) = \hat{V}(\infty)$. It follows of course that we also have $W(\infty) = \hat{W}(\infty)$. Complete details are given in section 6. When $r > 0$ it is not yet clear how the factors of $\hat{F}(s)$ should best be normalized, and a discussion is postponed pending further investigation.

6. STATE SPACE FORMULAS FOR THE PHASE MATCHING ALGORITHM

In this section we provide state space formulas for all steps in the algorithm up to the generation of $\hat{F}(s)$ (see figure 2). For the case $r = 0$ (non-singular-at-infinity power spectra), formulas for $\hat{V}(s)$, $\hat{W}(s)$ will also be given.

Step 1: Spectral Factorization (see eg. [10,11]):

Let

$$P(s) = Z(s) + Z(-s)^t \tag{6.1}$$

$$Z(s) = J + H(sI - F)^{-1}G \tag{6.2a}$$

$$Z(s) \text{ positive real of degree } m \tag{6.2b}$$

Let (P_+, K, V) be a solution triple for the positive real equations for $Z(s)^t$ with P_+ minimum and K with the same dimensions as G^t :

$$P_+F^t + FP_+ + K^tK = 0 \tag{6.3a}$$

$$P_+H^t = G - K^tV^t \tag{6.3b}$$

$$VV^t = J + J^t = R = P(\infty) \tag{6.3c}$$

$$P_+ = P_+^t > 0 \tag{6.3d}$$

Let (P_-, L, W) be a solution triple for the positive real equations for Z(s) with P_- minimum and L with the same dimensions as G :

$$P_-F + F^tP_- + LL^t = 0 \tag{6.4a}$$

$$P_-G = H^t - LW \tag{6.4b}$$

$$W^tW = R \tag{6.4c}$$

$$P_- = P_-^t > 0 \tag{6.4d}$$

Define V(s), W(s) by

$$V(s) = V + H(sI - F)^{-1}K^t \tag{6.5a}$$

$$W(s) = W + L^t(sI - F)^{-1}G \tag{6.5b}$$

Then V(s), W(s) are proper, stable, minimum phase and satisfy

$$P(s) = V(s)V(-s)^t = W(-s)^tW(s) \tag{6.6}$$

The normalization (3.3) can be accomplished by taking

$$W^t = V \qquad (\omega_0 = \infty) \tag{6.7a}$$

or

$$W^t - L^tF^{-1}G = V - HF^{-1}K^t \qquad (\omega_0 = 0) \tag{6.7b}$$

Below we shall assume the normalization (3.5) with $\omega_0 = \infty$, ie.

$$\lim_{\omega\to\infty} V(-j\omega)^{-1}W(j\omega)^t = I \tag{6.7c}$$

Note that the linear system with transfer matrix V(s) is a forwards innovations representation of P(s) with state covariance matrix P_+ and the linear system with transfer matrix $W(-s)^t$ is a backwards innovations representation of P(s) with state covariance matrix P_-.

Definition: [7] An innovations representation of the power spectrum P(s) will be called stochastically balanced if its state covariance matrix is Σ, where

$$\Sigma = \text{diag}\,(\sigma_i)\ ,\ i = 1,2,\ldots,m \tag{6.8a}$$

and

$$\sigma_i \text{ is the } i^{th} \text{ canonical correlation coefficient of } P(s) \tag{6.8b}$$

Note that the existence of stochastically balanced realizations is assured by [4].

Step 2: The Phase Matrix: With V(s), W(s) given as above and the normalization (6.7c), then

$$F(s) = V(-s)^{-1}W(s)^t = F_+(s) + F_-(s) \tag{6.9a}$$

where

$$F_+(s) = K(sI - F^t)^{-1}L \tag{6.9b}$$

and F(s) is the minimal all-pass extension of $F_+(s)$ computed via Theorem 4.2 using $D = I$, assuming the normalization (6.7c) (see appendix for proof of (6.9)). Furthermore, (K,F^t,L) is a balanced realization of $F_+(s)$ if and only if both V(s) and W(s) are stochastically balanced.

Given (6.9) this is seen as follows:
(K,F^t,L) is balanced if and only if $\Sigma = \text{diag}(\sigma_i)$, where σ_i are the Hankel singular values of $F_+(s)$, satisfies

$$F^t\Sigma + \Sigma F + LL^t = 0 \tag{6.10a}$$

$$F\Sigma + F^t\Sigma + K^tK = 0 \tag{6.10b}$$

Clearly this is the case if and only if $P_+ = P_- = \Sigma$ in (6.3a), (6.4a). Recalling that the Hankel singular values of $F_+(s)$ are the canonical correlation coefficients of P(s), we see that V(s), W(s) are stochastically balanced.

Step 3: Model Reduction by Balanced Truncation: Assume (K,F^t,L) is balanced and Σ is given as above. For some integer k, $r \le k \le m$, partition Σ as

$$\Sigma = \text{diag}(I_r, \Sigma_2, \Sigma_3) \tag{6.11a}$$

$$\Sigma_2 = \text{diag}(\sigma_{r+1}, \ldots, \sigma_k) \tag{6.11b}$$

$$\Sigma_3 = \text{diag}(\sigma_{k+1}, \ldots, \sigma_m) \tag{6.11c}$$

$$1 > \sigma_r \ge \ldots \ge \sigma_k > \sigma_{k+1} \ge \ldots \ge \sigma_m > 0 \tag{6.11d}$$

Note that as before, r may be zero. Partition (K,F^t,L) conformally with Σ. Truncating these matrices, define

$$\hat{F} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \quad \hat{K}^t = \begin{bmatrix} K_1^t \\ K_2^t \end{bmatrix} \quad \hat{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{6.12}$$

Then

$$\hat{F}_+(s) = \hat{K}(sI - \hat{F}^t)^{-1}\hat{L} \tag{6.12b}$$

is the k^{th} order approximant of $F_+(s)$ using balanced truncation [6].

Step 4: All-Pass Extension: Now calculate the all-pass extension $\hat{F}(s)$ of $F_+(s)$ using Theorem 4.2, which gives a state variable representation for $\hat{F}(s)$. The most important things to note are that

i) r is preserved.

ii) K_1 and L_1 are preserved, so that $D = I$ is still a solution of (4.11).

Step 5: Factorization: We now restrict ourselves to the case $r = 0$, which is to say $P(\infty)$ is non-singular

and V, W are non-singular. Note that this means the subscript 1 blocks are non-existent and the normalization (6.7c) is equivalent to (6.7a). According to Theorem 4.3 the factors of $\hat{F}(s)$ are

$$\hat{V}(s) = T^t(I - \tilde{C}\Gamma^{-1}(sI - \hat{F})^{-1}K_2^t) \tag{6.13a}$$

$$\hat{W}(s) = (I - L_2^t(sI - \hat{F})^{-1}\tilde{B})T \tag{6.13b}$$

with

$$\tilde{C} = K_2\Sigma_2 + L_2^t \tag{6.14a}$$

$$\tilde{B} = \Gamma^{-1}(\Sigma_2 L_2 + K_2^t) \tag{6.14b}$$

$$\Gamma = \Sigma_2^2 - I \tag{6.15}$$

To normalize, we want $\hat{V}(\infty) = V(\infty)$, so we take $T = W = V^t$ (see (6.7a,c)). Now if G, H of step 1 are partitioned conformally with Σ, see (6.11), then it is easy to verify from (6.3b), (6.4b) that

$$G_2 = -\tilde{B}T \tag{6.16a}$$

$$H_2 = -T^t\tilde{C}\Gamma^{-1} \tag{6.16b}$$

Thus we can re-write (6.13) as

$$\hat{V}(s) = V + H_2(sI - \hat{F})^{-1}K_2^t \tag{6.17a}$$

$$\hat{W}(s) = W + L_2^t(sI - \hat{F})^{-1}G_2 \tag{6.17b}$$

An approximate Kalman filter for the forward direction is $\hat{V}(s)^{-1}$, and one for the backward direction is $\hat{W}(-s)^{-t}$.

Observe that the reduced order left and right spectral factors ($\hat{V}(s)$, $\hat{W}(s)$) obtained by phase matching using balanced truncation when $P(\infty)$ is non-singular are just truncations of the original stochastically balanced left and right spectral factors of P(s) (V(s), W(s)). That is, when $P(\infty)$ is non-singular and we use balanced trucation, we obtain the same reduced spectral factors as would be obtained using the technique of Desai & Pal [7]. Alternatively, we see that the algorithm of [7] is a special case of phase matching.

7. CONCLUSION

We have presented the phase matching algorithm for the approximation of multivariable, continuous time power spectra, generalizing the scalar algorithm of [2,3]. We have also shown that when the power spectrum is non-singular at infinity, the stochastic approximation technique of [7] is equivalent to phase matching using balanced truncation [6].

Contrary to the sentiments of [1,2,3], non-singular-at-infinity power spectra are seen to be far easier to handle than singular-at-infinity power spectra. Indeed a number of questions remain unresolved for singular-at-infinity power spectra, including the appropriate normalization of the spectral and the phase matrix factors and also the preservation of normalizations under Hankel norm model reduction.

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APPENDIX

We now prove (6.9). Note that we know from Theorem 4.1 that F(s) is a minimal all-pass matrix, and consequently is the minimal all-pass extension of $F_+(s)$ computed via Theorem 4.2 using D = I. Thus we must prove (6.9b).

$$\begin{aligned}
&V(-s)K(sI - F^t)^{-1}L \\
&\quad = [V + H(-sI - F)^{-1}K^t]K(sI - F^t)^{-1}L && \text{by (6.5a)} \\
&\quad = VK(sI - F^t)^{-1}L + H(-sI - F)^{-1}K^tK(sI - F^t)^{-1}L \\
&\quad = H(-sI - F)^{-1}[(-sI - F)P_+ \\
&\qquad + P_+(sI - F^t)](sI - F^t)^{-1}L \\
&\qquad + VK(sI - F^t)^{-1}L && \text{by (6.3a)} \\
&\quad = (VK + HP_+)(sI - F^t)^{-1}L - H(sI + F)^{-1}P_+L \\
&\quad = G^t(sI - F^t)^{-1}L - H(sI + F)^{-1}P_+L && \text{by (6.3b)} \\
&\quad = W(s)^t - W^t - H(sI + F)^{-1}P_+L
\end{aligned}$$

Hence

$$\begin{aligned}
V(-s)^{-1}W(s)^t &= K(sI - F^t)^{-1}L \\
&\quad + V(-s)^{-1}[W^t + H(sI + F)^{-1}P_+L] \qquad (A.1)
\end{aligned}$$

Since $V(-s)^{-1}W(s)^t$ is all-pass, it is proper. Thus $V(-s)^{-1}[W^t + H(sI + F)^{-1}P_+L]$ is proper, and is completely unstable, since V(s) is minimum phase and F is stable. Hence

$$[V(-s)^{-1}W(s)^t]_+ = K(sI - F^t)L \qquad (A.2)$$