LINEAR CONTROLLER APPROXIMATION: A METHOD WITH BOUNDS

B. D. O. Anderson

Department of Systems Engineering, Australian National University, Canberra, Australia

Abstract. The problem of approximating a high order controller by a low order one is tackled with the aid of a frequency-weighted Hankel norm approximation procedure. Bounds are obtained which weight those arising in the unweighted Hankel norm approximation procedure by a factor depending solely on the weighting function.

Keywords. Controllers; approximation theory; controller reduction; system order reduction.

INTRODUCTION

This paper is concerned with the task of simplifying a linear, time-invariant compensator which is connected in closed-loop with a given plant. Though the ideas apply to multivariable plants, for simplicity of presentation, we shall suppose that the plant and compensator are single-input, single-output. We shall describe (a) formulation of the approximation problem (b) solution of the approximation problem using an extension of the Hankel-norm approximation procedure of Adamjan, Arov and Krein (1971) (c) bounds on the approximation, extending ideas of Glover (1984).

Of course, the motivation for the problem comes from the fact that in modern control design methods, such as linear-quadratic design, the order of the compensator resulting from such a procedure is comparable with that of the plant, and often much higher than experience suggests is needed.

FORMULATION OF THE APPROXIMATION PROBLEM

Suppose that the plant and controller transfer functions are P(s), C(s) respectively, and define

$$\bar{G} = P(1 + PC)^{-1} \tag{1}$$

Suppose that the interconnection of plant and controller[1] is stable, so that each transfer function $(1 + PC)^{-1}$, $(1 + PC)^{-1}P$ and $C(1 + PC)^{-1}$ is stable. Then it is not hard to check that if C is replaced by $\hat{C} = C + \tilde{C}$, where $\tilde{C}$ is stable, the closed-loop remains stable provided that

$$\sup_{\omega} |\tilde{C}(j\omega)\bar{G}(j\omega)| = ||\tilde{C}\bar{G}||_\infty < 1 \tag{2}$$

This observation suggests that one should seek to approximate C(s) by $\hat{C}(s)$ where the quality of the approximation is measured by the L_∞ norm

$$J = ||[C - \hat{C}]\bar{G}||_\infty \tag{3}$$

and the unstable part (in an additive decomposition) of $\hat{C}$ is identical with the unstable part of C. [Actually, it may be that a better result could be achieved by allowing these unstable parts to not be the same.] Of course, the stable part of $\hat{C}$ has McMillan degree less than that of the stable part of C, when the approximation is seeking a reduced order controller.

Further insight into the appropriateness of (3) can be obtained in the following way. Suppose the controller is used as a series compensator in a unity negative feedback loop. If C is chosen to ensure a high loop gain at low frequencies and if at the same time $|P(j\omega)|$ is not especially large at low frequencies , then we will have $|\bar{G}| \approx |C^{-1}|$ = small at low frequencies. Likewise, at very high frequencies where $|P(j\omega)|$ is small, $|\bar{G}|$ will be small. Thus there is some likelihood that $|\bar{G}|$ will take its largest values in the vicinity of the unity gain crossover frequency - a region where we would normally expect that approximation had to be good.

The above argument suggests that one approach to the controller approximation problem is to design a high order controller, and then obtain a lower order controller, with the same unstable part: obviously, the smaller J is, the better the approximation, so it is legitimate to seek that lower order controller (of prescribed order and unstable part) which minimizes J.

A second, quite independently justified, argument which leads to the posing of a weighted minimization problem in respect of controller reduction is as follows. Assume as above that the controller is used as a series compensator in a unity negative feedback loop. Suppose further that there is process noise (input noise for the plant) and measurement noise (contaminating the output of the plant). With knowledge of the noise spectra, the spectrum of the stationary random process at the input to the controller (which exists after transients have settled down) will be computable. Call it $\Phi(j\omega)$. Since $\Phi(j\omega)$ measures the intensity at a particular frequency of the signals which the controller handles, it seems reasonable to require any approximation to the controller to be more accurate where $\Phi(j\omega)$ is large. With $\bar{G}(j\omega)$ a spectral factor of $\Phi(j\omega)$, this leads us to seek to minimize some norm associated with $(C-\hat{C})\bar{G}$, and the norm (3) is one such candidate.

As it turns out, we do not have a precise solution to the problem of minimizing J, and we consider in the next section a related problem involving the minimization of $(C-\hat{C})\bar{G}$ with a different norm.

Another indirect approach is provided by Enns (1984), who uses a modification of balanced approximation to incorporate weighting.

HANKEL-NORM FREQUENCY-WEIGHTED APPROXIMATION

To begin, let us suppose C is stable. Suppose $\bar{G}$ in (3) is modified (if necessary) so that $\bar{G}$ and $\bar{G}^{-1}$ are both proper. (This can be done without changing the essential nature of the approximation problem.) Also, let us note that in (3) $\bar{G}$ can be replaced by any $\check{G}$ with the property that $|\check{G}(j\omega)| = |\bar{G}(j\omega)|$ for all ω, and the approximation problem is quite unchanged. Let us assume that we choose $\check{G}$ to have all poles and zeros in Re(s) > 0. The only possible difficulty could arise in case $\bar{G}(j\omega_0) = 0$ for some ω_0, and then it is necessary to perturb the problem slightly, to avoid this occurrence. For future reference let us also describe by $\hat{G}$ that minimum phase, stable transfer function for which $|\hat{G}(j\omega)| = |\bar{G}(j\omega)|$ for all ω.

Then a weighted Hankel-norm approximation procedure, developed in Latham and Anderson (1985), can be summarised as follows.

Let C, $\hat{C}$ have dgrees n, k with k<n. Let $\check{G}(s) = \hat{G}(-s)$. Noting that $||[C - \hat{C}]\bar{G}||_\infty = ||[C - \hat{C}]\check{G}||_\infty$, and motivated by the usefulness of optimal Hankel-norm approximation for obtaining (non optimal) L_∞ norm approximations, we shall minimize $||[C-\hat{C}]\check{G}||_H$.

Let X be a degree k Hankel-norm approximation of $C\check{G}$. (We permit X to be nonstrictly proper if this makes $||C\check{G} - X||_\infty$ smaller, see Glover (1984); this reference also explains the construction of an algorithm for finding X.) Define

$$\hat{C} = [X\check{G}^{-1}]_- \tag{4}$$

where $[Z]_-$ denotes taking the constant part and strictly proper, stable part of Z. Then, as shown in Latham and Anderson (1985), if C is stable,

$\hat{C}$ has degree k and is stable (5a)

$||[C - \hat{C}]\check{G}||_H$ is minimized over all stable $\hat{C}$ of degree k (5b)

If C is unstable, we copy its unstable part in $\hat{C}$, and approximate only the stable part of C with a lower order transfer function, which becomes the stable part of $\hat{C}$.

Our main concern in this paper is to find bounds on the L_∞ bound of the weighted error $(C - \hat{C})\check{G}$. In the unweighted case, an extensive theory generating bounds is available in Glover (1984).

CONSTRUCTION OF A BOUND

Let $[Z]_+$ denote the strictly proper, unstable part of Z. Let $E = [C\check{G}]_- - X = [(C-\hat{C})\check{G}]_-$. Bounds on $||E||_\infty$ are available from Glover (1984) in terms of the Hankel singular values σ_i of $[C\check{G}]_-$, thus

$$||E||_\infty \le \sigma_{k+1} + \dots + \sigma_n \tag{6}$$

Now

$$\begin{aligned} ||[C-\hat{C}]\bar{G}||_\infty &= ||[C-\hat{C}]\check{G}||_\infty \\ &\le ||E||_\infty + ||[(C-\hat{C})\check{G}]_+||_\infty \\ &= ||E||_\infty + ||[(E\check{G}^{-1})_-\check{G}]_+||_\infty \end{aligned} \tag{7}$$

To simplify the calculation bounding the second term in (7), let us assume temporarily that we are working in the discrete time domain. Then we can establish several simple results:

Lemma 1. Let $E(z) = \sum_{i=0}^{\infty} E_i z^{-i}$ be rational with poles in $|z| < 1$, let $\check{G} = \sum_{i=-\infty}^{0} H_i z^{-i}$ with poles and zeros in $|z| > 1$. Then

$$\begin{aligned} ||[E\check{G}^{-1}]_-||_2 &\le ||z^{-1}E||_H||\check{G}^{-1}||_2 \\ &\le ||E||_\infty||\check{G}^{-1}||_2 \end{aligned} \tag{8}$$

Proof. Evidently $||[E\check{G}^{-1}]_-||_2 = ||z^{-1}[E\check{G}^{-1}]_-||_2 = ||[(z^{-1}E)\check{G}^{-1}]_-||_2$. By a standard property of the Hankel norm, see Kung and Lin (1981), equation (2.6), $||[(z^{-1}E)\check{G}^{-1}]_-||_2 \le ||z^{-1}E||_H||\check{G}^{-1}||_2$. The second inequality follows because $||z^{-1}E||_H \le ||z^{-1}E||_\infty = ||E||_\infty$, see Kung and Lin (1981).

Lemma 2. With quantities as above

$$||[(E\check{G}^{-1})_-\check{G}]_+||_2 \le ||\hat{G}||_H||[E\check{G}^{-1}]_-||_2 \tag{9}$$

Proof. Let $(E\check{G}^{-1})_- = A = \sum_{i=0}^{\infty} A_i z^{-i}$. Let $G = \sum_{i=0}^{\infty} G_i z^i$. Let $B = \sum_{i=0}^{\infty} A_i z^i$ and observe that $\hat{G} = \sum_{i=0}^{\infty} G_i z^{-i}$, since $|\check{G}| = |\hat{G}|$, and poles and zeros of $\check{G}$ and $\hat{G}$ are the reciprocals of one another. Now

$$\begin{aligned} [(E\check{G}^{-1})_-\check{G}]_+ &= [\sum_{i=0}^{\infty} A_i z^{-i} \sum_{i=0}^{\infty} G_k z^k]_+ \\ &= \text{strictly positive powers of } [\sum_{i=0}^{\infty} A^i z^{-i} \sum_{k=0}^{\infty} G_k z^k] \end{aligned}$$

Evidently

$$||[E\check{G}^{-1})_-\check{G}]_+||_2 = ||\text{strictly negative powers of } [\sum_{i=0}^{\infty} A_i z^i \sum_{k=0}^{\infty} G_k z^{-k}]||_2$$

and by the standard property of the Hankel-norm, we then have

$$\begin{aligned} ||[(E\check{G}^{-1})_-\check{G}]_+||_2 &\le ||\sum_{k=0}^{\infty} G_k z^{-k}||_H||\sum_{i=0}^{\infty} A_i z^i||_2 \\ &= ||\hat{G}||_H||[E\check{G}^{-1}]_-||_2. \end{aligned}$$

Observe that Lemmas 1, 2 together imply that

$$\begin{aligned} ||[(E\check{G}^{-1})_-\check{G}]_+||_2 &\le ||z^{-1}E||_H||\check{G}^{-1}||_2||\hat{G}||_H \\ &\le ||E||_\infty||\check{G}^{-1}||_2||\hat{G}||_H \end{aligned} \tag{10}$$

However, this is of no immediate help in (7), since there is no obvious relation between the L^2 and L^∞ bounds. In pursuit of such a relation, we shall use the following lemma, a proof of which is in the Appendix.

Lemma 3. Let $M(z) = \alpha_0 + \sum_{i=1}^{N} \alpha_i z(1-\beta_i z)^{-1}$, $|\beta_i|<1$. Suppose that the β_i are prescribed and distinct, the α_i are unknown, M(z) is real rational and that $||M||_2^2 \le k$. Then

$$||M||_\infty^2 \le k[1 + ||m'(e^{j\omega})B^{-1}m(e^{-j\omega})||_\infty] \tag{11}$$

where

$$B = \left[\frac{1}{1-\beta_i^*\beta_j}\right] \quad m'(e^{j\omega}) = \left[\frac{1}{e^{-j\omega}-\beta_1} \cdots \frac{1}{e^{-j\omega}-\beta_N}\right] \qquad (12)$$

In case $\alpha_0=0$, (10) is replaced by

$$||\mu||_\infty^2 \le k\ ||m'(e^{j\omega})B^{-1}m(e^{-j\omega})||_\infty \qquad (13)$$

The Lemma applies as follows. The poles of $\check{G}$ are known, and it is not hard to see that they are identical with the poles of $[E\check{G}^{-1})_-\check{G}]_+$. These poles all lie in $|z|>1$. Provided they are distinct, Lemma applies with α_0 to yield

$$||[E\check{G}^{-1}]_-\check{G}]_+||_\infty \le \mu(\check{G})||[E\check{G}^{-1}]_-\check{G}]_+||_2 \qquad (14)$$

where μ^2 is the overbound on the ratio $||\check{G}||_\infty^2/||\check{G}||_2^2$, computable for $\check{G}$ and depending only on the poles of $\check{G}$, as described in Lemma 3.

There follows from (7), (10) and (14) the overbounds we have been seeking:

$$||[C-\hat{C}]\bar{G}||_\infty \le ||E||_\infty + \mu||z^{-1}E||_H||\hat{G}^{-1}\ ||_2||\hat{G}||_H \qquad (15a)$$

$$\le [1 + \mu(\check{G})||\hat{G}^{-1}||_2||\hat{G}||_H]||E||_\infty \qquad (15b)$$

[Observe that $||\check{G}||_2 = ||\hat{G}||_2$]

While (15a) is the tighter bound, (15b) may be more useful in that it separates the error bound into a product of two parts, the "normal" bound which is the bound on $||E||_\infty$, and the second part which depends solely on the weighting. Notice that when the weighting is constant, $||\hat{G}||_H$ will be zero, and there is no additional cost in the approximation procedure which gets reflected in the larger bound.

Notice also that the product $||\hat{G}^{-1}||_2||\hat{G}||_H$ is a form of condition number; the greater the frequency variation in $|\hat{G}|$, the greater this product is likely to be.

Lemma 3 is restricted in the requirement that $M(z)$, equivalently $\check{G}(z)$, have distinct poles. Doubtless it could be extended to cope with multiple poles. Alternatively, should one encounter $\check{G}(z)$ with multiple poles, one could vary one very slightly.

Finally, we remark that it is possible to characterize $\mu(\check{G})$, $||\hat{G}^{-1}||_2$ and $||\hat{G}||_H$ in terms of s-domain quantities, rather than z-domain quantities. There seems numerical benefit in doing so.

EXAMPLES

To illustrate the computation of the quantity $[1 + \mu(\check{G})||\hat{G}^{-1}||_2\hat{G}||_H]$, let us take the s-domain $\hat{G}(s) = (s^2+2s+1)(s^2+s\alpha s+1)^{-1}$, $\alpha<1$, which under the bilinear tranformation $s = (z-1)(z+1)^{-1}$, becomes

$$\hat{G}(z) = \frac{1}{\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)z^{-2}}$$

In Appendix B, we set out calculation leading to

$$\mu(\check{G}) = \sqrt{\frac{2}{\alpha}} \quad ||\hat{G}^{-1}||_2 = \sqrt{\frac{1+\alpha^2}{2}} \quad ||\hat{G}||_H = \frac{1-\alpha}{2\alpha}$$

With $\alpha=0.1$, observe that $G(j0)=G(j\infty)=1$ while $G(j1)=10$. Thus there is a weighting in the vicinity of $\omega=1$ of a factor of 10. With this α, one has

$$\mu(\check{G})||\hat{G}^{-1}||_2||\hat{G}||_H = 14.3$$

With $\alpha=0.01$, the corresponding figure is 500.

Consider now

$$C(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$$

With $\hat{G}(s) = (s^2 + 2s + 1)(s^2 + 2\alpha s + 1)^{-1}$, $\alpha=0.1$, the procedure of Section 3 leads to a first order weighted approximation

$$\hat{C}(s) = \frac{0.9867s + 1.1262}{s + 2.9475}$$

The smallest singular value of $(C\check{G})_-$ is 0.0133 and the bound from the previous section is given by

$$||(C-\hat{C})\hat{G}||_\infty \le 14.3 \times 0.0133$$

$$= .1902$$

Actual evaluation for the functions themselves shows that

$$||(C-\hat{C})\hat{G}||_\infty = 0.050$$

Figures 1 and 2 show as a function of frequency the magnitude of the weighted error $(C-\hat{C})\hat{G}$ and unweighted error $(C-\hat{C})$. We remark that if $\hat{C}$ were determined with no weighting, i.e. $\hat{G}=1$, $C-\hat{C}$ will turn out to have constant magnitude. The effect of the higher weighting around $\omega=1$ is clear from Figure 2.

As a second example, adopt the same $\hat{G}$, but

$$C(s) = \frac{(s^2 + 0.2s + 1.01)(s^2 + 0.2s + 9.01)}{(s^2 + 0.2s + 4.04)(s^2 + 0.2s + 16.02)}$$

With the same weighting function $\hat{G}$, the third order approximation is

$$C(s) = \frac{3.5317s^3 - 3.6926s^2 + 51.9880s - 79.2582}{s^3 + 4.9209s^2 + 16.3429s + 77.1334}$$

$$= \frac{3.5317(s^2 + .4182s + 15.332)(s - 1.4637)}{(s^2 + 0.0828s + 15.9425)(s + 4.8382)}$$

The smallest singular value of $(C\check{G})_-$ is 2.5317 and the bound becomes

$$||(C-\hat{C})\hat{G}||_\infty \le 14.3 \times 2.5317$$

$$= 36.203$$

Direct evaluation yields

$$||(C-\hat{C})\hat{G}||_\infty = 11.9411$$

Figures 3 and 4 show the weighted and unweighted errors as a function of frequency.

CONCLUSIONS

We have shown how the problem of reducing the order of a high order controller can be viewed as a frequency weighted approximation problem. Though a direct solution of this problem is not given, the solutions of a related Hankel-norm approximation problem can be given, together with the construction of bounds on the L_∞ error resulting.

Evidently, one can use a frequency weighted balanced approximation instead of a Hankel-norm approximation to determine a reduced order controller (Enns, (1984)). In the absence of frequency weighting, bounds are available for balanced approximations which are greater than those for Hankel-norm approximations, suggesting that the latter may be generally superior. No bounds are available for frequency-weighted balanced approximation so that a comparison cannot be made.

The formula obtained for the bound in this paper suggests that the greater the variation in the weighting function, the larger will be the bound.

REFERENCES

Adamjan, V.M., D.Z. Arov and M.G. Krein (1977). Analytic properties of Schmidt pairs for a Hankel operator and the generlized Schur-Takagi problem. Math. USSR Sbornik, 15, 31.

Enns, D. (1984). Model reduction for control system design. PhD. Thesis, Department of Electrical Engineering, Stanford University.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. Int. J. Control, 39, 1115.

Kung, S.-Y., and D.W. Lin (1981). Optimal Hankel-norm model reductions: multivariable systems. IEEE Trans. Auto. Control, AC-26, 832.

Latham, G., and B.D.O. Anderson (1985). Frequency-weighted optimal Hankel-norm approximation of stable transfer functions. Syst. and Contr. Letters, 5, 229.

APPENDIX A

Proof of Lemma 3

Before proving Lemma 3 itself, we require a preliminary result.

Lemma A. Let x,y be vectors in C^n and let A be a positive definite hermitian matrix. Let k be positive real. Then

$$\max_x |x^*y|^2 \text{ subject to } x^*Ax \le k$$

is solved by $x = \frac{A^{-1}y}{\sqrt{(y^*A^{-1}y)}}\sqrt{k}$ and is $ky^*A^{-1}y$.

Proof. It is obvious that at the optimum, $x^*Ax = k$. Set $\bar{x} = A^{\frac{1}{2}}x$. Then we are required to maximize

$$x^*yy^*x = \bar{x}^*(A^{-\frac{1}{2}})^*yy^*A^{-\frac{1}{2}}\bar{x}$$

subject to $\bar{x}^*\bar{x} = k$. The maximum is $k\lambda_{max}[(A^{-\frac{1}{2}})^*yy^*A^{-\frac{1}{2}}] = ky^*A^{-\frac{1}{2}}A^{-\frac{1}{2}*}y = ky^*A^{-1}y$. The associated $\bar{x}$ is

$$\bar{x} = \frac{(A^{-\frac{1}{2}})^*y}{\sqrt{(y^*A^{-1}y)}}\sqrt{k}$$

or

$$x = \frac{A^{-1}y}{\sqrt{(y^*A^{-1}y)}}\sqrt{k}$$

Now we turn to the proof of Lemma 3.

With

$$M(z) = \alpha_0 + \sum_{i=1}^{N} \alpha_i z(1-\beta_i z)^{-1}$$

$$= \alpha_0 + \sum_{\ell=1}^{\infty}(\sum_{i=1}^{N}\alpha_i\beta_i{}^{\ell-1})z^{\ell}$$

it follows that

$$||M||_2^2 = |\alpha_0|^2 + \sum_{i,j}\sum_{\ell=1}^{\infty}\alpha_i^*\alpha_j(\beta_i^*)^{\ell-1}\beta_j{}^{\ell-1}$$

$$= |\alpha_0|^2 + \sum_{i,j}\alpha_i^*\alpha_j\frac{1}{1-\beta_i^*\beta_j}$$

$$= [\alpha_0^*\alpha_1^*\ldots\alpha_N^*][1 \dotplus B][\alpha_0\alpha_1\ldots\alpha_N]'$$

Also

$$||M||_\infty^2 = \sup_\omega|[\alpha_0^*\alpha_1^*\ldots a_N^*][1\ \frac{1}{e^{j\omega}-\beta_1^*}\ldots\frac{1}{e^{j\omega}-\beta_N^*}]'|^2$$

Set $\underline{a} = [\alpha_0\alpha_1\ldots\alpha_N]'$, and $p(j\omega) = [1\ m'(e^{j\omega})]'$. Let us seek

$$\max_{\underline{a}}\max_{\omega}\underline{a}^*p(e^{-j\omega})p'(e^{j\omega})\underline{a}$$

subject to $\underline{a}^*[1 \dotplus B]\underline{a} \le k$. It is easily verified using Lemma A that for any fixed ω,

$$\max_{\underline{a}} a^*p(e^{-j\omega})p'(e^{j\omega})a$$

subject to $\underline{a}^*[1 \dotplus B]\underline{a} \le k$ is given by

$$kp'(e^{j\omega})(1 \dotplus B)^{-1}p(e^{-j\omega})$$

$$= k[1 + m'(e^{j\omega})B^{-1}m(e^{-j\omega})]$$

It then follows that with $\underline{a}^*[1 + B]a \le k$,

$$||M||_\infty^2 \le \max_a\max_\omega\underline{a}^*p(e^{-j\omega})p'(e^{j\omega})a$$

$$\le k[1 + \max_\omega m'(e^{j\omega})B^{-1}m(e^{-j\omega})]].$$

The case $\alpha_0 = 0$ follows with trivial variation.

APPENDIX B

Evaluation of $\mu(\check{G})$. There holds

$$\hat{G}(z) = \frac{1}{\frac{1}{2}(1+\omega)+\frac{1}{2}(1+\omega)z^{-2}} \qquad \check{G}(z) = \frac{1}{\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)z^2}$$

and so β_1, $\beta_2 = \pm jr$ where $r^2 = (1-\alpha)(1+\alpha)^{-1}$. The matrices B and B^{-1} becomes

$$B = \begin{bmatrix}\frac{1}{1-r^2} & \frac{1}{1+r^2}\\ \frac{1}{1+r^2} & \frac{1}{1-r^2}\end{bmatrix} \quad B^{-1} = \frac{(1-r^4)^2}{4r^2}\begin{bmatrix}\frac{1}{1-r^2} & -\frac{1}{1+r^2}\\ -\frac{1}{1+r^2} & \frac{1}{1-r^2}\end{bmatrix}$$

and

$$m'(e^{j\omega})B^{-1}m(e^{-j\omega}) = \frac{(1-r^4)^2}{4r^2}\begin{bmatrix}\frac{1}{e^{-j\omega}-jr} & \frac{1}{e^{-j\omega}+jr}\end{bmatrix}$$

$$\times\begin{bmatrix}\frac{1}{1-r^2} & -\frac{1}{1+r^2}\\ -\frac{1}{1+r^2} & \frac{1}{1-r^2}\end{bmatrix}\begin{bmatrix}\frac{1}{e^{j\omega}+jr}\\ \frac{1}{e^{j\omega}-jr}\end{bmatrix}$$

$$= \frac{2(1-r^2)}{1 + r^2 - 2r\cos 2\theta}$$

$$= \frac{2(1-r^4)}{1 + 2r^2\cos 2\omega + r^4}$$

and

$$||m'(e^{j\omega})B^{-1}m(e^{-j\omega})||_\infty = 2\,\frac{1+r^2}{1-r^2} = \frac{2}{\alpha}$$

Evaluation of $||\hat{G}^{-1}||_2$. Observe that $\hat{G}^{-1} = \frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)z^{-2}$. Hence

$$||\hat{G}^{-1}||_2^2 = \tfrac{1}{4}(1+\alpha)^2 + \tfrac{1}{4}(1-\alpha)^2$$
$$= \tfrac{1}{2}(1+\alpha^2).$$

Evaulation of $||\hat{G}||_H$. Since $\hat{G}(z) = \dfrac{\frac{2}{1+\alpha} z^2}{z^2 + \frac{1-\alpha}{1+\alpha}}$

the following is a minimal state-variable realization

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1-\alpha}{1+\alpha} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} -\frac{2(1-\alpha)}{(1+\alpha)^2} \\ 0 \end{bmatrix} \quad D = \frac{2}{1+\alpha}$$

The solutions, P, Q of $P - APA' = BB'$ and $Q - A'QA = CC'$ are

$$P = \frac{(1+\alpha)^2}{4\alpha}I, \quad Q = \frac{(1-\alpha)^2}{\alpha(1+\alpha)^2}I$$

Then we have

$$||\hat{G}||_H = \lambda_{max}^{\frac{1}{2}}(PQ) = \frac{1-\alpha}{2\alpha}$$

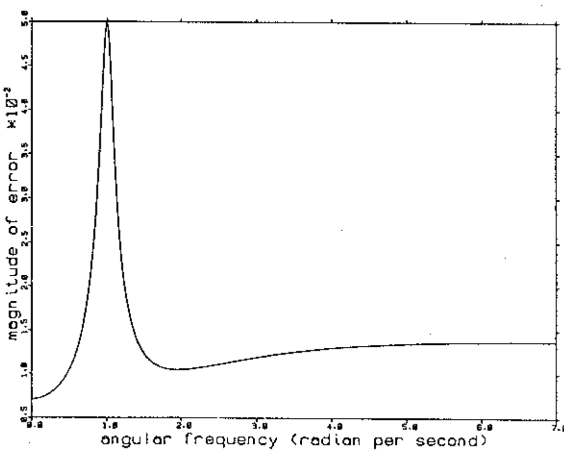

Figure 1: Weighted error for first example

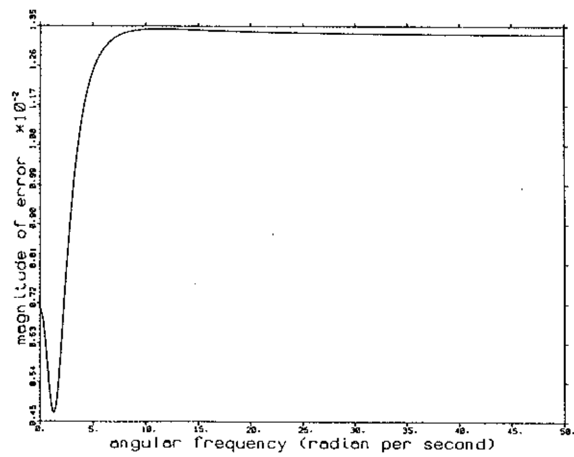

Figure 2: Unweighted error for first example

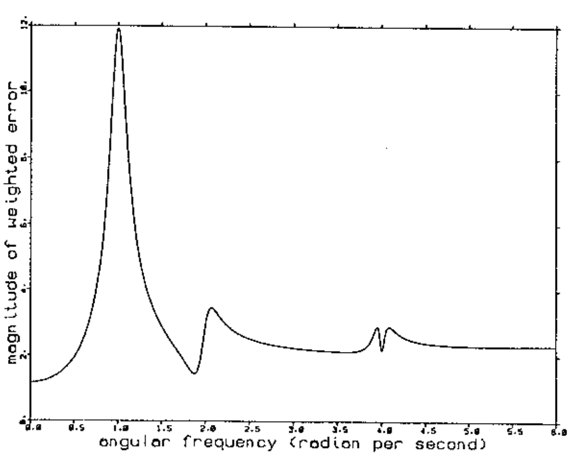

Figure 3: Weighted error for second example

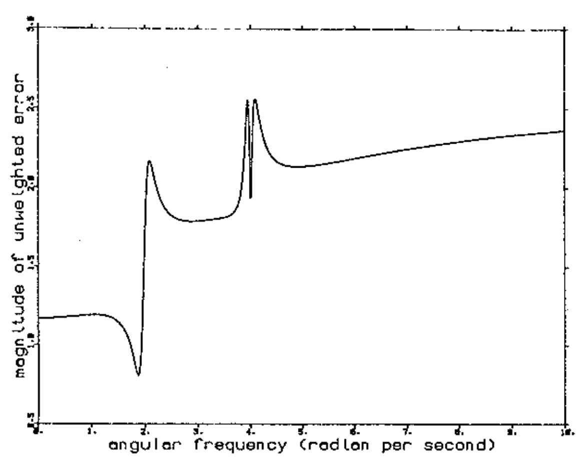

Figure 4: Unweighted error for second example