# POSITIVE REAL CONDITIONS FOR ADAPTIVE CONTROL ARE NOT REALLY NECESSARY

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Abstract. Model reference adaptive control with constrained complexity controllers is studied and stability results obtained. The use and advantages of regression vector filtering are also explained in connection with the stability argument.

Keywords. Adaptive control; adaptive systems; stability

## INTRODUCTION

This paper considers the problem of model reference control for linear discrete-time systems. The principal concern is with the situation where exact following of the model is impossible, because of constraints imposed on the controller complexity. These constraints may in turn arise from a (mistaken) assumption that the plant order is less than is really the case.

In Johnson, Anderson and Bitmead (1984), a model reference scheme of this type was described which was based on direct adaptive control. This reference also explained how filtering of the regression vector could be used. In this paper, we carry these ideas forward in several respects:

(a) A stability analysis is provided. The equations are nonlinear and forced, and we argue first that associated linearised unforced equations are exponentially stable, given suitable conditions on the input and a reasonably close approximation to an exact solution of the model reference problem.

(b) The use of regression vector filtering is motivated in two ways: as an aid to promoting stability, and as a means of approximately securing minimization of a frequency weighted performance index involving the error between the reference model and plant outputs. In Johnson, Anderson and Bitmead (1984), simulation evidence of the properties established here was provided. The simulations are not repeated here, but their existence should be noted.

## 2. DEVELOPMENT OF THE ADAPTIVE ALGORITHM

The true plant is described by

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) \qquad (2.1)$$

where $u(\cdot)$, $y(\cdot)$ are the input and output, $A(0) = 1$, $B(0) \neq 0$, $q^{-1}$ is the unit delay operator. Later, we shall introduce a modification in the form of a disturbance signal, which can be used to account for nonlinearity, etc.

The plant is to be controlled so that its response matches, at least as far as possible, the output $z(\cdot)$ of the reference model.

$$C(q^{-1})z(k) = q^{-\bar{d}}D(q^{-1})r(k) \qquad (2.2)$$

where $r(\cdot)$ is the reference model input, $C(0) = 1$, $D(0) \neq 0$, $\bar{d} \geq d$. Quality of matching is measured by a performance index

$$J = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{m} \{H(q^{-1})\,[z(k) - y(k)]\}^2 \qquad (2.3)$$

where $H(q^{-1})$ and $H^{-1}(q^{-1})$ are stable causal transfer functions, with $H(q^{-1})$ chosen to emphasize the importance of various frequency components.

It is assumed also that $B(q^{-1})$ and $C(q^{-1})$ have all their roots inside the unit circle, as is standard, (Goodwin and Sin (1984)).

To develop the adaptive system equations, we shall proceed in several stages.

<u>Known System with Exact Matching</u>

Suppose that $A(q^{-1})$ and $B(q^{-1})$ are known, and that we use an observer with stable characteristic polynomial $P(q^{-1})$ of degree n. Let $M(q^{-1})$, $N(q^{-1})$ with $M(0) = 1$ be polynomial in $q^{-1}$ solving

$$C(q^{-1})P(q^{-1}) = M(q^{-1})A(q^{-1}) + q^{-d}N(q^{-1}) \qquad (2.4)$$

and define also

$$W(q^{-1}) = M(q^{-1})B(q^{-1}) \qquad (2.5)$$

To obtain a unique solution to (2.4) let us require that

$$\deg N < \deg A \qquad (2.6)$$

Then the arrangement of Fig. 1 achieves exact matching. Fig. 1b, though more complicated than its equivalent Fig. 1a, displays the observer structure. Note that the transfer function from $r_1$ to y is easily verified to be $z^{-d}C^{-1}$ after cancellation of $P(z^{-1})B(z^{-1})$, which is stable. With exact matching, the minimum J in (2.3) is zero, irrespective of $H(\cdot)$.

Defining

$$\deg A = n,\ \deg B = m,\ \deg C = p,\ \deg D = r,\quad \deg P = s,\ \deg M = t,\ \deg N = v \qquad (2.7a)$$

we notice that (2.4) forces, in addition to (2.6),

$$\max\,(n+t),\ d+v) \geq p+s \qquad (2.7b)$$

In addition, generically M and N must have enough variable parameters to adjust all the coefficients on the right of (2.4). This requires

$$t+\bar{v}+1 \geq \max(n+t,\ d+v) \qquad (2.7c)$$

Adaptive system with possibility of exact matching, no regression vector filtering

Now we assume that $A(q^{-1})$ and $B(q^{-1})$ are unknown. Define as the regression vector

$$X(k) = [\bar{u}(k)\ \bar{u}(k-1)\ \dots\ \bar{u}(k-t-m)\ \bar{y}(k)\ \bar{y}(k-1)\ \dots\ \bar{y}(k-v)]^T \qquad (2.8)$$

Block diagram manipulation will show that

$$W(q^{-1})\bar{u}(k) + N(q^{-1})\bar{y}(k) = q^d C(q^{-1})y(k)$$

so that with $W(q^{-1}) = \sum_0^{m+t} w_i q^{-i}$, $N(q^{-1}) = \sum_0^{v} n_i q^{-i}$ and

$$\theta = [w_0\ w_1\ \dots\ w_{t+m}\ n_0\ \dots\ n_v]^T \qquad (2.9)$$

there holds

$$C(q^{-1})y(k) = X^T(k-d)\theta \qquad (2.10)$$

The adaptive control algorithm proceeds as follows. Measure $y(k)$; update $\hat{\theta}(k-1)$ to $\hat{\theta}(k)$ by

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{\gamma(k)X(k-d)}{1+\gamma(k)X^T(k-d)X(k-d)}[C(q^{-1})y(k) - X^T(k-d)\hat{\theta}(k-1)] \qquad (2.11)$$

with $\gamma(k) \in (0,2)$ such that the first entry of $\hat{\theta}(k)$, viz $\hat{w}_0(k)$, will not be zero; using $\hat{\theta}(k)$, take

$$u(k) = -\frac{1}{\hat{w}_0(k)}\left[\sum_{i=1}^{t+m}\hat{w}_i(k)\bar{u}(k-i)\right] + \sum_{j=1}^{s} p_j\bar{u}(k-j) - \frac{1}{\hat{w}_0(k)}\left[\sum_{\ell=0}^{v}\hat{n}_\ell(k)\bar{y}(k-\ell)\right] + \frac{1}{\hat{w}_0(k)}\left[\sum_{h=0}^{r} d_k r(k-h-\bar{d}+d)\right] \qquad (2.12)$$

This equation has a simple interpretation: $u(k)$ is constructed from $\bar{u}(k-i)$, $i=1,..t+m$, $\bar{y}(k-\ell)$, $\ell=0,..,v$ and $r_i(k)$ in the manner depicted by Fig. 1b, save that estimates of the coefficients of W, N are used in place of the true value of those coefficients.

With $\tilde{\theta}(k) = \theta - \hat{\theta}(k)$, (2.10) and (2.11) give the parameter error equation

$$\tilde{\theta}(k) = [I - \frac{\gamma(k)X(k-d)X^T(k-d)}{\gamma(k)X^T(k-d)X(k-d)}]\tilde{\theta}(k-1) \qquad (2.13)$$

and arguments, set out in for example Johnson, Anderson and Bitmead (1984), by now almost standard, establish that $|z(k) - y(k)| \to 0$ and, with a persistency of excitation condition on $X(\cdot)$, that $\tilde{\theta}(k) \to 0$ exponentially fast. Using argument as in Anderson and Johnson (1982), the persistency of excitation condition on $X(\cdot)$ follows from a sufficient richness condition on $r(\cdot)$ and the additional assumption that $A(\cdot)$, $B(\cdot)$ are coprime. Again, J in (2.3) becomes zero.

Adaptive systems with possibility of exact matching, with regression vector filtering

Following Johnson, Anderson and Bitmead (1984), let $F(q^{-1})$, $G(q^{-1})$ have all roots within $|q| < 1$, $F(0) = 1$, $G(0) \neq 0$ and define $Z(k)$, $Y(k)$ by

$$[F(q^{-1})I]Z(k) = [G(q^{-1})I]X(k) \qquad (2.14)$$

$$Y(k) = [G^{-1}(q^{-1})I]Z(k) = [F^{-1}(q^{-1})I]X(k) \qquad (2.15)$$

The adaptive algorithm described above is varied so that $Z(k)$, which is a filtered version of the earlier regression vector $X(k)$, is used in place of $X(k)$ for parameter up-date purposes. This is done as follows. Let

$$F(q^{-1})\tau(k) = C(q^{-1})y(k) \qquad (2.16)$$

Some manipulation shows that, with $F(q^{-1}) = \sum_{i=0}^{\rho} f_i q^{-i}$, $G(q^{-1}) = \sum_{i=0}^{\rho} g_i q^{-i}$, then

$$Z^T(k-d)\tilde{\theta}(k-1) = g_0[C(q^{-1})y(k) - X^T(k-d)\hat{\theta}(k-1)] + \sum_{i=1}^{\rho}[g_i - g_0 f_i][\tau(k-i) - Y^T(k-d-i)\hat{\theta}(k-1)] \qquad (2.17)$$

Since all quantities on the right are known a priori or measurable by time k, the quantity on the left is known. Then (2.11) can be replaced by

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{\gamma(k)Z(k-d)}{1+\gamma(k)Z^T(k-d)Z(k-d)}Z^T(k-d)\tilde{\theta}(k-1) \qquad (2.18)$$

Otherwise, the adaptive algorithm is the same.

Once again, following Johnson, Anderson and Bitmead (1984) and Anderson and Johnson (1982) one can establish that $|y(k) - z(k)| \to 0$ and with coprime $A(\cdot)$ and $B(\cdot)$ and a persistently exciting condition on $Z(k)$, or a sufficient richness condition on $r(k)$, convergence of $\tilde{\theta}(k)$ and $y(k) - z(k)$ to zero is exponential. Finally, the optimum J is zero.

Adaptive Systems with no exact matching possible

We now restrict the controller dimension so that exact asymptotic tracking i.e. $|z(k) - y(k)| \to 0$ is impossible, because the true dimension of the plant is larger than what we care to try to model.

Let us suppose however that there exists a tuned stabilizing controller, defined by polynomials $W^*(q^{-1})$ and $N^*(q^{-1})$, such that with this controller, the closed loop is stable and J in (2.3) is minimized. Of course, $W^*$ and $N^*$ will now in general depend on the external reference signal $r(\cdot)$ and there is an existence question; if $r(\cdot)$ is almost periodic, they will certainly exist. The degrees $\alpha,\beta$ of $W^*$, $N^*$ may be those which would allow exact solution of the tracking problem were the plant of some known lesser order. The coefficients of $W^*$,$N^*$ will be unknown in the adaptive case, but $\alpha,\beta$ are assumed known.

Refer now to figure 2. The nominal control signal is

$$u^*(k) = -\frac{1}{w_0^*}\left[\sum_{i=1}^{\alpha} w_i^*\bar{u}(k-i) + \sum_{\ell=0}^{\beta} n_\ell^*\bar{y}(k-\ell) - \sum_{h=0}^{r} d_h r(k-h-\bar{d}+d)\right] + \sum_{j=1}^{s} p_j\bar{u}(k-j) \qquad (2.19)$$

The actual control $u(k)$ differs from $u^*(k)$, and is defined by (2.12) with the summation limits t+m and v replaced by $\alpha,\beta$:

$$u(k) = -\frac{1}{\hat{w}_0(k)}\left[\sum_{i=1}^{\alpha}\hat{w}_i(k)\bar{u}(k-i) + \sum_{\ell=0}^{\beta}\hat{n}_\ell(k)\bar{y}(k-\ell) - \sum_{h=0}^{r} d_h r(k-h-\bar{d}+d)\right] + \sum_{j=1}^{s} p_j\bar{u}(k-j) \qquad (2.20)$$

Let us now sum up the assumptions and the adaptive algorithm. We work with the plant (2.1), with $A(0) = 1$, $B(0) \neq 0$, and the model (2.2) with $C(0) = 1$, $D(0) \neq 0$, $\bar{d} \geq d$, $C(q^{-1})$ stable. We no longer explicitly require $B(q^{-1})$ stable; but note that if, for example, $\alpha$ and $\beta$ are such that exact matching is possible, then the requirement that the closed-loop be stable with no hidden modes implies that $B(q^{-1})$ must be stable. In principle though, $B(q^{-1})$ does not have to be stable.

The adaptive algorithm now becomes:
(i) measure $y(k)$
(ii) update $\hat{\theta}(k-1)$ to $\hat{\theta}(k)$ by

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{Y(k)Z(k-d)}{1 + Y(k)Z^T(k-d)Z(k-d)}v(k) \quad (2.21)$$

where $Z(k)$ is the filtered regression vector, defined in terms of the standard regression vector $X(k)$ by

$$[F(q^{-1})I]Z(k) = [G(q^{-1})I]X(k) \quad (2.22a)$$

where now

$$X(k) = [\bar{u}(k) \ldots \bar{u}(k-\alpha)\ \bar{y}(k) \ldots \bar{y}(k-\beta)] \quad (2.22b)$$

($\bar{u}$, $\bar{y}$ are defined via Fig. 2); also, $v(k)$ is given by

$$v(k) = g_0[C(q^{-1})y(k) - X^T(k-d)\hat{\theta}(k-1)] + \sum_{i=1}^{\rho} [g_i - g_0 f_i][\tau(k-i) - Y^T(k-d-i)\hat{\theta}(k-1)] \quad (2.23)$$

with $Y(\cdot)$ and $\tau(\cdot)$ as in (2.15) and (2.16)
(iii) apply $u(k)$, as defined in (2.20).

## 3. FORMULATION OF AN ERROR MODEL

Let us begin by making a simple observation.

Lemma 3.1. In the adaptive algorithm stated at the end of the preceding section, and with

$$\theta^* = [w_0^*\ w_1^* \ldots w_\alpha^*\ n_0^* \ldots n_\beta^*] \quad (3.1)$$

$$\tilde{\theta}(k) = \theta^* - \hat{\theta}(k) \quad (3.2)$$

there holds

$$u(k) - u^*(k) = \frac{1}{w_0^*} X^T(k)\tilde{\theta}(k) \quad (3.3)$$

Proof is omitted due to space limitations.
Now with reference to Fig. 2, let

$$y(k) = \tilde{y}(k) + y^*(k) \quad (3.4)$$

where $y^*(k)$ is the portion of the output due to $r(k)$, and $\tilde{y}(k)$ is the portion due to $\tilde{u}(k) = u(k) - u^*(k)$. Modulo decaying exponentials, there holds (as is evident from the figure)

$$\tilde{y}(k) = \frac{q^{-d}B(q^{-1})P(q^{-1})}{W^*(q^{-1})A(q^{-1}) + z^{-d}B(q^{-1})N(q^{-1})}\tilde{u}(k) \quad (3.5)$$

$$y^*(k) = \frac{q^{-\bar{d}}D(q^{-1})B(q^{-1})P(q^{-1})}{W^*(q^{-1})A(q^{-1}) + z^{-d}N^*(q^{-1})B(q^{-1})}r(k) \quad (3.6)$$

Define the transfer function

$$K(q^{-1}) = \frac{P(q^{-1})B(q^{-1})}{W^*(q^{-1})A(q^{-1}) + z^{-d}N^*(q^{-1})B(q^{-1})} \quad (3.7)$$

To make connection with the preceding section, let us observe that there we had, see (2.4) and (2.5),

$$C(q^{-1})P(q^{-1}) = M(q^{-1})A(q^{-1}) + q^{-d}N(q^{-1})$$
$$W(q^{-1}) = M(q^{-1})B(q^{-1})$$

When inserted into $K(q^{-1})$, these identities lead after stable pole-zero cancellation to

$$K(q^{-1}) = \frac{1}{C(q^{-1})} \quad (3.8)$$

In the nonexact matching case, we still expect $1 - K(q^{-1})C(q^{-1})$ to have small magnitude on $|z|$ B 1, and to be stable. For notice that (as is earlier checked) $y^*(k) - z(k) = [K(q^{-1})C(q^{-1}) - 1]z(k)$, and we expect $y^*(k) - z(k)$ to be small in relation to $z(k)$.

Returning to the inexact matching case, observe, using (3.5), (3.6) and (3.3), that

$$\tilde{y}(k) = K(q^{-1})[X^T(k-d)\tilde{\theta}(k-d)] \quad (3.9)$$

$$y^*(k) = q^{-\bar{d}}K(q^{-1})D(q^{-1})r(k) \quad (3.10)$$

Notice that (2.20) and the fact that

$u(k) - \sum_{j=1}^{s} p_j\bar{u}(k-j) = \bar{u}(k)$ yield

$$q^{-\bar{d}}D(q^{-1})r(k) = X^T(k-d)\hat{\theta}(k-d) \quad (3.11)$$

so that

$$y^*(k) = K(q^{-1})[X^T(k-d)\hat{\theta}(k-d)] \quad (3.12)$$

Taking (3.4), (3.9) and (3.10) together, this means that

$$y(k) = K(q^{-1})X^T(k-d)\theta^* \quad (3.13)$$

A crucial quantity in the parameter update equation (2.21) is $v(k)$, given in (2.23). We assert:

Lemma 3.2 With quantities as defined above,

$$v(k) = K(q^{-1})C(q^{-1})[Z^T(k-d)\tilde{\theta}(k-1] + [K(q^{-1})C(q^{-1}) - 1][Z^T(k-d)\hat{\theta}(k-1)] \quad (3.14)$$

$$= KC[Z^T(k-d)\tilde{\theta}(k-1)] + [KC-1]\frac{G}{F}q^{-\bar{d}}Dr(k) + [KC-1][\frac{1}{F}\{\sum_{i=1}^{\rho}[g_iX(k-d-i) - f_iZ(k-d-i)]^T[\hat{\theta}(k-d) - \hat{\theta}(k-d-i)]\} + Z^T(k-d)\{\hat{\theta}(k-1) - \hat{\theta}(k-d)\}] \quad (3.15)$$

Let us sum up the change to the parameter error update equation. With exact matching, we obtain from (2.8)

$$\tilde{\theta}(k) = [I - \frac{Y(k)Z(k-d)Z^T(k-d)}{1+Y(k)Z^T(k-d)Z(k-d)}]\tilde{\theta}(k-1) \quad (3.16)$$

Without exact matching, there obtains using (2.21) and (3.16)

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - \frac{Y(k)Z(k-d)}{1+Y(k)Z^T(k-d)Z(k-d)}K(q^{-1})C(q^{-1}) \times [Z^T(k-d)\tilde{\theta}(k-1)]$$
$$\times [Z^T(k-d)\tilde{\theta}(k-1)] - \frac{Y(k)Z(k-d)}{1+Y(k)Z^T(k-d)Z(k-d)}[KC-1] \times \{\frac{G}{F}q^{-\bar{d}}Dr(k) + \frac{1}{F}\sum_{i=1}^{\rho}[g_iX(k-d-i) - f_iZ(k-d-i)]^T[\hat{\theta}(k-d) - \hat{\theta}(k-d-i)] + Z^T(k-d)[\hat{\theta}(k-1) - \hat{\theta}(k-d)]\} \quad (3.17)$$

The changes can be summarised as follows:
The homogeneous part of (3.16) has an identity operator replaced by $K(q^{-1})C(q^{-1})$ which should at least be close to 1 in some way. A forcing term appears. The smaller $K(q^{-1})C(q^{-1})-1$ is, the smaller is this term. Part of the term is due to the external input $r(\cdot)$. A further part has magnitude in part defined by the rate of variation of $\hat{\theta}$. If $\gamma(k)$ is small, this further part is $O[\gamma^2(k)]$.

## 4. STABILITY OF THE ADAPTIVE SYSTEM

### System with regression vector filtering

Let us assume that $A(\cdot)$ and $B(\cdot)$ are coprime. Equation (2.18) or (3.17) is the most important equation. The argument that $||Z(k)||$ is bounded, and that $|Z^T(k-d)\tilde{\theta}(k-1)| \to 0$ has been given in Johnson, Anderson and Bitmead(1984). With $Z(k-d)$ persistently exciting, then $||\tilde{\theta}(k)|| \to 0$ exponentially fast, Anderson and Johnson (1982). We shall now consider the conditions for $Z(k-d)$ to be persistently exciting. Notice that as $||Z^T(k-d)\tilde{\theta}(k-1)|| \to 0$, $||\tilde{\theta}(k) - \tilde{\theta}(k-1)|| \to 0$ and so the controller coefficients become more and more slowly varying. For any constant controller defined by $W(q^{-1})$, $N(q^{-1})$, there results, as a little calculation shows

$$X(k-d) = [A\ q^{-1}A \ldots q^{-(m+t)}A\ q^{-d}B \ldots q^{-(d+v)}B]^T \times \frac{q^{-d}D}{WA+z^{-d}NB} z(k) \qquad (4.1)$$

If k is very large, and W,N are time-varying, but varying very slowly, (4.1) is still virtually true. We shall state conditions for the persistency of excitation of $Z(k-d)$ and $X(k-d)$ which are valid for any W,N. (Given satisfaction of the condition it then follows of course that W,N asymptotically agree with the values giving exact matching.

Arguing as in Anderson and Johnson (1982) if the signal $[D/(WA+z^{-d}NB)]r$ is sufficiently rich, then $X(k-d)$ will be persistently exciting provided there exist no nonzero constants $\gamma_i$, $\delta_j$ for which

$$[\gamma_0\ \gamma_1 \ldots \gamma_{m+t}\ \delta_1 \ldots \delta_v][A\ q^{-1}A \ldots q^{-(m+t)}A\ q^{-d}B \ldots q^{-(d+v)}B]^T = 0 \qquad (4.2)$$

Since $B(q^{-1})$ and $A(q^{-1})$ are coprime, this can be shown to be impossible. Since G,F are both stable polynomials, it is not hard to check that $Z(k-d) = (\frac{G}{F}I)X(k-d)$ is persistently excitng when $X(k-d)$ is persistently exciting, while $[D/(WA + Z^{-d}NB)]r$ is sufficiently rich if Dr is sufficiently rich. See for example Boyd and Sastry (1984) for a quick argument.

Consequently, if Dr contains at least m+t+1+v Max (m+t+n+1, d+v+m+1) complex frequencies, or r(k) satisfies a related time domain condition, of a type set out in, for example, Anderson and Johnson (1982), Z(k-d) is persistently exciting.

### Stability of the error model with filtering and inexact matching

We turn now to an examination of (3.17).

Let $Z^*(k-d)$ be the value of $Z(k-d)$ obtained when a controller is inserted which is fixed, and defined by $\theta^*$. Consider temporarily the equation

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - \frac{\gamma(k)Z^*(k-d)}{1+\gamma(k)Z^{*T}(k-d)Z^*(k-d)} \times K(q^{-1})C(q^{-1})[Z^{*T}(k-d)\tilde{\theta}(k-1) \qquad (4.3)$$

and suppose $r(k)$ is such that $Z^*(k-d)$ is persistently exciting. (Conditions guaranteeing this are discussed later.)

Refer now to Figure 3. The upper part of the figure (above the dashed line) denotes an exponentially stable system. Hence the system with input $s_1(k)$, output $s_2(k)$ (without the bottom loop connected) is BIBO stable. With a break in the bottom loop between $s_2(k)$ and $s_3(k)$, we see that the gain from $s_3(k)$ to $s_1(k)$ will be small (in the $\ell_p$ sense for any $1 \le p \le \infty$) if KC is stable and $||KC-1||$ for an appropriate norm is small. Then the loop gain from $s_3(k)$ to $s_2(k)$ will be less than 1 if $||KC-1||$ is small, and then (4.3) will be movably exponentially stable.

There are several other ways of considering this set-up. First, suppose that $\gamma(k)$ is sufficiently small that we can neglect terms in $O(\gamma^2)$ in comparision with $O(\gamma)$ terms. Then (4.4) has the same stability properties as

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - \gamma(k)Z^*(k-d)K(q^{-1})C(q^{-1}) \times [Z^{*T}(k-d)\tilde{\theta}(k)] \qquad (4.4)$$

and a sufficient condition for exponential stability, given persistency of excitation of $Z^*(\cdot)$, is that KC be strictly positive real. If $|K(e^{j\omega})C(e^{j\omega}) - 1|$ is less than 1 for all $\omega$, the strict positive real condition holds. It is quite possible however that the condition $Re[K(e^{j\omega})C(e^{j\omega})] > 0$ fails for large $\omega$, i.e. $\pi-\omega$ small. Then we can appeal to results of [5] to conclude that if $\gamma$ is small, and if the failure of $Re[K(e^{j\omega})C(e^{j\omega})] > 0$ occurs in a frequency band where $Z^*(k-d)$ is not frequency rich, then stability is still retained. For $Z^*(\cdot)$ comprising a linear combination of sinusoids, this second idea can be sharpened by appealing to a discrete time version of results of Riedle and Kokotovic (1984): if

$$\bar{A} = I - \lim_{k\to\infty} \frac{1}{k} \sum_{1}^{k} \gamma(j)Z^*(j-d)[K(q^{-1})C(q^{-1})]Z^{*T}(j-d) \qquad (4.5)$$

has eigenvalues inside the unit circle and $\gamma$ is suitably small, then exponential stability holds. For constant $\gamma(k)$, and $Z^*(\cdot)$ of the form nominated, the evaluation is easy to perform, and shows that the bulk of the frequency content of $Z^*(\cdot)$ needs to be where $Re[K(e^{j\omega})C(e^{j\omega})] > 0$.

Next, with $X^*(k-d)$ defined similarly to $Z^*(k-d)$, consider

$$\begin{aligned}\tilde{\theta}(k) = \tilde{\theta}(k-1) &- \frac{\gamma(k)Z^*(k-d)}{1+\gamma(k)Z^{*T}(k-d)Z^*(k-d)} K(q^{-1})C(q^{-1}) \times [Z^{*T}(k-d)\tilde{\theta}(k-1)] \\ &- \frac{\gamma(k)Z^*(k-d)}{1+\gamma(k)Z^{*T}(k-d)Z^*(k-d)} \{\frac{1}{F(q^{-1})} \times \sum_{i=1}^{\rho} [g_i X^*(k-d-i) - f_i Z^*(k-d-i)]^T \\ &\times [\tilde{\theta}(k-d)-\tilde{\theta}(k-d-i)] + Z^{*T}(k-d)[\tilde{\theta}(k-1)-\tilde{\theta}(k-d)]\}\end{aligned} \qquad (4.6)$$

This is a linear equation for $\tilde{\theta}(k)$ still, with the right side of (4.6) differing from the right side of (4.3) by $O(\gamma^2(k))$. It readily follows that with $\{\gamma(k)\}$ suitabily small, (4.6) inherits the stability properties of (4.3).

Now consider (4.6) with $Z^*(\cdot)$, $X^*(\cdot)$ replaced by $Z(\cdot)$ and $X(\cdot)$. Since $Z^*(k-d) - Z(k-d)$ and $X^*(k-d) - X(k-d)$ are $O(\max_{j\leq k} ||\tilde{\theta}_j||)$, as examination of Figure 2 will suggest, it follows that (4.6) represents a linearised version of the following equation:

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - \frac{\gamma(k)Z(k-d)}{1+\gamma(k)Z^T(k-d)Z(k-d)}K(q^{-1})C(q^{-1})$$
$$\times [Z^T(k-d)\tilde{\theta}(k-1)] - \frac{\gamma(k)Z(k-d)}{1+\gamma(k)Z^T(k-d)Z(k-d)}$$
$$\times \{\frac{1}{F(q^{-1})}\sum_{i=1}^{\beta}[g_iX(k-d-i) - f_iZ(k-d-i)]^T$$
$$\times [\tilde{\theta}(k-d) - \tilde{\theta}(k-d-i)]$$
$$+ Z^T(k-d)[\tilde{\theta}(k-1) - \tilde{\theta}(k-d))]\} \qquad (4.7)$$

Accordingly, we can expect exponential convergence at least if the actual values of $||\tilde{\theta}(0)||$ are not too great. Last, when we consider (3.17), which differs from (4.7) by virtue of the inclusion of the forcing term, results on total stability as set out in, for example Anderson and Johnstone (1981) guarantee that (3.17) will enjoy BIBO behaviour in the sense that if the conditions are fulfilled guaranteeing exponential stability of the homogeneous version (4.7) of (3.17), and if in addition the forcing term is small (which will be the case if $|[K(e^{j\omega})C(e^{j\omega})-1][G(e^{j\omega})/F(e^{j\omega})]D(e^{j\omega})|$ is small in the region where r(k) has significant frequency content), then for some constant M,

$$\lim_{k\to\infty}\sup ||\tilde{\theta}(k)|| \leq M \lim_{k\to\infty}\sup ||(KC-1)\frac{G}{F}q^{-\bar{d}}Dr(k)|| \qquad (4.8)$$

Let us sum up again the condition we have imposed:
(a) $Z^*(k-d)$ is persistently exciting
(b) $||KC-1||$ is small, at least $|K(e^{j\omega})C(e^{j\omega})-1|$ is small where $Z^*$ has significant frequency content
(c) $\gamma(k)$ is small
(d) $||\tilde{\theta}(0)||$ is not large
(e) $|[K(e^{j\omega})C(e^{j\omega})-1][G(e^{j\omega})/F(e^{j\omega})]D(e^{j\omega})|$ is small where r has significant frequency content.

The condition on persistency of excitation of $Z^*(k-d)$ requires some comment. First, notice that $Z^*(k-d)$ will be persistently exciting if and only if $X^*(k-d)$ is persistently exciting, with $X^*(k-d)$ given by the following variant on (4.1):

$$X^*(k-d) = [A\ q^{-1}A\ldots q^{-\alpha}A\ q^{-d}B\ldots q^{-(d+\beta)}B]$$
$$\times \frac{q^{-\bar{d}}D}{W^*A+z^{-d}N^*B}r(k) \qquad (4.9)$$

Now there are two ways to think about the requirement on $X^*(k-d)$. First, $X^*(k-d)$ will be persistently exciting if there is no solution of (4.2) with m+t, v replaced by $\alpha,\beta$ and if $[D/(W^*A+z^{-d}N^*B)]r$ is sufficiently rich. Since, in general, either $\alpha<m+t$ or $\beta<v$ or both, we merely need Dr to contain $\max(\alpha+1+n,\ d+v+m+1)$ distinct complex frequencies (or fulfill a related time domain condition). This statement of the condition unfortunately involves m and n, which may be large. The second way to consider the issue is to postulate that

$$Dr(k) = \sum_{i=1}^{M} v_i e^{j\omega_i k} \qquad \omega_i \neq \omega_j \qquad v_i \neq 0 \qquad (4.10a)$$

$$[\gamma_0\ldots\gamma_\alpha\ \delta_0\ldots\delta_\beta]X^*(k-d) = 0 \qquad (4.10b)$$

which implies, with $W(z^{-1})$ the plant transfer function,

$$\left|\sum_{i=0}^{\alpha}\gamma_i z^{-i} + \sum_{j=0}^{\beta}\delta_i z^{-i}W(z^{-1})\right|_{z=e^{j\omega_i}} = 0 \qquad (4.11)$$

for i=1,..M. This states that the plant transfer function can be matched at M distinct frequencies with a transfer function of denominator degree $\alpha$ and numerator degree $\beta$. For generic $\omega_i$, for $M > \alpha+\beta+1$ and for a $W(z^{-1})$ which actually cannot be so represented, (4.12) cannot be fulfilled.

## 5. THE CONVERGENCE REGION

In the nonexact matching case, one cannot expect the error $\tilde{\theta}(k)$, satisfying (3.17), to approach zero as $k\to\infty$ (although in particular situations it may). We recall also that the nominal $\theta^*$ was defined by requiring minimisation of the index J in (2.3). Let us temporarily drop this requirement on $\theta^*$, and seek to describe conditions ensuring that the (asymptotic) time-averaged value of $\tilde{\theta}(k)$ is zero, which means that $\theta^*$ must represent a central value for the fluctuating estimates $\hat{\theta}(k)$. We shall compare this value to the one used earlier, which causes minimization of J (for a particular choice of $H(\cdot)$.

We shall make the following assumptions: (a) r(k) is a linear combination of sinusoids (b) $||Z(k)||$ is bounded, as are solutions of (3.17) (c) A discrete-time version of standard theorems for differential equations in Hale (1969) holds, so that solutions of (3.17) as $k\to\infty$ become almost periodic (in an appropriate discrete-time sense), and possess time-averages (d) A discrete-time version of averaging theorems for differential equations in Hale (1969) holds, when $\gamma$ is suitably small and constant.

Separate unpublished work of the authors has demonstrated, unsurprisingly, the validity of (c) and (d). With the above assumptions in force, we make the following assertion:

<u>Theorem</u> 5.1. Let $\theta_1^*$ define the controller, assumed stabilizing, which causes the average value of the solution of (3.17), with $O(\gamma^2)$ terms neglected, to be zero. Let $\theta_2^*$ define the (fixed) controller, assumed stabilizing which minimizes the index

$$J = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\frac{G(q^{-1})}{F(q^{-1})}C(q^{-1})[z(k)-y(k)]]^2 \qquad (5.1)$$

Then $\theta_1^*$ causes

$$\overline{[Z(k-d)[K(q^{-1})C(q^{-1})-1]Z^T(k-d)\theta_1^*} = 0 \qquad (5.2)$$

and $\theta_2^*$ causes

$$\overline{[K(q^{-1})C(q^{-1})Z(k-d)[K(q^{-1})C(q^{-1})-1]Z^T(k-d)\theta_2^*} = 0 \qquad (5.3)$$

with $K(\cdot)$ depending in (5.2) and (5.3) on $\theta_1^*$, $\theta_2^*$. Note that in case $K(e^{j\omega})C(e^{j\omega})-1$ is small in the region where $Z(\cdot)$ has significant frequency content, (5.2) and (5.3) are evidently approximately equivalent. In Johnson, Anderson and Bitmead (1984) equality of $\theta_1^*$ and $\theta_2^*$ was conjectured.

## 6. ADDITION OF A DISTURBANCE SIGNAL

Referring to figure 2, let us suppose now that $y(\cdot)$ represents the plant output perturbed by a disturbance or noise signal $\eta(\cdot)$. We find that instead of (3.4) we have

$$y(k) = \tilde{y}(k) + y^*(k) + y_\eta(k) \tag{6.1}$$

with $\tilde{y}, y^*$ as before, and

$$y_\eta(k) = \frac{W^*(q^{-1})A(q^{-1})}{P(q^{-1})B(q^{-1})}K(q^{-1})\eta(k) \tag{6.2}$$

Similarly one can determine a correction to $u(k)$. It follows that the regression vector $X(k)$ and then the filtered regression vector $Z(k)$ are modified. If the frequency content of $\eta(\cdot)$ is mainly in a band where $r(\cdot)$ has little frequency content, then the regression vector filtering will be able to suppress some of the effect of the effect of $\eta(\cdot)$.

The disturbance also affects $v(k)$, the normal error signal driving the parameter update. Following the proof of Lemma 3.2, where we start with the expression (2.23) for $v(k)$, we derive in lieu of (3.14),

$$\begin{aligned} v(k) &= K(q^{-1})C(q^{-1})[Z^T(k-d)\tilde{\theta}(k-1)] \\ &\quad + [K(q^{-1})C(q^{-1})-1][Z^T(k-d)\hat{\theta}(k-1)] \\ &\quad + G(q^{-1})F^{-1}(q^{-1})C(q^{-1})y_\eta(k) \end{aligned} \tag{6.3}$$

and (3.17) is modified by the addition on the right side of

$$\begin{aligned} \lambda(k) &= \frac{\gamma(k)Z(k-d)}{1+\gamma(k)Z^T(k-d)Z(k-d)}G(q^{-1})F^{-1}(q^{-1}) \\ &\quad \times \frac{W^*(q^{-1})A(q^{-1})}{P(q^{-1})B(q^{-1})}K(q^{-1})C(q^{-1})\eta(k) \end{aligned} \tag{6.4}$$

If $\sup_k \left\| \frac{G(q^{-1})}{F(q^{-1})}\eta(k) \right\|$ is small enough, the filtered regression vector will be little different, and the extra driving term to (3.17) will be small enough, to continue the possibility of appealing to BIBO stability results, so that an adjustment to (4.6) will apply, with the modification reflecting the disturbance.

## 7. CONCLUSION

This paper has attempted to give theoretical underpinnings to the simulation results reported in Johnson, Anderson and Bitmead (1984). We have been concerned with a nonlinear, forced set of equations, and we have attempted to clarify the intermingled roles of persistency of excitation, strict positive realness, and regression vector filtering in the stability of the algorithm studied. The key conditions are that the gain in the parameter update algorithm be kept small, and the filtered regression vector have the bulk of its frequency content in the region where good model following is possible, even with controller constraints.

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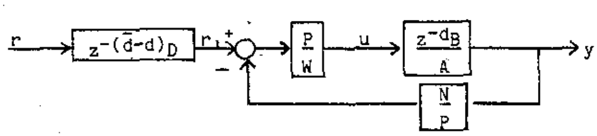


Fig. 1a.

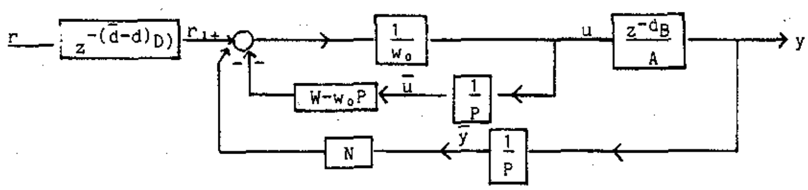


Fig. 1b.

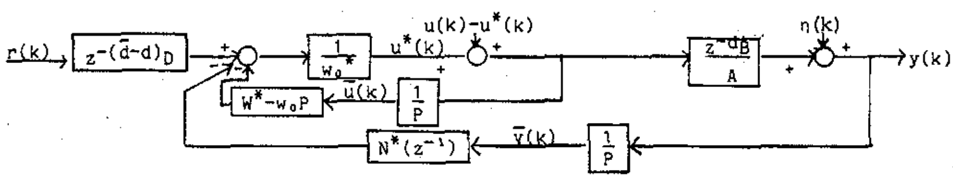


Fig. 2

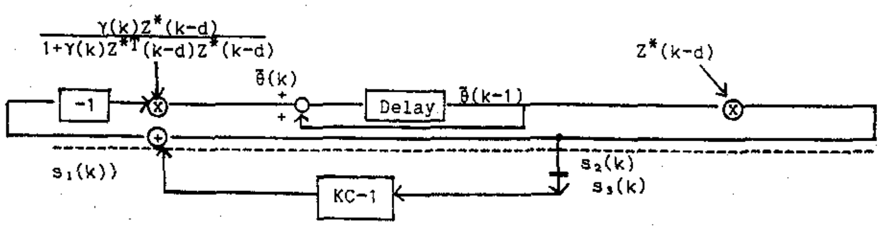


Fig. 3.