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CONTROL OF DECENTRALIZED SYSTEMS WITH DISTRIBUTED CONTROLLER COMPLEXITY

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ABSTRACT

The problem is considered of describing for a two-channel linear system what feedback around the second channel will reduce the McMillan degree of the resulting one-channel system. Primary attention is given to systems where the channel inputs and outputs are scalar. The results are applied to decentralized system control.

1. INTRODUCTION

Consider a linear time-invariant finite-dimensional system with two scalar inputs u_1, u_2 and scalar outputs y_1, y_2. We study the problem of defining a linear, time-invariant, finite-dimensional controller from y_2 to u_2 so that the McMillan degree of the resulting transfer function t_{11} from u_1 to y_1 is less than the McMillan degree of the original system; this means that the feedback controller round channel 2 must introduce pole-zero cancellations (which reduce the McMillan degree) of such a number as to exceed the McMillan degree of the feedback controller itself.

One posible application of the ideas includes the design of decentralized controllers to stabilize an unstable large-scale system. The Corfmat-Morse[1] approach to decentralized controller design roughly speaking puts all the controller complexity round one channel, with memoryless feedback aroung the remaining channels. The idea of this paper would allow consideration, for a two-channel 6th-order system, of whether a 3rd-order controller could be put round the second channel, so that the resulting one channel system looked 3rd-order, and could therefore be controlled by a 3rd-order controller. This distributing of the controller complexity could be a worthwhile objective.

The paper is mainly concerned with systems where the channel inputs and outputs are scalar. For this case, the analysis is done in the frequency domain. It is shown that the zeros of the cross-coupling terms in the system transfer fucntion are the only candidates for uncontrollable or unobservable closed loop poles. Then a controller is constructed which actually introduces these uncontrollable or unobservable modes. The main step in the controller construction is reduced to solving a certain polynomial equation. This problem is shown to be equivalent to a rational interpolation problem, an extension of the partial realization problem[2]. The design procedure is illustrated by means of some low order examples. The vector channel case is also shortly discussed, using state variable ideas. A complete paper with proofs is available from the authors.

2. FREQUENCY DOMAIN APPROACH - SCALAR CHANNELS

Consider the scheme of Figure 1, in which the open loop system is defined by

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = W(s) \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} n_{11}/d & n_{12}/d \\ n_{21}/d & n_{22}/d \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \qquad (2.1)$$

and the feedback controllers are defined by

$$u_i(s) = -\frac{p_i(s)}{q_i(s)} y_i(s) + v_i(s), \quad i = 1,2 \qquad (2.2)$$

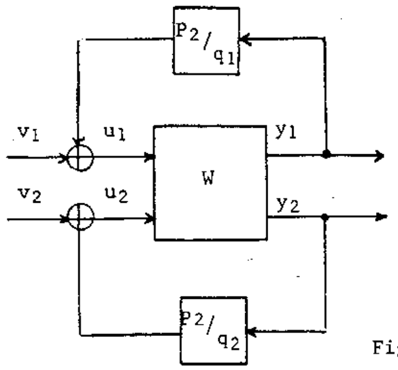

Figure 1

The quantities $u_i(s)$ and $y_i(s)$ for $i=1,2$ are scalar, the quantities $n_{ij}(s)$, $p_i(s)$, $q_i(s)$ and $d(s)$ are all polynomial, $W(s)$ is assumed proper, and $d(s)$ is the characteristic polynomial of $W(s)$, and so is the least common mulitple of the denominators of each entry of $W(s)$ and $\det W(s)$. This means that for the polynomial $\ell(s) \triangleq d(s) \cdot \det W(s)$, there holds

$$d(s)\ell(s) = n_{11}(s)n_{22}(s) - n_{12}(s)n_{21}(s) \qquad (2.3)$$

and

$$\gcd[n_{11}(s), n_{12}(s), n_{21}(s), n_{22}(s), d(s), \ell(s)] = 1 \qquad (2.4)$$

Suppose that $W(s)$ is prescribed, that we first select $p_2(s)/q_2(s)$ and then select $p_1(s)/q_1(s)$. The aim of control includes, but may not be limited to, stabilization. Now after $p_2(s)/q_2(s)$ has been selected and the channel 2 controller inserted, there is an effective transfer function from $u_1(s)$ to $y_1(s)$ which is

$$t_{11}(s) = \frac{n_{11}(s)}{d(s)} - \frac{n_{12}(s)}{d(s)} \frac{p_2(s)/q_2(s)}{1 + [n_{22}(s)/d(s)][p_2(s)/q_2(s)]}$$

$$\times \frac{n_{21}(s)}{d(s)} \qquad (2.5)$$

Algebraic manipulation and use of (2.3) leads to

$$t_{11}(s) = \frac{\ell(s)p_2(s)+n_{11}(s)q_2(s)}{n_{22}(s)p_2(s)+d(s)q_2(s)} \qquad (2.6)$$

One approach to controller design could revolve round choosing p_2/q_2 so that t_{11} (after cancellation) is simpler in some way than $W(s)$. In this section, we discuss how $p_2(s)/q_2(s)$ may be selected so as to minimize the McMillan degree of $t_{11}(s)$, without unstable pole-zero cancellations being introduced in (2.6). Put another way, we are seeking p_2/q_2 so as to maximise the number of asymptotically stable uncontrollable-from-u_1, and/or unobservable-from-y_1 modes.

It is obvious that a cancellation occurs in (2.6) at the common zeros of $\ell(s)$, $n_{11}(s)$, $n_{22}(s)$, $d(s)$, no matter how p_2/q_2 is chosen. This means, that the zeros of the greatest common divisor of $\ell(s)$, $n_{11}(s)$, $n_{22}(s)$, $d(s)$, denoted by $\sigma(s)$, are decentralized fixed modes[3]. In fact, all decentralized fixed modes must be zeros of $\sigma(s)$[4]. Define the polynomials $\bar{\ell}(s):=\ell(s)/\sigma(s)$, $\bar{n}_{11}(s):=n_{11}(s)/\sigma(s)$, $\bar{n}_{22}(s):=n_{22}(s)/\sigma(s)$, $\bar{d}(s):=d(s)/\sigma(s)$ and assume $\sigma(s)$ is stable (has all its zeros in the left half plane). Then the problem can be restated as follows: Choose p_2 and q_2 such that

$$t_{11}(s) = \frac{\bar{\ell}(s)p_2(s)+\bar{n}_{11}(s)q_2(s)}{\bar{n}_{22}(s)p_2(s)+\bar{d}(s)q_2(s)} \qquad (2.7)$$

has least degree without unstable pole-zero cancellations being introduced in (2.7).

It would be nice to also demand that $p_2(s)/q_2(s)$ be proper. As it turns out, generically this will be so, but we may not always enforce the constraints.

Notice that if $p_2(s)/q_2(s)$ is proper the generic McMillan degree of $t_{11}(s)$ is deg $\bar{d}$ + deg q_2. If $p_2(s)/q_2(s)$ is not proper, the generic McMillan degree of t_{11} may be less than, but not greater than, deg $\bar{d}$ + deg p_2.

By definition of $\sigma(s)$ and (2.3) it is clear, that $\sigma(s)^2$ divides $n_{12}(s)n_{21}(s)$. The following first result of this section states that the remaining zeros of $n_{12}(s)n_{21}(s)$, i.e. the zeros of the polynomial $n_{12}(s)n_{21}(s)/\sigma(s)^2$ are crucial for further cancellations to occur in (2.7).

<u>Proposition 2.1</u>: With quantities as defined above, suppose that in (2.7), $p_2(s)$ and $q_2(s)$ are coprime. Suppose $\rho(s)$ is the greatest common divisor of the numerator and the denominator in (2.7). Then $\rho(s)$ divides $n_{12}(s)n_{21}(s)/\sigma(s)^2$.

Proposition 2.1 shows that stable pole-zero cancellations in (2.6) are restricted to the decentralized fixed modes (these cancel irrespective of the choice of p_2 and q_2) and to the stable part of $n_{12}(s)n_{21}(s)/\sigma(s)^2$.

As a preliminary to presenting the converse of Proposition 2.1, we state:

<u>Lemma 2.2</u>: Let $v(s)$ be a divisor of $n_{12}(s)n_{21}(s)/\sigma^2(s)$. Suppose that v is factored as $v(s) = v_1(s)\cdot v_2(s)$, where $\gcd(v_1, \bar{n}_{22}, \bar{d}) = 1$, $\gcd(v_2, \bar{n}_{11}, \bar{\ell}) = 1$, and $\gcd(v_1, v_2) = 1$. [This is always possible, since $\gcd(\bar{n}_{11}, \bar{n}_{22}, \bar{d}, \bar{\ell}) = 1$.] If further $v_1(s), \bar{n}_{22}(s)$ are coprime, $v_2(s), \bar{\ell}(s)$ are coprime, and $p_2(s)$, $q_2(s)$ are chosen such that

$$\bar{n}_{22}(s)p_2(s) + \bar{d}q_2(s) = v_1(s)\alpha(s) \qquad (2.8)$$

$$\bar{\ell}(s)p_2(s) + \bar{n}_{11}(s)q_2(s) = v_2(s)\beta(s) \qquad (2.9)$$

for some $\alpha(s)$, $\beta(s)$, then $v(s)$ is a common factor of $\bar{n}_{22}(s)p_2(s) + \bar{d}(s)q_2(s)$ and $\bar{\ell}(s)p_2(s) + \bar{n}_{11}(s)q_2(s)$.

<u>Remark 2.3</u>: For the above result we need v_1, $\bar{n}_{11}$ and v_2, $\bar{\ell}$ to be coprime. There is however no guarantee that this condition is a priori satisfied. Fortunately, we can enforce this condition by employing a preliminary control. Suppose the feedforward control

$$y_2' = y_2 + fu_2, \text{ f real} \qquad (2.10)$$

is applied to the system (2.1). Then the transfer function $W(s)$ turns into

$$\begin{bmatrix} n_{11}(s)/d(s) & n_{12}(s)/d(s) \\ n_{21}(s)/d(s) & [n_{22}(s) + fd(s)]/d(s) \end{bmatrix}$$

and the polynomial is $\ell(s)$ modified into

$$\ell'(s) = \ell(s) + fn_{11}(s).$$

From this and the definition of $v_1(s)$, $v_2(s)$, it is clear that we can always choose f such that v_1, $\bar{n}_{22}+f\bar{d}$ and v_2, $\bar{\ell}+f\bar{n}_{11}$ are coprime. Indeed almost any real f will do the job. Note that the control (2.10) followed by the control

$$u_2(s) = (p_2(s)/q_2(s))y_2'(s)$$

is equivalent to the control

$$u_2(s) = \frac{p_2(s)}{q_2(s) - fp_2(s)} y_2(s).$$

Lemma 2.2 together with Remark 2.3 reduces the problem to the one of finding p_2 and q_2 satisfying (2.8), (2.9). This last step is accomplished by the following lemma.

<u>Lemma 2.4</u>: Let all quantities be defined as before and let μ:=deg$v$. There exists a solution $p_2(s)$, $q_2(s)$ to (2.8), (2.9) for some $\alpha(s)$, $\beta(s)$, where the polynomials p_2, q_2 satisfy deg p_2, deg $q_2 \le \mu/2$ in case μ is even and deg p_2, deg $q_2 \le (\mu-1)/2$ in case μ is odd. Furthermore, q_2 is not identically zero in case $\bar{n}_{22}$, v_1 and $\bar{\ell}$, v_2 are coprime.

Some further remarks are in order now.

<u>Remark 2.5</u>: Lemma 2.4 does not state that p_2 and q_2 are coprime. Indeed, there are examples where the constructive proof of Lemma 2.4 leads to non-coprime polynomials p_2 and q_2. Of course, there is no sense in implementing a non-coprime controller. Therefore let us consider this case in more detail.

Assume p_2 and q_2 satisfy (2.8), (2.9) for some α, β, and $p_2 = \gamma(s)p_2'(s)$, $q_2 = \gamma(s)q_2'(s)$, where deg $\gamma \ge 1$ and p_2', q_2' are coprime. Using the controller given by p_2' and q_2', there is only a cancellation of order μ-deg γ in (2.6). However, the degree of the controller is also reduced by deg γ. Hence the effective degree of t_{11} is unaffected by using the controller p_2'/q_2' instead of p_2/q_2.

<u>Remark 2.6</u>: Lemma 2.4 does not ensure that p_2/q_2 is proper. Indeed, there are examples for which no proper solutions exist (c.f. Example 3.2 below). Generically however, the leading coefficient of q_2 is non-zero and we have a proper controller. Note also, that by the preliminary control (2.10), a non-proper controller is generically transformed into a proper one.

Remark 2.7: Lemma 2.4 does not assure - besides the cancellations at $\sigma(s)v(s)$ - that there are no further cancellations in $t_{11}(s)$. However, these occur only in nongeneric cases and only at the zeros of $n_{12}n_{21}/\sigma^2 v$ (c.f. Proposition 2.1 and Example 3.3).

Putting together the results of Lemma 2.2, Lemma 2.4, Remark 2.3 and Remark 2.5, we obtain the following main result of this paper.

Theorem 2.8: Consider the arrangement described at the beginning of this section. Suppose that $\sigma(s)$ is stable and of degree r and that $v(s)$ is a stable divisor of $n_{12}(s)n_{21}(s)/\sigma(s)^2$ of degree μ. Then there exist coprime polynomials $p_2(s)$ and $q_2(s)$ of degree no greater than $\mu/2$ in case μ is even and $(\mu-1)/2$ in case μ is odd such that the transfer function

$$t_{11}(s) = \frac{\ell(s)p_2(s)+n_{11}(s)q_2(s)}{n_{22}(s)p_2(s)+d(s)q_2(s)} \tag{2.11}$$

has McMillan degree no greater than degd-r-μ/2 in case μ is even and degd-r-(μ-1)/2 in case μ is odd. Except in non-generic cases p_2/q_2 is proper and there are only stable pole-zero cancellations occurring in (2.11).

Proposition 2.1 provides a partial converse, in that it shows that the number of stable pole-zero cancellations in (2.11) is restricted by r plus the degree of the stable part of $n_{12}(s)n_{21}(s)/\sigma(s)^2$. Especially if $n_{12}(s)n_{21}(s)$ has no stable roots, then it is not possible to find a feedback controller $p_2(s)/q_2(s)$ such that the McMillan degree of t_{11} is smaller than the McMillan degree of W.

Remark 2.9: Note that generically it cannot be recommended that one applies Theorem 2.8 with μ even. If μ is even and $v'(s)$ is a divisor of $v(s)$ of degree μ-1, and $v'(s)$ is cancelled in $t_{11}(s)$ using a controller of the order

$$\frac{(\mu-1)-1}{2} = \frac{\mu}{2} - 1$$

(the odd version of Theorem 2.8), then the McMillan degree of $t_{11}(s)$ is deg d-r-μ/2. Hence the same goal can be accomplished by a controller of reduced dimension.

Remark 2.10: We have seen that the questions (2.8), (2.9) are crucial for the solution of the decentralized control problem. In this remark we show that these equations are equivalent to an interpolation problem. In order to keep the presentation simple, let us assume that $v_1(s)$, $\bar{n}_{22}(s)$ are coprime and $v_2(s)$, $\bar{\ell}(s)$ are coprime. Also, suppose that $v_1(s)$ and $v_2(s)$ both have only simple zeros $s_1,s_2,\ldots,s_\xi$ and $s_{\xi+1},s_{\xi+2},\ldots,s_\mu$, respectively. If further $p_2(s)$ and $q_2(s)$ are coprime, then (2.8), (2.9) may be rewritten as

$$\frac{p_2(s_i)}{q_2(s_i)} = c_i \quad , \quad i = 1,2,\ldots,\mu \tag{2.12}$$

where

$$c_i = \begin{cases} -\dfrac{\bar{d}(s_i)}{\bar{n}_{22}(s_i)} & i = 1,2,\ldots,\xi \\[2ex] \dfrac{\bar{n}_{11}(s_i)}{\bar{\ell}(s_i)} & i = \xi+1,\xi+2,\ldots,\mu \end{cases}$$

Note that $\bar{n}_{22}(s_i) \neq 0$, $\bar{\ell}(s_i) \neq 0$, $q_2(s_i) \neq 0$ because of the coprimeness assumptions. Hence the polynomial equations (2.8), (2.9) are equivalent to the rational interpolation problem (2.12). Characterisations of solvability of this type of problem are e.g. given in [5]. Equations (2.12) will help us when trying to generalize the scalar results to vector problems.

3. EXAMPLES

In this section Theorem 2.8 will be illustrated by means of low order examples.

Example 3.1: Consider

$$\begin{bmatrix} \dfrac{-8(2s^3+3s^2+7s+3)}{(s-1)(s-2)(s-3)} & \dfrac{(s+1)(s+2)}{(s-1)(s-2)(s-3)} \\[3ex] \dfrac{-(s+3)(s+4)}{(s-1)(s-2)(s-3)} & \dfrac{1}{(s-1)(s-2)(s-3)} \end{bmatrix}$$

It is not hard to verify that detW(s)= $s/(s-1)(s-2)(s-3)$. Thus d(s), which is the least common multiple of the denominators of all entries of W(s) and det W(s), is given by $d(s) = (s-1)(s-2)(s-3)$, while $n_{11}(s) = -8(2s^3+3s^2+7s+3)$, $n_{12}(s) = (s+1)(s+2)$, etc. Also, $\ell(s) = s$ and $\sigma(s) = GCD(\ell,n_{11},n_{22}d) = 1$. Since $n_{12}(s)n_{21}(s) = -(s+1)(s+2)(s+3)(s+4)$ we may take $v(s) = n_{12}(s)n_{21}(s)$ with $\mu = 4$. The controller then should have degree $\mu/2 = 2$ and the closed loop system is of order no greater than 3+2-4 = 1. However, as explained in Remark 2.9, it is advisable to take v(s) of degree $\mu = 3$, e.g. $v(s) = (s+2)(s+3)(s+4)$. With this choise, the controller will be of degree $(\mu-1)/2 = 1$ and the closed loop systems has degree no more than 3+1-3 = 1. We make the latter choice on v(s) and assume the controller to be of the form $p_2(s)=as+b$, $q_2(s)=cs+d$. Since n_{22} and v are coprime, we may use $v_1(s) = v(s)$, $v_2(s) = 1$ in Lemma 2.2. Then (2.9) is always satisfied for some $\beta(s)$, and $\alpha(s)$ is of degree *no more than 1. Assuming* $\alpha(s) = es+f$, *(2.8) reads*

$$1(as+b)+(s-1)(s-2)(s-3)(cs+d)=(s+2)(s+3)(s+4)(es+f)$$

i.e.

$$cs^4+(-6c+d)s^3+(11c-6d)s^2+(a-6c+11d)s+(b-6d)$$
$$= es^4+(9e+f)s^3+(26e+9f)s^2+(24e+26f)s+24f.$$

Comparing coefficients of equal powers in s, we obtain the following system of 5 homogeneous linear equations in the unknowns a,b,c,d,e,f:

		c		-e		=	0
		-6c	+d	-9e	-f	=	0
		11c	-6d	-26e	-9f	=	0
a		-6c	+11d	-24e	-26f	=	0
	b		-6d		-24f	=	0

Taking (w.l.o.g.) c = 1, we get a = -240, b = -120, d = 8, e = 1, f = -7. Thus, we have to choose $p_2(s) = -120(2s+1)$, $q_2(s) = s+8$. With this choice of the controller, we get

$$t_{11}(s) = -8\,\frac{2s^4+19s^3+61s^2+74s+24}{s^4+2s^3-37s^2-158s-168}$$

$$= -8\,\frac{(s+2)(s+3)(s+4)(2s+1)}{(s+2)(s+3)(s+4)(s-7)} = 8\,\frac{2s+1}{s-7}$$

In this example, everything is generic. The reader is invited to follow the steps of the proof of Theorem 2.8 with this example.

Example 3.2: This example illustrates that the controller $p_2(s)/q_2(s)$ in Theorem 2.8 may be non-proper. Accordingly, this example is non-generic. Consider

$$W(s) = \begin{bmatrix} \dfrac{-(s+\frac{17}{8})}{s^3+3s^2+3s+2} & \dfrac{-(s+3)}{s^3+3s^2+3s+2} \\ \\ \dfrac{s^3+\frac{25}{8}s^2+3s+\frac{11}{8}}{s^3+3s^2+3s+2} & \dfrac{s^3+3s^2+3s+1}{s^3+3s^2+3s+2} \end{bmatrix}$$

Straightforward computations show that det $W = (s^3+3s^2+3s+2)^{-1}$. Hence $d(s) = s^3+3s^2+3s+2$, $\ell(s) = 1$, $n_{11}(s) = -(s+\frac{17}{8})$, $n_{12}(s) = -(s+3)$, etc. Suppose we want to achieve a cancellation in $t_{11}(s)$ of $v(s) = n_{12}(s)n_{21}(s) = \frac{1}{8}(8s^4+49s^3+99s^2+8s+33)$ [this is a stable polynomial]. The controller dimension will be 2. Thus we seek $p_2(s)$, $q_2(s)$ or degree 2 and $\alpha(s)$ of degree 1 such that

$$n_{22}(s)p_2(s)+d(s)q_2(s) = v(s)\alpha(s). \qquad (3.1)$$

Assuming $p_2(s) = as^2+bs+c$, $q_2(s) = ds^2+es+f$, $\alpha(s) = gs+h$, (3.1) reads

$$(a+d+g)s^5 + (3a+b+3d+e+\frac{49}{8}g+h)s^4$$

$$+ (3a+3b+c+3d+3e+f+\frac{99}{8}g+\frac{49}{8}h)s^3$$

$$+ (a+3b+3c+2d+3e+3f+\frac{83}{8}g+\frac{99}{8}h)s^2$$

$$+ (b+3c+2e+3f+\frac{33}{8}g+\frac{83}{8}h)s + (c+2f+\frac{38}{8}h) = 0$$

Equating the coefficients to zero, we obtain the following general solution (g,h arbitrary): $a=-g$, $b=-\frac{25}{8}g-h$, $c=-\frac{17}{8}g-\frac{17}{8}h$, $d=0$, $e=-g$, $f=-g-h$.

From this it is clear that a proper controller cannot be found. A nonproper controller is in principle given by $p_2(s) = s+\frac{17}{8}$, $q_2(s) = 1$ but this leads to $t_{11}(s)=0$.

Example 3.3: This example is to illustrate that in Theorem 2.8 there may occur pole-zero cancellations *additional to those related to v(s).* Consider

$$W(s) = \begin{bmatrix} \dfrac{2(2s^3-2s-1)}{s^3} & \dfrac{2(s+1)(s-1)}{s^3} \\ \dfrac{(s+1)^2}{s^3} & \dfrac{1}{s^3} \end{bmatrix}$$

One readily checks that det $W = -2/s^2$, hence $d(s) = s^3$, $\ell(s) = -2s$, $n_{11}(s) = 2(2s^3-2s-1)$, $n_{12}(s) = 2(s+1)(s-1)$, etc. Suppose we want to achieve a cancellation of $v(s) = (s+1)^3$ in $t_{11}(s)$ by using a controller of degree 1. Assuming $p_2(s) = as+b$, $q_2(s) = cs+d$, $\alpha(s) = es+f$, equation (2.8) reads

$$cs^4+ds^3+as+b = es^4 + (3e+f)s^3 + (3e+3f)s^2 + (e+3f)s + f$$

Comparing coefficients, it is seen that this is equivalent to $a = -2c$, $b = -c$, c arbitrary, $d = 2c$, $e = c$, $f = -c$. Taking w.l.o.g. $c = 1$, we obtain the solution $p_2(s) = -(2s+1)$, $q_2(s) = s+2$. But now

$$t_{11} = \frac{4(s+1)^3(s-1)}{(s+1)^3(s-1)} = 4$$

Hence an additional, undesigned cancellation occurs. Unfortunately, this cancellation is at the unstable location $s = 1$. But as already mentioned in Remark 2.7, examples like this are nongeneric.

4. STATE-VARIABLE VIEW; MULTICHANNEL AND VECTOR PROBLEMS

With the benefit of hindsight, the ideas of the preceding sections can be described by state-variables ideas. Such a description then will allow study of multichannel problems and problems where channel inputs and outputs can be vectors. We shall first consider two channel systems.
Suppose the system is given by

$$\dot{x} = Ax+B_1u_2+B_2u_2, \; y_1 = C_1x, \; y_2 = C_2x, \qquad (4.1)$$

where we permit the channel inputs and outputs to be nonscalar: $u_i \in R^{m_i}$ and $y_i \in R^{p_i}$. Using a controller of the form ($z \in R^{\ell}$)

$$\dot{z} = Kz+Gy_2 \quad u_2 = Hz+Fy_2 \qquad (4.2)$$

around the second channel, we find the closed loop system to be

$$\dot{x}_c = A_cx_c+B_cu_1 \quad y_1 = C_cx_c, \qquad (4.3)$$

where

$$x_c = \begin{bmatrix} x \\ z \end{bmatrix}, \quad A_c = \begin{bmatrix} A+B_2FC_2 & B_2H \\ GC_2 & K \end{bmatrix}, \quad B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$C_c = [\, C^1 \quad 0 \,].$$

In the following we assume (K,G) to be controllable and (K,H) to be observable. Then the following necessary condition for the controller (4.2) to introduce uncontrollable or unobservable modes in (4.3) holds true.

Proposition 4.1
(i) Let s_0 be an uncontrollable mode of (4.3). Then

$$rk \begin{bmatrix} A-s_0I & B_1 \\ C_2 & 0 \end{bmatrix} < n + p_2 \qquad (4.4)$$

(ii) Let s_0 be an unobservable mode of (4.3). then

$$rk \begin{bmatrix} A-s_0I & B_2 \\ C_1 & 0 \end{bmatrix} < n + m_2 \qquad (4.5)$$

Remark 4.2 If $m_1 = p_2$, $p_1 = m_2$ and s_0 is not an eigenvalue of A, the conditions (4.4) and (4.5) are equivalent to det $C_2(s_0I-A)^{-1}B_1 = 0$ and det $C_1(s_0I-A)^{-1}B_2 = 0$, respectively. Hence Proposition 4.1 generalizes Proposition 2.1 for the case that there are no fixed modes. Let us also note, that (4.4), (4.5) are satisfied for decentralized fixed modes [6].

Remark 4.3 In case $m_1 < p_2$ the necessary condition of Proposition 4.1 (i) is always satisfied. Conversely, if $m_1 > p_2$, then generically there exists no controller such that (4.3) is uncontrollable. Similar statements hold for Proposition 4.1 (ii). In case $m_1 > p_2$ and $p_1 > m_2$ (the control problem is generically not solvable) one can exchange the roles of the controller channels and obtain a more promising control problem.

Remark 4.4 Proposition 4.1 also effectively applies to problems with three or more channels. Suppose to fix ideas that there are just three channels, and one is studying the problem of selecting feedback around channel 3 to produce a reduction in the McMillan degree of the resulting two channel transfer function matrix. What is then needed are values s_0 such that

$$\mathrm{rk}\begin{bmatrix} s_0I-A & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} < n + p_3 \qquad (4.6)$$

or

$$\mathrm{rk}\begin{bmatrix} s_0I-A & B_3 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} < n + m_3 \qquad (4.7)$$

In the special case of scalar u_i, y_i for $i = 1, 2, 3$, it is clear that for generic A, B_i, C_i, no such s_0 will exist. This means that the basic approach to the design of decentralized controllers with a spread of controller complexity roughly equally among three (or more) channels will generally not be successful. For a special class of large scale systems, viz interconnected systems, results can however be obtained see [7].

Remark 4.5 A constructive approach to a converse of Proposition 4.1 leads to an interpolation problem similar to the one in Remark 2.10. However, since certain values of the controller transfer function matrix are constrained to be contained in some linear subspaces of $\mathbb{C}^{m_2 \times p_2}$, the multivariable version of the interpolation problem is much more involved than the scalar one. The details are omitted.

CONCLUSION

We have analyzed the task of designing a controller to connect around one channel of a two channel system to achieve a reduction in the McMillan degree of the resulting one channel system, through the introduction of stable pole-zero cancellations. For scalar channel systems, and vector channel systems with constraints on the dimensionality of certain vectors, one requires certain transfer functions to have stable zeros. For other vector channel systems, this is not a requirement.

For scalar channels, the design task proceeds by solving either a polynomial equation or a rational interpolation problem. Generically, the controller will be proper and only stable pole/zero cancellations are introduced. However examples show also that non-proper controllers might be necessary and that undesired unstable pole/zero cancellations might occur.

For vector channels, some preliminary results are obtained. Further research is necessary in this direction.

The interpolation problem in both its scalar transfer function and matrix transfer function version is clearly of independent interest and is currently under study.

For systems with more than two channels, the procedures of the paper are generally not possible. When they are possible, there is essentially no difference from the two channel case.

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