Proceedings of 24th Conference on Decision and Control
Ft. Lauderdale, FL • December 1985

STABILITY THEORY FOR ADAPTIVE SYSTEMS: METHODS OF AVERAGING AND PERSISTENCY OF EXCITATION

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ABSTRACT

A method of averaging is developed for the stability analysis of linear differential equations with small time-varying coefficients which do not necessarily possess a (global) average. The technique is then applied to determine the stability of a linear equation which arises in the study of adaptive systems where the adaptive parameters are slowly varying. The stability conditions are stated in the frequency-domain which shows the relation between persistent excitation and unmodeled dynamics.

1. INTRODUCTION

For a large class of adaptive feedback systems as well as for some output error identification schemes, a stability analysis in the neighborhood of the desired behavior leads to investigating the stability of the following homogeneous linear system of differential - operator equations (see e.g. [1] - [3])

$$\dot{\theta} = -\varepsilon\, u(t)\, H(u(\cdot)'\theta(\cdot)), \quad \forall t \in R_+ \tag{1.1}$$

where $\theta(0) \in R^p$ ε is a positive constant, $u(\cdot)$: $R_+ \to R^p$ is regulated and bounded, and H is a linear time-invariant operator whose transfer function $H(s)$ is proper, rational, and stable, i.e., all poles have negative real parts.

Linearization and Local Stability

In [2], for example, system (1.1) is obtained as a result of linearization of the adaptive system in the neighborhood of a "tuned" system, i.e., a system where the adaptive parameters are set to a constant value $\theta_* \in R^p$ and whose behavior is deemed acceptable. Hence, in (1.1) $\theta(t)$ is the vector of parameter errors between the parameter estimate at time t and the tuned value θ_*, $u(t)$ is the regressor vector from the tuned system (e g., filtered revisions of measured signals), and the scalar ε is the magnitude of the adaptation gain which essentially controls the rate of adaptation. The operator H depends on the actual system being controlled or identified and also on the tuned parameter setting θ_*.

It is shown in [2,3] that if the zero solution of (1.1) is uniformly asymptotically stable (u.a.s), then the adaptive system is locally stable, i.e., the adaptive system behavior will remain in a neighborhood of the desired behavior provided the initial parameter error $\theta(0)$ and the effect of external disturbances are sufficiently small. Although the results in [2,3] were arrived at using input-output properties [16], the local stability property also follows from the results on "total" stability [4].

Unmodeled Dynamics and Slow Adaptation

In the ideal case there are a sufficient number of adaptive parameters (the number p) such that the tuned parameter setting results in H(s) being strictly positive real (SPR), i.e., Re $H(j\omega) > 0$ $\forall \omega \in R_+$ Under these conditions, we have the following results (see e.g., [5]-[8], [1]): (1) the zero solution of (1.1) is stable, i.e., $\theta(t)$ is bounded but not necessarily constant; (2) if, in addition, u(t) is persistently exciting, then the zero solution is uniformly asymptotically stable (u.a.s.), thus, $\theta(t) \to 0$ exponentially fast as $t \to \infty$. The trouble starts when there are an insufficient number of parameters to obtain $H(s) \in$ SPR, as is the case in adaptive control when the plant has unmodeled dynamics (see e.g. [2, 7], [12]).

In this paper we will examine the stability of (1.1) when ε is small, u(t) is persistently exciting, and H(s) is not necessarily SPR, but only stable. Riedle and Kokotovic [9] refer to this case as "slow adaptation" and by using the methods of averaging described by Hale [10], they show that the stability of the zero solution of (1.1) is critically dependent on the spectrum of the excitation in relation to the frequency response $H(j\omega)$. With the same assumptions, Astrom [11] uses averaging techniques to analyze the interaction between unmodeled dynamics and external inputs in the counter-example posed by Rohrs et al. [12]. Both these analyses require the assumption that u(t) is almost periodic. In this case

*Supported by AFOSR under contract F49620 C-84-0054 while this author was a Visiting Fellow at the Australian National University.

#Supported as a Research Assistant by the National Fund for Scientific Research, Belgium, which support is acknowledged.

Riedle and Kokotovic [9] show that the zero solution of (1.1) is u.a.s. if

$$\lambda\Big(\sum_{\omega\in\Omega} [\alpha(\omega)\alpha(\omega)^*] \operatorname{Re} \hat{H}(j\omega) > 0 \qquad (1.2)$$

where Ω and $\{\alpha(\omega),\ \omega \in \Omega\}$ are, respectively, the Fourier exponents and coefficients of $u(t)$. Condition (1.2) can be considered as a generalized positivity condition, but unlike the SPR condition, Re $H(j\omega)$ is not required to be positive at all frequencies.

Averaging: Uses and Limitations

The main contribution of this paper is to extend the theory of averaging to include the case when $u(t)$ does not have a (generalized) Fourier series representation, but is only known to be regulated and bounded. Thus, $u(t)$ need not be almost periodic nor even possess a (global) average value. We also state stability conditions in the frequency-domain in a form similar to (1.2). Analagous results can be stated for the discrete-time system

$$\theta(t+1) = \theta(t) - \varepsilon\, u(t)\, H(u(\cdot)'\theta(\cdot)), \quad \forall t \in Z_+ \qquad (1.3)$$

where we only require $u(\cdot) \in \ell^p_\infty$ and H to be linear-time-invariant and stable. Averaging results for (1.3) with $H = 1$ and with $u(\cdot)$ not almost periodic can also be found in [13]; and this suggests the possibility of being able to dispense with the almost periodicity assumption on $u(\cdot)$ and analyzing (1.1) with a non-SPR operator H.

The averaging theory developed here, as well as averaging theory in general, has its uses and limitations for adaptive system. In the first place, the theory requires slow adaptation which can be counter-productive because performance can be below par for the long period of time it takes for the parameters to readjust. Secondly, averaging theory is a form of linearization, thus, the (nonlinear) adaptive system must be initialized in a (not necessarily small) neighborhood of the tuned system. On the positive side, however, we do obtain frequency domain conditions which explain the system behavior near the tuned solutions. In this sense, we can consider the results of averaging theory to be necessary conditions for good performance of adaptive systems.

To obtain the heralded goal of frequency-domain stability conditions, it may be inevitable to encounter linearization.

Somewhat less intuitively appealing results can be obtained without resorting to direct linearization or averaging, e.g., in [2,3], [14] and [15] the results arise from a combination of small gain theory and perturbation theory.

2. AVERAGING FOR LINEAR HOMOGENEOUS SYSTEMS

In this section we will consider the homogeneous linear system

$$\dot{x} = \varepsilon A(t)x \qquad (2.1)$$

Lemma (2.2)

Suppose in (2.1) that $A(\cdot): R_+ \to R^{n\times n}$ is regulated and bounded Then $\forall s,\tau \in R_+$, the transition matrix $\phi(s+\tau,s)$ of (2.1) is given by

$$\phi(s+\tau,s) = \exp[\varepsilon\tau\, \bar{A}_\tau(s)] + R(s,\varepsilon\tau) \qquad (2.3)$$

where

$$\bar{A}_\tau(s) = \frac{1}{\tau}\int_s^{s+\tau} A(t)\,dt \qquad (2.4)$$

is the sampled average value of $A(t)$ on the interval $s \le t \le s + \tau$, and

$$||R(\cdot,\ \varepsilon\tau)||_\infty \le r(\varepsilon\tau||A||_\infty) := (\varepsilon\tau||A||_\infty)^2 \exp(\varepsilon\tau||A||_\infty) \qquad (2.5)$$

Remarks

(1) Assuming that $A(t)$ is regulated and bounded is sufficient for the existence and uniqueness of solutions [17].

(2) Observe that Lemma (2.2) is valid $\forall s,\tau \in R_+$ and $\forall\varepsilon \in R$. In the sequel we use Lemma (2.2) only for the case when $\varepsilon > 0$ and $\varepsilon\tau$ is small.

The stability properties of (2.1) can be established by application of Lemma (2.2) as stated in Theorem (2.9) below. We first require

Definition [16] (2.6)

The function $\mu(\cdot): C^{n\times n} \to R$, defined by

$$\mu(M) = \lim_{\alpha\downarrow 0} (|I + \alpha M|_i - 1)/\alpha \qquad (2.7)$$

is called the measure of the matrix M, where $|\cdot|_i$ is the matrix norm on $C^{n\times n}$ induced by the vector norm $|\cdot|$ on C^n. For example, if $|\cdot|$ is the Euclidean norm then $\mu(M) = \max \lambda[(M+M^*)/2]$. For any norm on C^n we have the relation,

$$-\mu(-M) \le \operatorname{Re} \lambda(M) \le \mu(M), \quad \forall M \in C^{n\times n} \qquad (2.8)$$

Theorem (2.9)

Suppose $A(t)$ in (2.1) is regulated and bounded with the sequence of local average values $\{\bar{A}_T(kT),\ \forall k \in Z_+\}$. Then:

(i) If $\exists\, T > 0$ and $\alpha > 0$ such that

$$\mu[\bar{A}_T(kT)] \le -\alpha, \quad \forall k \in Z_+ \qquad (2.10)$$

then $\exists\, \eta > 0$ such that $\forall\varepsilon T \in (0,\eta)$ the zero solution of (2.1) is u.a.s.

(ii) If $\exists\, T > 0$ and $\alpha > 0$ such that

$$\mu[-\bar{A}_T(kT)] \le -\alpha, \quad \forall k \in Z_+ \qquad (2.11)$$

then $\exists\, \eta > 0$ such that $\forall\varepsilon T \in (0,\eta)$ the zero solution of (2.1) is completely unstable.

Remarks

(1) The proof (which is omitted) is based on Lemma (2.2) and the inequality [16]:

$$\exp\{-\varepsilon T\mu[-\bar{A}_T(kT)]\} \leq |\phi((k+1)T.kT) - R(kT,\varepsilon T)|$$

$$\leq \exp\{\varepsilon T\ \mu[\bar{A}_T(kT)]\}, \qquad \forall k \in Z_+ \quad (2.12)$$

where $\phi(\cdot,\cdot)$ and $R(\cdot,\cdot)$ are as defined in Lemma (2.2).

(2) Whenever (2.10) holds we have $|\exp\{\varepsilon T\bar{A}_T(kT)\}| < 1$, $\forall k \in Z_+$ which insures a contraction (for small εT) on the interval $s \leq t \leq s + T$. It is possible to weaken condition (2.10) and still have a contraction by just enforcing $|\exp\{\varepsilon T\ \bar{A}_T(kT)\}| < 1$ directly as is done by Coppel [18].

(3) Note that Thereom (2.9) can be stated in terms of stronger conditions on $\mu[\bar{A}_T(s)]$, $\forall s \in R_+$.

Using the same technique, but allowing $A(\cdot)$ (equivalently $\bar{A}_T(\cdot)$) to possess a global average, we obtain the following sharper result.

Theorem (2.13)

Suppose $A(t)$ in (2.1) is regulated, bounded, and has a (global) average $\bar{A} \in R^{nxn}$, i.e.,

$$\lim_{T\to\infty} \bar{A}_T(s) = \bar{A} \qquad (2.14)$$

uniformly $\forall s \in R_+$ with $\mathrm{Re}\ \lambda(\bar{A}) \neq 0$. Under these conditions:

(i) If $\exists\ \alpha > 0$ such that

$$\mathrm{Re}\ \lambda(\bar{A}) \leq -\alpha \quad (2.15)$$

then $\exists\ \varepsilon_o > 0$ such that $\forall\varepsilon \in (0, \varepsilon_o)$, the zero solution of (2.1) is u.a.s.

(ii) If $\exists\ \alpha > 0$ such that

$$\max \mathrm{Re}\ \lambda(\bar{A}) \geq \alpha \qquad (2.16)$$

then $\exists\varepsilon_o > 0$ such that $\forall\varepsilon \in (0, \varepsilon_o)$, the zero solution of (2.1) is unstable.

Discussion

The results in Theorem (2.9) and Theorem (2.13) generalize some results obtained by averaging methods such as those described by Hale [10], or as obtained by Coppel [18] using the notion of integral smallness. Theorem (2.13) is a classical result of averaging theory, except that as stated allows for functions which are not necessarily almost periodic. The class of functions allowed in theorem (2.13) - regulated, bounded, with an average - is not precisely characterized. Obviously it includes the class of asymptotically almost periodic functions of the form [19]

$$A(t) = A_p(t) + A_1(t) \qquad (2.17)$$

where $A_p(t)$ is almost periodic and $A_1(\cdot) \in L_1^{nxn}$.

Theorem (2.9) considers a larger class of functions -- those without an average -- at the expense of a weaker result: the stability -- instability boundary is not as sharp as in theorem (2.13).

3. FREQUENCY-DOMAIN STABILITY CONDITIONS

In this section we apply the results of Section 2 to the homogeneous linear system (1.1), i.e.,

$$\dot{\theta} = -\varepsilon\ u(t)\ H(u(\cdot)'\theta(\cdot)), \quad (3.1)$$

where H is a linear time-invariant operator with transfer function $H(s)$. We first show that for sufficiently small $\varepsilon > 0$, the stability analysis of (3.1) can be determined from the stability of an "averaged" system

$$\dot{\theta} = -\varepsilon\ \mathrm{avg}\{u(t)(Hu)'(t)\}\theta$$

where avg $\{\cdot\}$ has yet to be precisely defined. Using this result we then establish stability conditions in the frequency-domain involving the Fourier transform $H(j\omega)$ and the "spectral" content of $u(t)$, where this notion has also to be defined. Finally, we show that the appropriately defined spectral content of $u(t)$ necessarily requires that $u(t)$ have a persistency of excitation property, and that the dominant excitation should be at those frequencies where $\mathrm{Re}\ H(j\omega) > 0$.

To establish frequency domain stability conditions for (3.1) requires that $u(t)$ be restricted to those functions which have a Fourier series representation on any finite interval. A known class of such functions is defined as follows.

Definition [20]

A function $f(\cdot)$: $R_+ \to R^n$ is a C_δ^n function if it is regulated, bounded and $\exists$ a constant $\delta > 0$ such that any two points t_1, $t_2 \in R_+$ where $f(\cdot)$ is discontinuous are separated by at least an interval δ, i.e., $|t_1 - t_2| \geq \delta$.

Frequency-domain stability conditions ... (3.1) can now be stated.

Theorem (3.2)

Assume in (3.1) that:

(A1) H is linear with stable proper rational transfer function $H(s)$ and impulse response $h(t)$. Thus, $\exists$ a > 0 and $b > 0$ such that

$$|h(t) - h(0)\delta(t)| \leq a \exp(-bt), \quad \forall t \in R_+ \qquad (3.3)$$

(A2) $u(\cdot) \in C^p_\delta$ with piece-wise Fourier series representation

$$u(t) \sim \sum_{m \in Z} \alpha_k(\omega_m) e^{j\omega_m t}$$
$$\forall t \in [kT, (k+1)T], \ \forall k \in Z_+ \qquad (3.4)$$

for any $T \geq \delta$ where $\omega_m = 2\pi m/T$.

Define the matrix sequence in $R^{p \times p}$ by

$$R_T(k) = \sum_{m \in Z} \alpha_k(\omega_m)\alpha_k(\omega_m)^* \hat{H}(-j\omega_m),$$
$$\forall \ k \in Z_+ \qquad (3.5)$$

Under these conditions:

(i) If $\exists \ T \geq \delta$ and $\alpha > 0$ such that

$$\mu[-R_T(k)] \leq -(\alpha + 2(a/b^2)||u||_\infty/T),$$
$$\forall k \in Z_+ \qquad (3.6)$$

then $\exists \ \varepsilon_o > 0$ such that $\forall \varepsilon \in (0, \varepsilon_o)$ the zero solution of (3.1) is u.a.s.

(ii) If $\exists \ T \geq \delta$ and $\alpha > 0$ such that

$$\mu[R_T(kT)] \leq -(\alpha + 2(a/b^2)||u||^2_\infty/T),$$
$$\forall k \in Z_+ \qquad (3.7)$$

then $\exists \ \varepsilon_o > 0$ such that $\forall \varepsilon \in (0, \varepsilon_o)$ the zero solution of (3.1) is unstable.

Remarks

(1) The existence of the piece-wise Fourier series representation (3.4) for $u(t)$ is guaranteed by $u(\cdot) \in C^p_\delta$ [17]. The Fourier coefficients $\alpha_k(\omega_m)$ are the coefficients of the T-periodic function

$$u_k(t) = \sum_{m \in Z} \alpha_k(\omega_m) e^{j\omega_m t}, \ \forall t \in R_+ \qquad (3.8)$$

which is equal to $u(t)$ for $kT \leq t \leq (k+1)T$ and, in general, not equal to $\bar{u}(t)$ on any other interval. Thus, $u_k(t)$ is just $u(t)$, $\forall t \in [kT, (k+1)T]$, repeated with period T. Observe that the spectrum of $u_k(t)$ is what determines the stability-instability boundary and not the spectrum of $u(t)$. These will merge only when $u(\cdot)$ has a (global) average as assumed in Theorem (3.14) below.

(2) The matrix $R_T(k)$ can be equivalently expressed as the local average value of $u_k(t)(Hu_k)'_\infty(t)$, i.e.,

$$R_T(k) = \frac{1}{T} \int_{kT}^{(k+1)T} u_k(t)(Hu_k)'_\infty(t)dt \qquad (3.9)$$

where $(Hu_k)_\infty(t)$ is the "steady-state" part of $(Hu_k)(t)$, i.e.,

$$(Hu_k)_\infty(t) = \sum_{m \in Z} \hat{H}(j\omega_m)\alpha_k(\omega_m)e^{j\omega_m t}, \ \forall t \in R_+ \qquad (3.10)$$

(3) If we use the measure $\mu(M) = \max \lambda(M+M^*)/2$, then (3.6) and (3.7) become, respectively,

$$\lambda[Q_T(k)] \geq \alpha + 2(a/b^2)||u||^2_\infty/T, \ \forall k \in Z_+ \qquad (3.11)$$

and

$$\lambda[Q_T(k)] \leq -(\alpha + 2(a/b^2)||u||^2_\infty/T),$$
$$\forall k \in Z_+ \qquad (3.12)$$

where $Q_T(k)$ is the Hermitian part of $R_T(k)$, i.e.,

$$Q_T(k) = \sum_{m \in Z_+} \mathrm{Re}[\alpha_k(\omega_m)\alpha_k(\omega_m)^*]$$
$$\times \ \mathrm{Re}[H(j\omega_m)] \qquad (3.13)$$

(5) The "initial conditions" at $t = kT$ contribute to the term $2(a/b^2)||u||^2_\infty/T$ in (3.6)-(3.7) or (3.11)-(3.12). Hence, the average energy in $u_k(t)(Hu_k)'_\infty(t)$ must dominate long enough (T sufficiently large) to overcome these (possibly) negative effects.

As before, if $u(t)$ is further restricted such that $R_T(\cdot)$ has a global average, then we can sharpen the stability-instability boundary. For example, if $u(t)$ is almost periodic then a Fourier series representation exists $\forall t \in R_+$ and $R_T(\cdot)$ has an average [10]. The stability conditions for this case are stated as follows.

Theorem (3.14)

Suppose in (3.1) that $u(t)$ is almost periodic with Fourier series

$$u(t) \sim \sum_{\omega \in \Omega} \alpha(\omega) e^{j\omega t}, \quad \forall t \in R_+ \qquad (3.15)$$

where $\Omega \in R$ are the distinct Fourier exponents and $\{\alpha(\omega), \ \omega \in \Omega\}$ are the Fourier coefficients. Define the matrix $R \in R^{n \times n}$ by

$$R = \sum_{\omega \in \Omega} \alpha(\omega) \ \alpha(\omega)^* \ \hat{H}(-j\omega) \qquad (3.16)$$

If $\mathrm{Re} \ \lambda(R) \neq 0$ then $\exists \ \varepsilon_o > 0$ such that $\forall \varepsilon \in (0, \varepsilon_o)$ the zero solution of (3.1) is:

(i) u.a.s if $\mathrm{Re} \ \lambda(R) > 0$ (3.17)

(ii) unstable if $\max \mathrm{Re} \ \lambda(R) < 0$ (3.18)

Discussion

Theorem (3.14) is the result obtained in [9] when $u(t)$ is almost periodic. Theorem (3.2) is a generalization in that $u(\cdot) \in C^p_\delta$.

Observe that the stability-instability boundary determined by (3.11) - (3.13) exists if and only if

$$\lambda[Q_T(k)] \neq 0, \quad \forall k \in Z_+ \qquad (3.19)$$

By (3.13), this will hold if and only if for some finite integer $q \geq p$,

$$\text{rank}\,[\alpha_k(0), \alpha_k(\omega_1), \ldots, \alpha_k(\omega_q)] = p, \quad \forall k \in Z_+. \qquad (3.20)$$

Hence, Theorem (3.2) implicitly restricts $u(\cdot) \in C_\delta^p$ to those functions whose (time-varying) Fourier coefficients satisfy the rank condition above. This class of functions, however, are precisely those which can be categorized as persistently exciting:

<u>Definition</u> [1] (3.21)

A function $f(\cdot): R_+ \to R^n$ is <u>persistently exciting</u> over an interval h if it is regulated, bounded, and $\exists$ constants $h > 0$ and $\beta > 0$ such that

$$\min \lambda\left(\frac{1}{h}\int_s^{s+h} f(t)f(t)'dt\right) \geq \beta, \quad \forall s \in R_+ \qquad (3.22)$$

Denote such functions by $u(\cdot) \in PE^n(h,\beta)$.

Hence, we immediately see that if u(t) in (3.1) is in $PE^p(h,\beta)$ and C_δ^p then (3.20) will hold for $\forall T \geq h \geq \delta$. It is important to emphasize, however, that even if u(t) is PE, u.a.s. of the zero solution of (3.1) is guaranteed if (3.11) holds. The implication then is that u(t) must have a dominant spectrum at those frequencies where $Re[H(j\omega_m)] > 0$. Thus, we can view (3.11) as a <u>generalized positivity condition</u> on the operator H.

As pointed out by Riedle and Kokotovic [9], condition (3.11) is significantly weaker than the usual positivity conditions on H. For example, a strictly proper transfer function H(s) is strictly positive real (SPR) if it is exponentially stable and $\exists$ constant $\rho > 0$ such that [16]:

$$Re\,[\hat{H}(j\omega)] \geq \rho|\hat{H}(j\omega)|^2, \quad \forall \omega \in R_+ \qquad (3.23)$$

This condition must hold at <u>every</u> frequency, whereas (3.11) requires $Re\,[H(j\omega_m)] > 0$ at those <u>discrete</u> frequencies in R_+ where the magnitude of the input spectrum is large. Conversely, at those frequencies in R_+ where $Re[H(j\omega_m)] < 0$, the magnitude of the input spectrum should be small. Since (3.16) will fail if $Re\,H(j\omega) < 0$, $\forall \omega \in R_+$, it follows that $Re\,H(j\omega) > 0$ at some frequencies, hence, the motivation to refer to (3.11) as a positivity condition.

Although condition (3.11) is weaker than condition (3.23), we do pay the Piper. Suppose H(s) is SPR and (3.23) holds. If u(t) is persistently exciting than Theorem (3.2) states that the zero solution of (3.1) is u.a.s. for sufficiently small $\epsilon > 0$. However, from other arguments (see e.g. [1]) we know that under these same conditions the zero solution of (3.1) is u.a.s. for <u>all</u> $\epsilon > 0$. Thus, Theorem (3.2) is conservative in this case. However, when H(s) is not SPR, but (3.23) holds at some frequencies, Theorem (3.11) is now applicable whereas the results in [1] do not apply. In fact in this latter case when ϵ gets too large the zero solution of (3.1) can be unstable, even if (3.11) holds. For example, if in (3.1) $u(t) = \sin(0.35t)$ and $H(s) = 1/(s^2 + 2s + 2)$ then condition (3.11) is satisfied. The simulations in Figure 3.1 with $\theta(0) = 1$ show that the zero solution is u.a.s. for $\epsilon = 4$ but is completely unstable for $\epsilon = 8$.

4. <u>EFFECT OF UNMODELED DYNAMICS</u>

In this section we will consider the system

$$\dot{\theta} = -\epsilon\, u(t)\, H(u(\cdot)'\theta(\cdot)), \quad \theta(0) \in R^p \qquad (4.1)$$

where H as before is linear with stable transfer function H(s). In addition we assume that H(s) has the decomposition

$$\hat{H}(s) = \hat{H}_o(s) + \hat{\Delta}(s) \qquad (4.2)$$

where $\hat{H}_o(s)$ is SPR, i.e., $\exists\, \rho > 0$ such that

$$Re\,\hat{H}_o(j\omega) \geq \rho|\hat{H}_o(j\omega)|^2, \quad \forall \omega \in R_+ \qquad (4.3)$$

and where $\hat{\Delta}(s)$ represents unmodeled dynamics such that

$$|\hat{\Delta}(j\omega)| \leq \delta_H(\omega), \quad \forall \omega \in R_+ \qquad (4.4)$$

We also assume that u(t) satisfies the conditions of Theorem (3.2) in that $\exists\, T > 0$ such that u(t) has the piece-wise Fourier series representation

$$u(t) \sim \sum_{m \in R} \alpha_k(\omega_m) e^{j\omega_m t}, \quad kT \leq t \leq (k+1)T \qquad (4.5)$$

where $\omega_m = 2\pi m/T$ and $k \in Z_+$. We also decompose

$$\alpha_{kT}(\omega_m) = \overset{\circ}{\alpha}_{kT}(\omega_m) + \beta_{kT}(\omega_m), \quad \forall m,k \in Z \qquad (4.6)$$

where $\overset{\circ}{\alpha}_{kT}(\omega_m)$ is due to predetermined inputs and $\beta_{kT}(\omega_m)$ is due to disturbances bounded by

$$|\beta_{kT}(\omega_m)| \leq \delta_u(\omega_m), \quad \forall m \in Z_+ \qquad (4.7)$$

Hence, the functions $\omega \to \delta_H(\omega)$ and $m \to \delta_u(\omega_m)$ represent, respectively, bounds on the effect of unmodeled dynamics in H(s) and unknown elements of u(t) as a function of frequency. Combining the above assumptions with Theorem (3.2) and using (3.16) gives:

<u>Lemma</u> (4.8)

The zero solution of (4.1) is u.a.s. if $\epsilon > 0$ is sufficiently small and if $\exists\, \alpha > 0$ and $T > 0$ such that $\forall k \in Z_+$,

$$\lambda\{\sum_{m \in Z_+} \rho|\hat{H}_o(j\omega_m)|^2\, Re[X^o_{kT}(\omega_m)]\} \geq q_{kT} + \alpha + J/T \qquad (4.9)$$

where J is found from (3.14) and where,

$$q_{kT} = \sum_{m \in Z_+} (\delta_H(\omega_m)[|X^o{}_{kT}(\omega_m)| + \mu_{kT}(\omega)|\hat{H}_o(j\omega_m)| \qquad (4.10)$$

with

$$X^{\circ}_{kT}(\omega_m) = \alpha_{kT}(\omega_m)\ \alpha_{kT}(\omega_m)^* \tag{4.11}$$

$$\mu_{kT}(\omega_m) = \delta_u(\omega_m)[\delta_u(\omega_m) + |\alpha^{\circ}_{kT}(\omega_m)|]$$

Discussion

Condition (4.9) shows that the dominant excitation must act in the frequency range where $H(s) \approx H_o(s) \in$ SPR. Moreover, there must be enough excitation and positivity in $\mathrm{Re}\, H_o(j\omega)$ in this range to overcome initial conditions (the 1/T term) and the effect of unmodeled dynamics and unknown disturbances (the q_{kT} term). Typically, the disturbances and unmodeled effects occur at high frequencies and the known efforts in $\hat{H}_o(s)$ and $\alpha^{\circ}_{kT}(\omega_m)$ at low frequencies. For example, if there is a frequency ω_c such that

$$\alpha^{\circ}_{kT}(\omega),\ \hat{H}_o(j\omega) \text{ small for } \omega >> \omega_c \tag{4.12}$$

$$\delta_H(\omega),\ \delta_u(\omega) \text{ small for } \omega << \omega_c$$

then condition (4.8) holds if $\forall k \in Z_+$,

$$\lambda\{\sum_{\omega_m \le \omega_c} \rho|\hat{H}_o(j\omega_m)|^2\ \mathrm{Re}[X^{\circ}_{kT}(\omega_m)]\} >> \sum_{\omega_m \ge \omega_c} \delta_H(\omega_m)\delta^2(\omega_m) \tag{4.13}$$

Observe that robustness conditions (4.8) or (4.13) are dependent on the input signal spectrum as as well as the unmodeled dynamics. In non-adaptive linear systems the robustness conditions only involve system dynamics.

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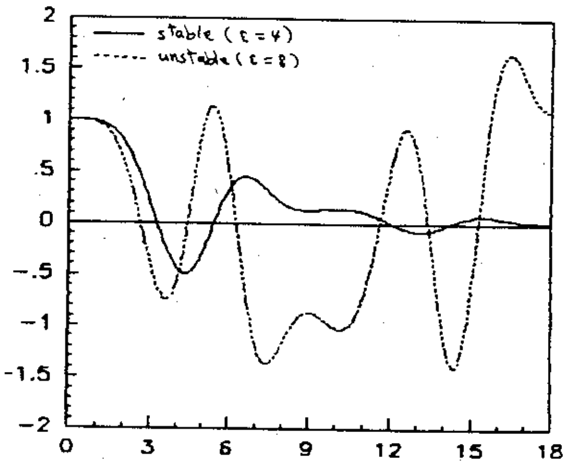

Figure 3.1