TP2 - 4:00

Proceedings of 23rd Conference on Decision and Control
Las Vegas, NV, December 1984

ROBUST ADAPTIVE CONTROL: CONDITIONS FOR LOCAL STABILITY

by

Robert L. Kosut* and Brian D.O. Anderson

Integrated Systems, Inc.
101 University Avenue
Palo Alto, CA 94301

Australian National University
Canberra, Australia ACT2601

ABSTRACT

The question is examined of when an adaptive control system is robust to unmodeled dynamics and unknown bounded disturbances. Conditions are presented that ensure the existence of such robustness properties, but only locally; i.e., restrictions are placed on the behavior of signals in the ideal, perfectly tuned adaptive system. Local L_∞-stability is investigated when certain tuned signals are assumed to be persistently exciting.

1. INTRODUCTION

Theoretical investigations on the stability of adaptive control systems have focused almost entirely on developing conditions that guarantee global stability, e.g., [1]-[3]. These results are global in the sense that initial conditions and external signal magnitudes need only be bounded. Specific bounds are not required. In addition, the results provide sufficient conditions. One of the conditions is that a particular subsystem operator be strictly passive with finite gain or, in the case of linear-time-invariant systems, the operator is strictly positive real (SPR). This condition results from application of the Passivity Theorem; specifically, the adaptive system can be reconfigured into two subsystems: a 'feedback' subsystem (the adaptation law) that is passive, and a 'feedforward' subsystem which is required to be SPR. This condition turns out to be quite restrictive. In the first place, the SPR condition necessitates that the system transfer function (in the scalar case) have a unitary relative degree. As pointed out by Rohrs, et al. [4], it is virtually impossible to guarantee unitary relative degree for an actual system. Secondly, the SPR condition has extremely limited robustness to unmodeled dynamics [5].

In this paper, conditions are developed that guarantee the existence of local stability and robustness properties of the adaptive system, i.e., conditions which take into acount the size of initial parameter error and external signal magnitudes. These conditions are imposed on certain subsystem operators, which have a time-varying dependence on signals that arise from an ideal fictitious system where the adaptive gains are perfectly tuned to the unknown plant to be controlled. The mechanism for local stability which is examined here is that of persistent excitation [7], [8]. Under these conditions, we develop a specific bound on model error which ensures conditions for local stability.

---

* Partially supported by the Air Force Office of Scientific Research (AFOSR) under contract F49620-83-C-0107.

2. NOTATION

Let L_p^n denote the set of Lebesgue integrable functions $x(.): R_+ \to R^n$ with finite norm $||x||_p := (\int_0^\infty ||x(t)||^p dt)^{1/p}$ for p [1,∞) and $||x||_\infty := \sup_{t\geq 0} |x(t)|$, where $|.|$ denotes a norm on R^n. Similarly, let L_{pe}^n denote the extension of L_p^n consisting of functions $x(.)$ such that $x_T \in L_p^n$, $\forall T \geq 0$, where $x_T(t)$ denotes the truncation of $x(t)$ at T, i.e., $x_T(t) = x(t)$ for $t \leq T$, and $x_T(t) = 0$ for $t > T$. The norm on L_{pe}^n is denoted by either $||x_T||_p$ or $||x||_{Tp}$.

3. ADAPTIVE ERROR SYSTEM

In this section we present an adaptive error system which is representative of a large class of adaptive control systems. The error system will be presented in two forms: a parameter variational form and a full variational form. The parameter variational form was developed in detail in [5b] and is used for global stability analysis. The full variational form, to be developed here, is used for local stability analysis.

3.1 Parameter Variational Error System

To facilitate the development of the error system, consider the simple model reference adaptive controller (MRAC) depicted in Fig. 3-1 with:

Uncertain Plant

$$y = d + Pu \tag{3.1a}$$

d := external disturbance + plant initial conditions

Reference Model

$$y_r = H_r \tag{3.1b}$$

r := reference command

Adaptive Control

$$u = -(\theta_1, \theta_2)\begin{pmatrix} y \\ -r \end{pmatrix} = -\theta' z \tag{3.1c}$$

θ := adaptive gains, z := regressor

Adaptation Law

$$\dot{\theta} = Bze, \quad B = B' > 0 \tag{3.1d}$$

$e := y - y_r$

Define the adaptive gain error by

$$\tilde{\theta} := \theta - \theta_* \qquad (3.2)$$

where $\theta_* = (\theta_{*1}, \theta_{*2})'$ is a constant vector of tuned gains; i.e., the values that would be selected if the plant P were known. Using (3.2) we can rewrite (3.1c) as

$$u = -\theta_*' z - v \qquad (3.3)$$

$$v := \tilde{\theta}' z$$

where v is the adaptive control error signal. An equivalent representation of (3.1) is given by the adaptive error system depicted in Figure 3-2 and described by:

$$e = e_* - H_{ev} v \qquad (3.4a)$$

$$z = z_* - H_{zv} v \qquad (3.4b)$$

$$v = z'\tilde{\theta} \qquad (3.4c)$$

$$\tilde{\theta} = \tilde{\theta}_0 + Lze \qquad (3.4d)$$

where (e_*, z_*) are the outputs of the tuned system, as shown in Figure 3-3; $\tilde{\theta}_0$ is the initial value of the adaptive gain error, and H_{ev}, H_{zv}, and L are the interconnection operators. For the simple MRAC case considered here (Fig. 3-1), the tuned signals are:

$$e_* = (1 + P\theta_{*1})^{-1} d$$
$$+ [(1 + P\theta_{*1})^{-1} P\theta_{*2} - H_r] r \qquad (3.5a)$$

$$z_* = (e_* + y_r, -r)' \qquad (3.5b)$$

and the interconnections have transfer functions,

$$H_{ev}(s) = (1 + P(s)\theta_{*1})^{-1} P(s) \qquad (3.6a)$$

$$H_{zv}(s) = [(1 + P(s)\theta_{*1})^{-1} P(s), 0]' \qquad (3.6b)$$

$$L(s) = (1/s)B \qquad (3.6c)$$

Although the error system (3.4) has been developed here for a very simple MRAC system, the form of (3.4) is generic and applies to practically all single-input-single-output adaptive controllers and filters [5]. Moreover, the extension of (3.4) to the multivariable case requires only that v and e are vectors and that H_{ev} and H_{zv} are multivariable of compatible dimensions. Specifically, (3.4c) and (3.4d) are replaced by

$$v = Z'\tilde{\theta} \qquad (3.4c)'$$

$$\tilde{\theta} = \tilde{\theta}_0 + LZe \qquad (3.4d)'$$

where Z is a block diagonal matrix of appropriate dimensions such that

$$Z = \text{diag}\,(z_1, \ldots, z_m) \qquad (3.4e)'$$

where the regressor vector is

$$z' = (z_1', \ldots, z_m') \qquad (3.4f)'$$

Hence, each adaptive control channel is given by

$$u_i = -\theta_i' z_i, \quad i = 1, \ldots, m \qquad (3.4g)'$$

In this paper, the ensuing analysis will be illustrated by using the error system of (3.4). The extension to the multivariable case follows immediately.

One of the very useful features of this error system is that the nonlinear effect of the adaptive algorithm can be analyzed separately from the analysis of the tuned system. The tuned system represents an ideal which could be achieved with the given structure of the adaptive control. Hence, the algebraic design procedure is separated from the nonlinear stability analysis. It is convenient, therefore, to view e_*, z_*, and $\tilde{\theta}_0$ as 'inputs' to the error system. The assumption, naturally, is that e_* and z_* are well behaved with e_* small. Note that $\tilde{\theta}_0$ need not be small. In the classic case, assuming perfect model following and no disturbances in the tuned system, $e_*(t) = 0$. If the disturbances are of a special kind then $e_*(t) \to 0$, i.e., the tuned system exhibits servo action. The more realistic case, however, is when $e_* \in L_\infty$ due to bounded disturbances which cannot be asymptotically rejected.

3.2 Global Stability Conditions

Conditions for global stability require that $H_{ev}(s) \in$ SPR. This arises because proofs of global stability utilize passivity theory (e.g., [9], p. 182). A detailed analysis can be found in [1]-[5]. Unfortunately, though of theoretical significance, these type of results do not offer any practical engineering guidelines. The major reason is that $H_{ev}(s) \in$ SPR is not robust with respect to even mild modeling error, particularly high frequency unmodeled dynamics [4]. Since $H_{ev}(s) \in$ SPR implies that the relative degree of $H_{ev}(s)$ cannot exceed one, it follows that applying this restriction to (3.6a) imposes a unitary relative degree restriction on P(s) as well. This is unrealistic, even in this simple example.

Another view of the restrictiveness of $H_{ev} \in$ SPR is from robustness theory, e.g., [15]. Suppose that P(s) in (3.1a) can be expressed as belonging to the set of transfer functions

$$P(s) = (1 + \Delta(s))\bar{P}(s) \qquad (3.7a)$$

$$|\Delta(j\omega)| \le \delta(\omega), \quad \forall \omega \in R \qquad (3.7b)$$

Hence, $\bar{P}(s)$ is a nominal model of P(s) and $\Delta(s)$ represents modeling error, e.g., high frequency unmodeled dynamics. We can now write

$$H_{ev} = \bar{H}_{ev} + \tilde{H}_{ev} \qquad (3.8a)$$

where the nominal is,

$$\bar{H}_{ev} = (1 + \bar{P}\theta_{*1})^{-1} \bar{P} \in \text{SPR} \qquad (3.8b)$$

and the deviation induced by modeling error is,

$$\tilde{H}_{ev} = \bar{H}_{ev}\,[1 + (1 + \bar{P}\theta_{*1})^{-1}\,(1 + \bar{P}\theta_{*1})^{-1}\Delta \qquad (3.8c)$$

It is shown in [5] that the largest tolerable $\delta(\omega)$ in (3.7b) to ensure $H_{ev} \in$ SPR is bounded by

$$\delta(\omega) < 1\,. \qquad (3.9)$$

Again, this is unrealistic and is violated even by the most mild form of unmodeled dynamics. Note that (3.9) and the unitary relative degree restriction both necessarily arise from the SPR condition.

3.3 Full Variational Form

The error system (3.4) can be transformed to the following variational form which is more useful for local stability analysis, i.e.,

$$x = x_L - Gf(x) \tag{3.10a}$$

where the quantities above are defined below by

$$x := \begin{pmatrix} \tilde{e} \\ \tilde{z} \\ \tilde{\theta} \end{pmatrix} := \begin{pmatrix} e - e_* \\ z - z_* \\ \theta - \theta_* \end{pmatrix}, \quad f(x) := \begin{pmatrix} \tilde{z}' & \tilde{\theta} \\ \tilde{z} & \tilde{e} \end{pmatrix} \tag{3.10b}$$

$$x_L := \begin{pmatrix} \tilde{e}_L \\ \tilde{z} \\ \tilde{\theta}_L \end{pmatrix} := \begin{pmatrix} -H_{ev} z_*' \tilde{\theta}_L \\ -H_{zv} z_*' \tilde{\theta}_L \\ (I + LM)^{-1}\tilde{\theta}_0 + K_* e_* \end{pmatrix} \tag{3.10c}$$

$$G := \begin{bmatrix} H_{ev}(1 - z_*' KN) & H_{ev} z_*' K \\ H_{zv}(1 - z_*' KN) & H_{zv} z_*' K \\ KN & -K \end{bmatrix} \tag{3.10d}$$

with

$$N := z_* H_{ev} + e_* H_{zv} \tag{3.10e}$$

$$M := N z_*' \tag{3.10f}$$

$$K := (I + LM)^{-1} L \tag{3.10g}$$

and where L has the transfer function,

$$L(s) = \frac{1}{s} B \tag{3.10h}$$

with B from the adaptive algorithm (3.1d).

The model (3.7) is arrived at by separating the nonlinear cross product terms in $f(x)$ from the linear terms in x_L. We shall refer to x_L as the response of the linearized system. This is almost identical to the linearized system studied by Rohrs, et al. [4a], which was arrived at by a 'final approach analysis.' Note that in this case the linearized system is the input to the nonlinear system (3.10a). The operators K and G are linear and time-varying due to their dependence on the tuned signals (e_*, z_*). This model (3.10) will now be utilized to develop local stability conditions.

4. CONDITIONS FOR LOCAL STABILITY

If the linearized response x_L in (3.10) is small, and if the nonlinear term $G f(x)$ is suitably restricted, then intuitively, x would be attracted to some neighborhood of x_L. The following theorem makes this notion precise.

Theorem 4.1

(i) If $\exists$ constant g_∞ such that,

$$\gamma_\infty(G) \le g_\infty < \infty \tag{4.1}$$

and if

$$\|x_L\|_\infty \le \varepsilon_\infty (1 - g_\infty \varepsilon_\infty/2), \quad \varepsilon_\infty \in (0, 2/g_\infty) \tag{4.2}$$

then,

$$\|x\|_\infty \le \varepsilon_\infty \tag{4.3}$$

(ii) In addition, if, for some $p \in [1,\infty)$, $\exists$ constant g_p such that,

$$\gamma_p(G) \le g_p < 2/\varepsilon_\infty \tag{4.4}$$

then

$$\|x\|_p \le (1 - g_p \varepsilon_\infty/2)^{-1} \|x_L\|_p \tag{4.5}$$

Proof

Theorem 4.1 is based on the linearization theorem of [9, p. 131]. The proof, as specialized here, is in Appendix A.

Remarks

(1) Theorem 4.1, part (i), asserts that the error outputs x of the adaptive error system are L_∞-bounded in an ε-neighborhood of the linearized response, provided that the linearized response is in L_∞ and is small enough (4.2), and that $G \in L_\infty$-stable (4.1). Condition (4.3) shows that the actual response can be arbitrarily close to the linearized response, if $\|x_L\|_\infty$ is small enough (4.2). Since Theorem 4.1, part (i), provides sufficient conditions, instability does not follow if $x_L \in L_\infty^n$ but exceeds the magnitude constraint (4.2).

(2) The results in part (ii) are stronger than in part (i) since they can only be applied when $x_L \in L_p^n$ for some $p \in [1,\infty)$. Looking at (3.10), this can only occur if $z_* e_* \in L_p^n$ which, in practical situations, almost never occurs due to the presence of disturbances in L_∞. Hence, part (ii) of Theorem 4.1 does not offer any practical advice and we will focus only on part (i).

(3) From (4.2), the largest upper bound on $\|x_L\|_\infty$ is $1/2g_\infty$ which occurs when $\varepsilon_\infty = 1/g_\infty$.

(4) Although Theorem 4.1 part (i) provides conditions for local L_∞-stability, these do not immediately provide a region of attraction, i.e., bounds on e_*, z_*, and $\tilde{\theta}_0$. These bounds in turn are determined from the set of allowable reference commands, plant initial conditions, and disturbances. Since e_* and z_* are bounded by predetermined performance goals of the tuned system, it follows that $\tilde{\theta}_0$ is the unknown driving factor governing the size of $\|x_L\|_\infty$. That the initial parameter error vector occupies this position of villainy should come as no surprise. For example, if $\tilde{\theta}_0$ is small (order ε_∞) then the adaptive system stays near the tuned system for small (order ε_∞) inputs e_*.

(5) No claims are made in Theorem 4.1 part (i) about the mechanism that provides $x_L \in L_\infty^n$ and $G \in L_\infty$-stable. However, it follows from the definition of the tuned system (3.2) that $e_* \in L_\infty$, $z_* \in L_\infty^n$ and H_{ev}, $H_{zv} \in L_\infty$-stable, thus, M in (3.10f) is L_∞-stable. Hence, a term by term inspection of G (3.10d) and x_L (3.10c) reveals that $x_L \in L_\infty$ and $G \in L_\infty$-stable, if and only if:

$$(I + LM)^{-1}\tilde{\theta}_0 \in L_\infty^n, \quad \forall \tilde{\theta}_0 \in R^n \tag{4.6a}$$

and

$$K \in L_\infty\text{-stable} \tag{4.6b}$$

More precisely, we have the following result.

Lemma 4.1

Suppose that a tuned solution $\theta_* \in R^n$ exists. Let $\xi(t)$ denote the solution at time t of the differential equation,

$$\dot{\xi}(t) = -B(M\xi)(t) + w(t) \qquad t \geq 0 \qquad (4.7)$$

Then, $x_L \in L_\infty^n$ and G_ϵ L_∞-stable if and only if $\xi \in L_\infty^n$ for all $\xi(0) \in R^n$ and $w \in L_\infty^n$.

Proof:

Using the definitions of K and M in (3.10), if $\xi(0) = 0$ then (4.7) represents $K : w \mapsto \xi$. Thus, $K \in L_\infty$-stable if $w \mapsto \xi \in L_\infty$-stable. Also, if $\xi(0) = \tilde{\theta}_o$ and $w = z_* e_*$ then $\tilde{\xi} = \tilde{\theta}_L$ from (3.10c). Hence, without any further restrictions on $\tilde{\theta}_o \in R^n$ or $z_* e_* \in L_\infty^n$, the result of Lemma 4.1 is established.

Remarks

Lemma 4.1 identifies the L_∞-stability of the system (4.7) as being crucial to obtaining local stability conditions from Theorem 4.1. This condition is not sufficient. Even if the conditions of Lemma 4.1 are satisfied the adaptive system is locally stable <u>provided that</u> $\|x_L\|_\infty$ <u>is small enough</u>, i.e., (4.2) must hold. Nonetheless, establishing (4.9) is a first step.

Recall that $\|x_L\|_\infty$ is small if $\|\tilde{\theta}_L\|_\infty$ is small, hence it is necessary to control the size of these signals. Comparing θ_L in (3.10c) to (4.7), some of these magnitude conditions can be secured by lowering the adaptation gain, i.e., the norm of the matrix B in (3.1d) or (3.10h).

5. PERSISTENT EXCITATION

In this section we examine persistent excitation as a mechanism to provide L_∞-stability of (4.7), and hence, local L_∞-stability of the adaptive system (3.10).

Definition [8]

A regulated function $f(.) : R_+ \to R^n$ is <u>persistently exciting</u>, denoted $f \in PE$, if positive constants α_1, α_2, and α_3 such that

$$\alpha_1 I_n \leq \int_s^{s+\alpha_3} f(t)\, f(t)'dt \leq \alpha_2 I_n, \quad \forall s \in R_+ \qquad (5.1)$$

The relationship between persistent excitation and stability of (4.7) is given as follows.

Lemma 5.1 [8]

Consider the differential equation:

$$\dot{\xi}(t) = -f(t)\,(Hf'\xi)(t) + w(t), \quad t \geq 0 \qquad (5.2)$$

If $f \in PE$ and $H(s) \in SPR$ then the map $(\xi(0), w) \mapsto \xi$ is exponentially stable, i.e., there exist positive constants m and λ such that,

$$|\xi(t)| \leq m e^{-\lambda t}|\xi(0)| + \int_0^t m e^{-\lambda(t-\tau)}|w(\tau)|d\tau \qquad (5.3)$$

Conditions for Robustness

The usefulness of applying Lemma 5.1 to determine robust stability conditions of (4.7) is made apparent by proceeding as in Section 3, i.e.,

$$H_{ev} = \bar{H}_{ev} + \tilde{H}_{ev} \qquad (5.4)$$

where $\bar{H}_{ev}$ is the nominal representation of H_{ev} and $\tilde{H}_{ev}$ is the deviation induced by modeling error. Combining (5.4) with (4.7), and using the definitions in (3.10) gives,

$$\dot{\xi} = -Bz_*\bar{H}_{ev}z_*'\xi + Q\xi + w \qquad (5.5)$$

where

$$Q := B(M - z_*\tilde{H}_{ev}z_*') \qquad (5.6)$$

If $\bar{H}_{ev}(s) \in SPR$ and $z_* \in SPR$, then (5.3) of Lemma (5.1) applied to (5.5) gives,

$$|\xi(t)| \leq m e^{-\lambda t}|\xi(0)| + \int_0^t m e^{-\lambda(t-\tau)}|(Q\xi)(\tau) + w(\tau)|d\tau \qquad (5.7)$$

Therefore, if Q has a sufficiently small gain then $(\xi(0), w) \mapsto \xi \in L_\infty$-stable, and hence, the adaptive system is locally L_∞-stable. Specific conditions are given as follows.

Theorem 5.1

Suppose $z_* \in PE$, $\bar{H}_{ev}(s) \in SPR$, and hence, from Lemma 5.1, $\xi(t)$ in (5.5) is bounded as shown in (5.7). Then, the adaptive system (3.10) is locally L_∞-stable if, for some $\alpha \in (0, \lambda)$,

$$\gamma_2(F) < 1 - q \qquad (5.8)$$

where

$$q = \frac{m}{\lambda - \alpha}\|e_*\|_\infty \|Bz_*\|_\infty \gamma_2(H^\alpha_{zv}) \qquad (5.9)$$

and the operator F has the integral form,

$$(Fu)(t) = \int_0^t m e^{-(\lambda-\alpha)(t-\tau)} z_*(\tau)(\tilde{H}^\alpha_{ev} z_*' u)(\tau)d\tau \qquad (5.10)$$

Proof: See Appendix B.

Remarks

(1) The α-superscript notation H^α means that if H has transfer function $H(s)$, then H^α has transfer function $H(s-\alpha)$. Thus, $\alpha \in (0,\lambda)$ is further limited so that $H_{ev}(s-\alpha)$ and $H_{zv}(s-\alpha)$ remain exponentially stable, otherwise the L_1-gains in (5.9), (5.10) are infinite.

(2) What Theorem 5.1 asserts is that if $z_* \in PE$, and if H_{ev} is close enough to being SPR, then under suitable small gain conditions (5.8), local stability can be guaranteed via Theorem 4.1. The crux of the matter is to establish that $\gamma_2(F)$ is sufficiently small despite a reasonably large $\tilde{H}_{ev}$.

Calculation of $\gamma_2(F)$

Intuitively, if the range of dominant frequencies of z_* is sufficiently separated from the frequencies where $\tilde{H}_{ev}(j\omega)$ is large, then $\gamma_2(F)$ would be small, e.g., z_*^{ev} is persistently exciting at 'low' frequencies where H_{ev} is approximately SPR. We will formalize this notion in Theorem 5.2 below. First, however, we need the following results from [10] for determining a large class of persistently exciting signals.

Definition 5.1 [10]

A function $f(.): R_+ \to R^n$ has a spectral line at frequency ω of amplitude $a_f(\omega) \in C^n$ if

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_s^{s+\tau} f(t)e^{-j\omega t}\, dt = a_f(\omega), \quad \forall s \in R_+ \tag{5.11}$$

when $a_f(\omega) \neq 0$, f has a spectral line at ω.

Lemma 5.2 [10]

Suppose $f \in L_\infty^n$ has spectral lines at frequencies $\omega_1, \ldots, \omega_p$ of amplitudes $a_f(\omega_1), \ldots, a_f(\omega_p)$. Then $f \in PE$, if

$$\text{rank}\,[a_f(\omega_1), \ldots, a_f(\omega_p)] = n \tag{5.12}$$

Theorem 5.2

Suppose that:

(A1) $z_* \in PE \Leftarrow z_*$ has spectral lines at frequencies $\omega_1, \ldots, \omega_p$ with rank $[a_{z_*}(\omega_1), \ldots, a_{z_*}(\omega_p)]^p = n$.

(A2) $\tilde{H}_{ev}(s) \in SPR$

Then, the adaptive system (3.10) is locally L_∞-stable if, for some $\alpha \in (0,\lambda)$, $\tilde{H}_{ev}(s-\alpha)$ is stable and bounded by,

$$|\tilde{H}_{ev}(j\omega-\alpha)| < \frac{1-q}{mp\,\sup|a_{z_*}(\omega k)|} \inf_k \frac{[(\omega-\omega_k)^2 + (\lambda-\alpha)^2]^{k/2}}{|a_{z_*}(\omega_k)|}, \quad \forall\omega \in R_+ \tag{5.13}$$

with m, λ from (5.7) and q from (5.9).

Proof

See Appendix C.

Discussion

(1) Theorem 5.2 provides an explicit upper bound on the amount by which H_{ev} can deviate from a nominal $\tilde{H}_{ev}$ which is SPR. Thus, if (5.13) holds and if $\|\tilde{z}_1^v\|_\infty$ is sufficiently small (4.2), then signals in the adaptive system are guaranteed to be bounded.

(2) *Unlike the global stability case where the* bound on the deviation $\tilde{H}_{ev}$ is severely restricted (3.9), the bound here can be quite large. Moreover, the bound can be determined from the spectral properties of z_*. Recall from Lemma 5.1 that m and λ in (5.13) are functions of the spectral properties of z_*.

(3) Since $\alpha \in (0,\lambda)$ in (5.13) can be arbitrarily small, and since $q \ll 1$ is likely due to $\|e_*\|_\infty$ being small, it follows that a reasonable approximation to the robustness test (5.13) is

$$|\tilde{H}_{ev}(j\omega)| < \frac{1}{mp\,\sup|a_{z_*}(\omega_k)|} \inf_k \frac{[(\omega-\omega_k)^2 + \lambda^2]^{1/2}}{|a_{z_*}(\omega_k)|}, \quad \forall\,\omega \in R_+ \tag{5.14}$$

6. CONCLUDING REMARKS

6.1 Test Procedures

The results of Theorem 5.2 can be of practical use since they provide the basis for developing robustness test procedures. The most obvious way to apply Theorem 5.2 is to determine the model error bound—the right hand side of (5.13)—by either analytical or empirical means. Once the bound is found, it remains to generate reasonable estimates of the model error and compare this to the bound.

An alternative procedure is to verify Lemma 4.1 by direct empirical means. In other words, we can utilize Theorem 5.2 to give qualitative guidelines on the required spectral characteristics of z_* and then simulate (4.7) for a variety of initial states $\xi(0)$ and inputs $w \in L_\infty^n$. This latter approach is not theoretically perfect, but is a practical means to gain understanding of the adaptive system behavior.

6.2 Other Mechanisms for Local Stability

Although we have focused on persistent excitation as a means to ensure local stability, this is by no means the only way. For example, if the adaptation algorithm (3.1d) is modified to include a retardation (see, e.g., [11], [12]) then $L(s)$ in (3.10h) will have the form [13]:

$$L(s) = \frac{a}{s+b} B, \quad B = B' > 0 \tag{6.1}$$

where (a,b) are positive constants. This means that if z_* is a constant vector, then the linearized system (3.10c) is L_∞-stable by passivity arguments [13]. (Note that it is not possible to prove (4.7) stable for z_* constant with $L(s) = (1/s)B$ as in (3.10h).) Hence, using (6.1) together with theorems on slowly varying systems (z_* stays near constant long enough), we can arrive at conditions for local L_∞-stability which are independent of persistent excitation (see [13] for preliminary results).

APPENDIX A
PROOF OF THEOREM 4.1

We first show that $f(x)$ in (3.10b) has the property that $\forall\varepsilon > 0$,

$$|x| < \varepsilon \Rightarrow |f(x)| < \frac{\varepsilon}{2} |x| \tag{A.1}$$

From (3.10b),

$$|f(x)| = (|\tilde{z}'\tilde{\theta}|^2 + |\tilde{z}\tilde{e}|^2)^{1/2}$$
$$\leq |\tilde{z}|\,(|\tilde{\theta}|^2 + |\tilde{e}|^2)^{1/2}$$
$$= |\tilde{z}|\,(|x|^2 - |\tilde{z}|^2)^{1/2}, \quad \text{by (3.7b)}$$

$$\leq \frac{1}{2}|x|^2, \text{ by holding } |x| \text{ fixed}$$

$$\leq \frac{\varepsilon}{2}|x|$$

by $|x| \leq \varepsilon$.

Now, assume temporarily that (A.1) holds for all $x \in R^n$, i.e.,

$$|f(x)| \leq (\varepsilon/2)|x|, \quad \forall |x| \leq \varepsilon \tag{A.2a}$$

$$|f(x)| \leq (\varepsilon/2)|x|, \quad \forall |x| > \varepsilon \tag{A.2b}$$

by (3.10a)

$$\begin{aligned} ||x||_{T\infty} &\leq ||x_L||_{T\infty} + ||Gf(x)||_{T\infty} \\ &\leq ||x_L||_{T\infty} + g_\infty ||f(x)||_{T\infty}, \text{ by (4.1)} \\ &\leq ||x_L||_{T\infty} + (g_\infty \varepsilon/2)\, ||x||_{T\infty} \end{aligned} \tag{A.3}$$

using the temporary assumption (A.2). Since $x_L \in L_\infty$ (4.2) and $||x_L||_{T\infty} \leq ||x_L||_\infty$,

$$\begin{aligned} ||x||_{T\infty} &\leq ||x_L||_\infty + (g_\infty \varepsilon/2)\, ||x||_{T\infty} \\ &\leq \varepsilon\,(1-g_\infty \varepsilon/2) + (g_\infty \varepsilon/2)\, ||x||_{T\infty} \end{aligned} \tag{A.4}$$

by (4.2). Since $g_\infty \varepsilon/2 < 1$ by assumption (4.2),

$$||x||_{T\infty} \leq \varepsilon \tag{A.5}$$

and hence, $||x||_\infty \leq \varepsilon$. Looking back over the proof we see that the temporary assumption (A.2) is never violated, i.e., the behavior of $f(x)$ for $|x| > \varepsilon$ (A.2b) is never needed under the assumptions of Theorem 4.1. This proves part (i).

Part (ii) proceeds analogously except now we use the L_p-norm. Note that part (ii) uses $||x||_\infty \leq \varepsilon$ as an assumption.

APPENDIX B
PROOF OF THEOREM 5.1

We use the exponential weighting techniques from [9], [14]. Let y^α denote the exponential weighting operation,

$$(y^\alpha)(t) := y^\alpha(t) := e^{\alpha t}\, y(t) \tag{B.1}$$

If $y = Hu$ then let H^α denote the map $u^\alpha \mapsto y^\alpha$. For example, if H has transfer function $H(s)$, then H^α has transfer function $H(s-\alpha)$.

Definition B.1 [14]

An operator $H : L^m_{2e} \to L^n_{2e}$ has decaying L_1-memory if $\exists$ a nonnegative, nonincreasing function $\beta(.) \in L_1$ such that

$$|(Hu)(t)|^2 \leq \int_0^t \beta(t-\tau)|u(\tau)|^2 d\tau, \quad \forall t \geq 0, \quad \forall u \in L^m_{2e}. \tag{B.2}$$

Lemma B.1 [14]

Suppose $H : L^m_{2e} \to L^n_{2e}$ has decaying L_1-memory. If, for some $\alpha > 0$, $H^\alpha \in L_2$-stable, then $H \in L_\infty$-stable.

Apply Lemma B.1 as follows: The exponentially weighted version of (5.7) is,

$$|\xi^\alpha(t)| \leq m e^{-(\lambda-\alpha)t}\xi(0) + \int_0^t m e^{-(\lambda-\alpha)(t-\tau)}[w^\alpha(\tau) + (Q^\alpha \xi^\alpha)(\tau)]d\tau \tag{B.3}$$

This, together with the definitions of F (5.10) and q (5.9) gives,

$$||\xi^\alpha||_{T2} \leq (m/\ \ 2(\lambda-\alpha)\ )|\xi(0)| + \frac{m}{\lambda-\alpha}||w^\alpha||_{T2} + (q + \gamma_2(F))\, ||\xi^\alpha||_{T2} \tag{B.4}$$

Using $q + \gamma_2(F) < 1$ from (5.8) gives,

$$||\xi^\alpha||_{T2} \leq [1-q-\gamma_2(F)]^{-1}\, [(m/\ \ 2(\lambda-\alpha)\ \ |\xi(0)| + \frac{m}{\lambda-\alpha}||w^\alpha||_{T2}] \tag{B.5}$$

Hence, $w^\alpha \mapsto \xi^\alpha \in L_2$-stable. Moreover, using (B.3) $w \mapsto \xi$ is exponentially stable and hence, from definition B.1, $w \mapsto \xi$ has decaying L_1-memory. Therefore, the conditions of Lemma B.1 are satisfied; consequently $w \mapsto \xi \in L_\infty$-stable.

APPENDIX C
PROOF OF THEOREM 5.2

We need to calculate the L_2-gain of F (5.10) under the assumptions of the theorem. From (5.10), $F : x \mapsto y$ has the form,

$$y = Fx = Gz_*\, Hz_*'x \tag{C.1}$$

where G and H have transfer functions,

$$G(s) = \frac{m}{s+\lambda-\alpha} \tag{C.2}$$

$$H(s) = \tilde{H}_{ev}(s-\alpha) \tag{C.3}$$

Since F is causal, the L_2-gain of F is,

$$\gamma_2(F) = \sup_{x \in L^2_n/0} \frac{||y||_2}{||x||_2} \tag{C.4}$$

$$= \sup_{x \in L^n_2/0} \frac{||y||_2}{||x||_2} \quad \text{(by Parseval's Theorem)} \tag{C.5}$$

where $y(j\omega)$ and $x(j\omega)$ are the Fourier transforms of y and x, respectively. Using assumption (A1) of Theorem 5.2,

$$y(j\omega) = \sum_{k=1}^{p}\sum_{r=1}^{p} G(j\omega)H(j\omega - j\omega_k)a_z(\omega_k)a_z(\omega_r)'x(j\omega-j\omega_k-j\omega_r) \tag{C.6}$$

Hence,

$$|y(j\omega)| \leq c(\omega)\, [\sum_{k=1}^{p}\sum_{r=1}^{p} |x(j\omega-j\omega_k-j\omega_r|^2\,]^{1/2} \tag{C.7}$$

where

$$c(\omega) = (\sup_k |a_{z_*}(\omega_k)|)(\sup_k |G(j\omega)H(j\omega - j\omega_k)|.|a_{z_*}(\omega_k)|) \tag{C.8}$$

Thus,

$$||y||_2 = (\int_{-\infty}^{\infty} |y(j\omega)|^2 d\omega)^{1/2}$$

$$\leq [\sup_{\omega} c(\omega)] [\int_{-\infty}^{\infty} \sum_{k=1}^{p} \sum_{r=1}^{p} |x(j\omega - j\omega_k - j\omega_r)^2 d\omega]^{1/2}$$

$$= p[\sup_{\omega} c(\omega)] \; ||x||_2 \qquad (C.9)$$

Using the definition of gain (C.5) together with (5.8) from Theorem 5.1 gives,

$$\gamma_2(F) \leq p \sup_{\omega} c(\omega) < 1 - q \qquad (C.10)$$

Condition (5.13) follows by substituting (C.2) and (C.3) into (C.10) and rearranging terms.

REFERENCES

[1] K.S. Narendra, Y.H. Lin and L.S. Valavani, 'Stable Adaptive Controller Design, Part II: Proof of Stability,' IEEE Trans. Aut. Contr., Vol. AC-25, pp. 440-448, June 1980.

[2] G.C. Goodwin, P.J. Ramadge and P.E. Caines (1980), 'Discrete Time Multivariable Adaptive Control,' IEEE Trans. Aut. Contr., Vol. AC-25, June 1980, pp. 449-456.

[3] Y.D. Landau, Adaptive Control: The Model Reference Approach, Marcel Dekker, 1979.

[4a] C. Rohrs, L. Valavani, M. Athans and G. Stein, 'Analytical Verification of Undesirable Properties of Direct Model Reference Adaptive Control Algorithms' Proc. 20th IEEE CDC, San Diego, CA, December 1981.

[4b] C. Rohrs, L. Valavani, M. Athans and G. Stein, 'Robustness of Adaptive Control Algorithm in the Presence of Unmodeled Dynamics,' Proc. 21st IEEE CDC, Orlando, FL, December 1982.

[5a] R.L. Kosut and B. Friedlander, 'Performance Robustness Properties of Adaptive Control Systems,' Proc. 21st IEEE CDC, Orlando, FL, December 1982.

[5b] R.L. Kosut and B. Friedlander, 'Robust Adaptive Control: Conditions for Global Stability,' IEEE Trans. on Aut. Contr., to appear.

[6] B. Wittenmark and K.J. Astrom, 'Implementation Aspects of Adaptive Controllers and Their Influence on Robustness,' Proc. 21st IEEE CDC, Orlando, FL, December 1982.

[7] B.D.O. Anderson and C.R. Johnson, Jr., 'Exponential Convergence of Adaptive Identification and Control Algorithms,' Automatica, Vol. 18, No. 1, 1982.

[8] B.D.O. Anderson, 'Exponential Stability of Linear Equations Arising in Adaptive Identification,' IEEE Trans. Aut. Contr., Vol. AC-22, No. 1, pp. 83-88, February 1977.

[9] C.A. Desoer and M. Vidysagar, Feedback Systems: Input-Output Properties, Academic, 1975.

[10] S. Boyd and S. Sastry, 'On Parameter Convergence in Adaptive Control,' Memorandum No. UCB/ERL M83/83, Electronics Research Lab., University of California, Berkeley, June 1983.

[11] G. Kriesselmeier and K.S. Narendra (1982), 'Stable Model Reference Adaptive Control in the Presence of Bounded Disturbances,' IEEE Trans. Aut. Contr., Vol. AC-27, No. 6, December 1982, pp. 1169-1175.

[12] P.A. Ioannou and P.V. Kokotovic (1983b), Adaptive Systems with Reduced Models, Springer-Verlag, 1983.

[13] R.L. Kosut, 'Robust Adaptive Control: Conditions for Local Stability,' Proc. 1982 ACC, pp. 565-569, San Francisco, CA, June 1983.

[14] M. Vidyasagar, 'L_∞-Stability Criteria for Interconnected Systems Using Exponential Weighting,' IEEE Trans. Circuits Syst., Vol. CAS-25, No. 11, Nov. 1978.

[15] J.C. Doyle and G. Stein, 'Multivariable Feedback Design: Concepts for a Modern Classical Synthesis,' IEEE Trans. Aut. Contr., AC-26, No. 1, pp. 4-17, February 1981.

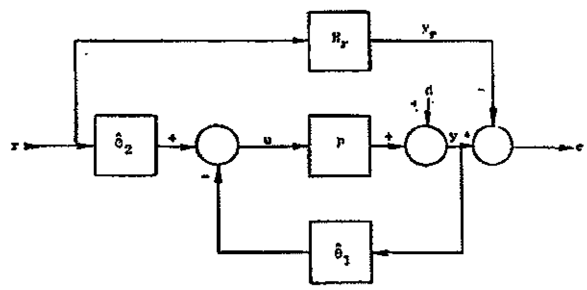

Figure 3-1. MRAC System

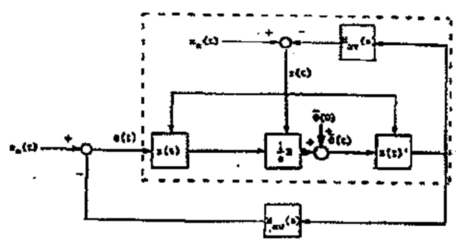
Figure 3-2. Adaptive Error System

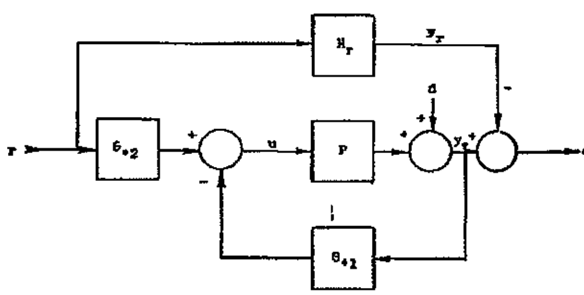

Figure 3-3. Tuned System