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FREQUENCY SHAPED LINEAR OPTIMAL CONTROL WITH TRANSFER FUNCTION RICCATI EQUATIONS*

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Abstract Standard linear optimal control theory is generalizeD using a spectral factorization approach to elucidate some effects of frequency shaped performance indices. The theory discusses robustness results which parallel those of standard optimal control design.Matrix transfer function Riccati equations emerge as a conceptual tool associated with the frequency shaped optimal designs.

Keywords Optimal control.

1 INTRODUCTION

Matrix Riccati equations are a key theoretical and numerical tool in optimal control problems associated with linear systems-having state space descriptions and quadratic performance indices. For time invariant linear systems described by state space equations in terms of matrices $\{A,B\}$ and quadratic control performance indices with parameters $\{R=R'>0,\ Q=Q'>0\}$, the associated steady-state Riccati equation is in continuous-time.

$$P A = A'P - P BR^{-1}B'P + Q = 0 \qquad (1.1)$$

The optimal state feedback gain matrix is

$$K = R^{-1}BP \qquad (1.2)$$

and the optimal index is expressed in terms of P.

This paper studies linear systems having transfer function descriptions and quadratic indices with frequency dependent weightings, using a spectral factorization approach. In particular, we study linear continuous time systems described by transfer function matrices $W(s) = A(s)^{-1}B(s)$ and quadratic control performance indices with parameters $\{R(s),Q(s)\}$, and in this way generalize in some directions the original theory which is stated in terms of constant $\{A,B,Q,R\}$ to a theory in terms of $\{A(s),B(s),Q(s),R(s)\}$. Surprisingly perhaps matrix transfer function Riccati equations emerge as a conceptual tool associated with an interesting subclass of such problems. Thus in the case of square, full rank, and minimum phase plants, the optimal controller has a decompositon in terms of a Riccati equation as

$$P^*(s)A(s) + A(s)^*P(s) \qquad (1.3a)$$

$$+ P^*(s)B(s)R_*^{-1}(s)B^*(s)P(s) - Q(s) = 0$$

$$K(s) = R^{-1}(s)B^*(s)P(s) \qquad (1.3b)$$

The notation $X^*(s) = x'(-s)$ is used throughout the paper where the prime denotes matrix transposition. For $s=j\omega$ with ω real, and X(s) real rational, $X^*(s)$ is the hermitian conjugate of X(s).

We stress that in the paper, computational procedures and robustness properties are derived from the spectral factorization formulations, rather than from the Riccati equations, although it is true that some of the computations or properties can be achieved working with augmented plants and augmented matrix Riccati equations.

An important motivation for studying control and estimation problems characterized by polynomial or transfer function matrices $\{A(s),B(s),\ Q(s),\ R(s)\}$ is to achieve good trades between performance for a nominal plant and robustness to variations from the nominal plant as discussed in many of the references.

2 FREQUENCY DOMAIN OPTIMAL CONTROL

Plant

In Laplace transform notation, the plant is defined by

$$y(s) = W(s)\ u(s) + W(s)\ v_1(s) \qquad (2.1)$$

where all transforms are defined in terms of the bilateral Laplace transform,

$$y(s) \triangleq \int_{-\infty}^{\infty} y(t)\ e^{-st}dt \qquad (2.2)$$

We assume that $v_1(t)$ is not under our control, we shall require that

$$v_1(t) = 0,\ t < t_0 \qquad (2.3)$$

The plant transfer function W(s) is assumed to be real rational with a left coprime matrix fraction description.

$$W(s) = A^{-1}(s)B(s) \qquad (2.4)$$

Also,

W(s) is strictly minimum phase, i.e. rank B(s) = normal rank B(s) for

$$\mathrm{Re}[s] > 0 \qquad (2.5a)$$

$\lim_{s\to\infty}$ sW(s) is finite, with rank equal to the normal rank of W(s) (2.5b)

Performance Index. Define a frequency domain cost function as

$$J \triangleq \int_{-\infty}^{\infty} [y^*(s)Q(s)y(s)+ u^*(s)R(s)u(s)]\Big|_{s=j\omega} d\omega \qquad (2.6)$$

$Q(s)$ is real rational, $Q(s) = Q^*(s)$, $Q(j\omega)$ exists for all

real ω with $Q(j\omega) > 0$ and $\lim_{s\to\infty} Q(s) < \infty$ (2.7a)

$R(s)$ is real rational, $R(s) = R^*(s)$, $R(j\omega)$ exists for all real ω with $R(j\omega) > 0$, and $\lim_{s\to\infty} R(s)$ is finite and nonsingular. (2.7b)

Also introducing the *stable*, *minimum phase* spectral factors of $R(s)$ and $Q(s)$. Let $R^{1/2}$ and $Q^{1/2}$ be rational and satisfy

$$R(s) = [R^{1/2}(s)]^*[R^{1/2}(s)] \quad Q(s)=[Q^{1/2}(s)]^*[Q^{1/2}(s)]$$

with $R^{1/2}(s)$, $R^{-1/2}(s)$ free of poles in $\mathrm{Re}[s] \geq 0$, and $Q^{1/2}(s)$ *free of poles in* $\mathrm{Re}[s] > 0$ and of constant rank in $\mathrm{Re}[s] > 0$.

Now we shall note a further restriction on $v_1(t)$. *We require that*

$$v_1(s) = R^{-1/2}(s)v_1'(s) \qquad (2.8)$$

where $v_1'(t)$ is a *time function zero outside the* interval $(t_0, 0)$.

Three other conditions on $W(s)$ are:
with $S=\{s_0:\|W(s_0)\|<\infty, \text{rank } W(s_0) = \text{normal rank } W(s)\}$

and $d=\dim$ [nullspace of $W(s_0)R^{-1/2}(s_0)$, $s_0=j\omega_0$, ω_0 real, any $s_0 \in S$], there exist d linearly independent constant α_i such that

$$W(s_0)R^{-1/2}(s_0)\alpha_i=0 \text{ for all } s_0 = j\omega_0,\ \omega_0 \text{ real},\ s_0 \in S. \qquad (2.9)$$

With $N(s)$, $D(s)$ polynomial and right coprime, and with $D(s)$ row proper, suppose that

$$Q^{1/2}(s) = N(s)D^{-1}(s),$$

$$D^{-1}(s)W(s)R^{-1/2}(s) = \bar{B}(s)\bar{A}^{-1}(s) \qquad (2.10)$$

for some right coprime polynomial pair $\bar{B}, \bar{A}$. The second assumption on $W(s)$, $Q(s)$, $R(s)$ is as follows:

Any right common divisor of $N(s)\bar{B}(s)$ and $\bar{A}(s)$ has all determinantal zeros in $\mathrm{Re}[s]<0$ (2.11)

A third assumption which allows one to initialize the states of the block $R^{-1/2}(s)$ to zero and achieve arbitrary initial states on the plant with $v_1(t)=0$ *for* $t > 0$ *is as follows:*

No zero of $R^{-1/2}(s)$ cancels a pole of $W(s)$ (2.12)

Control Laws.

We consider the class of rational transfer function control laws

$$u(s) = -K(s)y(s) + v(s) \qquad (2.13)$$

so that the closed loop system transfer function matrix is

$$W_{CL}(s) = W(s)[I + K(s)W(s)]^{-1}$$

$$= [I + W(s)K(s)]^{-1}W(s) \qquad (2.14)$$

$K(s)$ is a transfer matrix here which is constrained to satisfy the following conditions

$$W_{CL}^*(s)\, Q(s)\, W_{CL}(s) + [I-K(s)W_{CL}(s)]^* R(s)[I-K(s)W_{CL}(s)] = R(s) \qquad (2.15a)$$

$W_{CL}(s)$ has all poles in $\mathrm{Re}[s] < 0$ (2.15b)

$K(s)$ is proper (2.15c)

$K(s)W(s)$ has no pole-zero cancellation in $\mathrm{Re}[s] > 0$ (2.15d)

the controller has zero states at time t_0 (2.15e)

The "external control" can be chosen freely except that we require $v_2(t)=0$ for $t<0$.

Optimal Control Theory.

Theory concerning the optimal selection of $K(s)$ is developed in the following lemma. We defer for the moment *the existence question.*

Lemma 2.1

Suppose $W(s)$, $Q(s)$, $R(s)$ and $K(s)$ satisfy assumptions or inequalities (2.5), (2.8), (2.9) (2.11) and (2.15). Then $K(s)$ satisfies the following natural generalization of the return difference equation of linear optimal control:

$$W^*(s)Q(s)W(s) + R(s) = [I+W^*(s)K^*(s)]R(s)[I+K(s)W(s)] \qquad (2.16a)$$

$I + K(s)W(s)$ is strictly minimum phase, i.e

$[I+KW]^{-1}$ has all poles in $\mathrm{Re}[s] < 0$ (2.16b)

Moreover, $K(s)$ *leads to an asymptotically* stable closed-loop system and defines an optimal controller via (2.13) for $t > 0$. The optimal control law is

$$u(s) = -[K(s)\, y(s)], \quad v_2(s) = 0 \qquad (2.17)$$

Proof See Appendix.

3 CONSTRUCTION OF K(s), SPECTRAL FACTORIZATION AND ROBUSTNESS

Construction of K(s).

It proves convenient to work for much of the following with quantities

$$\bar{W}(s) = D^{-1}(s)W(s)R^{-1/2}(s) \qquad (3.1a)$$

$$\bar{K}(s) = R^{1/2}(s)K(s)D(s) \qquad (3.1b)$$

Then the search for $K(s)$ satisfying (2.1b) becomes one for $\bar{K}(s)$ satisfying

$$\bar{W}^* N^* N\bar{W} + I = (I + \bar{W}^*\bar{K}^*)(I + \bar{K}\bar{W}) \qquad (3.2)$$

Further, since $\bar{W} = D^{-1}WR^{-1/2}$ condition (2.9) on the nullspace *of* $WR^{-1/2}$ *carries over to a like* condition on $\bar{W}$. Suppose to begin with that $\bar{W}$ has full column rank. With $N\bar{W} = N\bar{B}\bar{A}^{-1}$, a right matrix fraction description of the strictly proper $N\bar{W}$ with the greatest right divisor of $N\bar{B}$ and $\bar{A}$ possessing a Hurwitz determinant, see (2.11), it is well known that a polynomial $\bar{C}$ can be found, see e.g. [10] such that

$$\bar{W}^* N^* N\bar{W}+I = [I+ (\bar{A}^*)^{-1}\bar{C}^*](I + \bar{C}\bar{A}^{-1}) \qquad (3.3)$$

and

$$\det(\bar{A} + \bar{C}) \text{ is Hurwitz}, \quad \bar{C}\bar{A}^{-1} \text{ is strictly proper} \qquad (3.4)$$

Accordingly, the task of finding $\bar{K}$ becomes one of solving $\bar{K}\bar{B} = \bar{C}$, *with* $\bar{B}$ *possessing full column* rank. We can characterize all solutions of this equation in the following way. (This characterization will prove part of the properties of K, and in turn will allow a sharper characterization). Let U be a polynomial unimodular matrix such that

$$U\bar{B} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \qquad (3.5)$$

with $\bar{B}_1$ square and nonsingular. Notice that $\bar{B}_1$ is a greatest common right divisor of the columns of $\bar{B}$. Then the set of all K satisfying

$\bar{K}\,\bar{B} = \bar{C}$ is given by

$$\bar{K} = [\bar{C}\bar{B}_l^{-1} \quad X]\, U^{-1} \qquad (3.6)$$

where X is an arbitrary rational matrix. If $\bar{C}, \bar{B}_l$ are right coprime, all $\bar{K}$ necessarily have the (determinantal) zeros of $\bar{B}_l$ as poles; otherwise, all $\bar{K}$ necessarily have as poles a subset of the zeros of $\bar{B}_l$. Also, there obviously exist $\bar{K}$ with poles confined to (a subset of) the zeros of $\bar{B}_l$.

Now suppose that $\bar{W}$ does not have full rank. There exists by (2.9) *an orthogonal constant P* such that $W'P = [\tilde{W}\ 0]$ where $\tilde{W}$ has full column rank. Premultiplying and postmultiplying (3.2) by P' and P yields

$$\begin{bmatrix} \tilde{W}^* \\ 0 \end{bmatrix} N^*N[\tilde{W} \quad 0] + I = [I + \begin{bmatrix} \tilde{W}^* \\ 0 \end{bmatrix} K^*P][I+P'\bar{K}[\tilde{W} \quad 0]]$$

We simply find $\tilde{K}$ such that

$$\tilde{W}^*N^*N\tilde{W} + I = [I+\tilde{W}^*\tilde{K}^*][I+\tilde{K}\tilde{W}] \qquad (3.7)$$

in the same manner as $\bar{K}$ is found, and then set

$$\bar{K} = P \begin{vmatrix} \tilde{K} \\ 0 \end{vmatrix} \qquad (3.8)$$

Properties of $\bar{K}$ and K

We shall prove a number of desired properties of K:

Lemma 3.1

Let $\bar{K}$ be determined via the above procedure, assuming (2.4), (2.5), (2.9) and (2.11) hold. Then there are no unstable pole-zero cancellations in forming $\bar{K}\bar{W}$ and KW. Moreover, $(I+\bar{K}\bar{W})^{-1}$ and $(I+KW)^{-1}$ are free of poles in $Re[s] > 0$, and $W_{CL}(s)$ has all poles in $Re[s] < 0$.

Proof See Appendix.

Passband robustness.

The loop gain defined by the plant and controller is

$$W_{OL}(s) = K(s)W(s) \qquad (3.9)$$

The spectral factorization (2.16) yields the return difference inequality

$$[I+W_{OL}(-j\omega)]'\ R(j\omega)[I+W_{OL}(j\omega)] \geq R(j\omega) \qquad (3.10)$$

so that if $R(s) = r(s)I$ for a scalar $r(s)$, then

$$[I+W_{OL}(-j\omega)]'\ [I+W_{OL}(j\omega)] \geq I \qquad (3.11)$$

Likewise, if we define

$$\bar{W}_{OL}(j\omega) = \bar{K}(j\omega)\bar{W}(j\omega) = R^{1/2}(j\omega)W_{OL}(j\omega)R^{-1/2}(j\omega) \qquad (3.12)$$

then

$$[I+\bar{W}_{OL}(-j\omega)]'\ [I+\bar{W}_{OL}(j\omega)] \geq I \qquad (3.13)$$

It is straightforward also to see that these inequalities hold with equality only for isolated ω if $Q^{1/2}W$ has full column rank almost everywhere.

A Minimum Phase Property of the Open-Loop Response.

We assert:

Lemma 3.2

For the optimal designs of section 2 $Z(s) = \bar{W}_{OL}(s)[I+\bar{W}_{OL}(s)]^{-1}$ is positive real, and $\bar{W}_{OL}(s)$ is minimum phase, but not necessarily strictly minimum phase; any $j\omega$ axis zero must be simple.

Proof See Appendix.

Properness of K(s):

It remains to be shown that we can find a proper K(s). For this purpose, let us employ a different constructive procedure for K(s) than that given earlier. This alternative procedure based on Forney (1975) constructs a K(s) of least McMillan degree which is proper, assuming there exists some proper solution. (Note that we were not in a position to present this procedure earlier, since to carry it through, we need to know that $\lim_{s\to\infty} sKW$ exists and is finite, as described above.)

Suppose that $\bar{B}(s)$ and $\bar{C}(s)$ have dimension $p \times m$ and $m \times m$ with $p \geq m$. Construct a minimal left polynomial basis [L M] such that

$$\overset{\leftarrow m \rightarrow \quad \leftarrow p \rightarrow}{p\,[L(s) \quad M(s)]} \begin{bmatrix} \bar{C}(s) \\ -D(s)\bar{B}(s) \end{bmatrix} = 0 \qquad (3.14)$$

The row degrees of [L M] are minimal and the highest degree coefficient matrix $[L^{(h)}\ M^{(h)}]$ has full (row) rank. Now if $L^{(h)}$ has full column rank, there exists, a proper $\bar{H}$ of least McMillan degree such that $\bar{H}D\bar{B} = \bar{C}$.

Assuming for the moment that $L^{(h)}$ has full column rank then it is clear that all poles of $\bar{H}D$ must be zeros of $\bar{B}$. With $\bar{K} = \bar{H}D$, $\bar{K}$ has poles which are zeros of $\bar{B}$, and $K = R^{-1/2}\,\bar{K}\,D^{-1} = R^{-1/2}\,\bar{H}$ is proper, and we are done.

4 MATRIX RICCATI EQUATION FOR OPTIMAL CONTROL

In this section it is shown how a Riccati matrix transfer function equation can be associated with the previous results. The Riccati equation is perhaps also of independent interest, but this aspect remains to be explored.

Associated with any K(s) *satisfying the spectral* factorization (3.2), let us introduce a transfer function $\bar{P}(s)$ satisfying the linear matrix equation

$$\begin{aligned} &\bar{P}^*(s)[\bar{A}(s) + \bar{B}(s)\bar{K}(s)] \\ &+ [\bar{A}^*(s) + \bar{K}^*(s)\bar{B}^*(s)]\bar{P}(s) \\ &= \bar{K}^*(s)\bar{K}(s) + N^*(s)N(s) \end{aligned} \qquad (4.1)$$

Here, we proceed by stages to show that (4.1) leads to the matrix transfer function Riccati equation as in (1.3).

Lemma 4.1

Let the conditions specified in the hypothesis of Lemma 2.1 hold and let K(s) be *defined as* described in the Lemma statement. Let N(s), $\bar{W}(s)$ and $\bar{K}(s)$ be as defined in (2.11) and (3.1) and let $\bar{W}(s)$ have a **left** coprime matrix fraction description $\bar{A}^{-1}(s)\,\bar{B}(s)$. Then there exists a rational $\bar{P}(s)$ such that (4.1) holds.

Proof (see Appendix)

As background, let us observe that if $\bar{K}(s)$ has a right coprime matrix fraction description $\bar{K}(s) = \bar{K}_N(s)\,\bar{K}_D^{-1}(s)$, then (4.1) implies and is *implied by*

$$\bar{K}_D^*(s)\ \bar{P}^*(s)[\bar{A}(s)\bar{K}_D(s) + \bar{B}(s)\bar{K}_N(s)]$$

$$+ [\bar{K}_D^*(s)\bar{A}^*(s) + \bar{K}_N^*(s)\bar{B}^*(s)]\bar{P}(s)K_D(s)$$

$$= \bar{K}_N^*(s)\, \bar{K}_N(s) + \bar{K}_D^*(s)\, N^*N\, \bar{K}_D(s) \qquad (4.2)$$

which causes us to look at the polynomial equation

$$G^*(s)[\bar{A}(s)\bar{K}_D(s) + \bar{B}(s)\bar{K}_N(s)]$$

$$+ [\bar{K}_D^*(s)\bar{A}^*(s) + \bar{K}_N^*(s)\bar{B}^*(s)]G(s)$$

$$= \bar{K}_N^*(s)\bar{K}_N(s) + \bar{K}_D^*(s)N^*N\,\bar{K}_D(s) \qquad (4.3)$$

Any solution of (4.3)(whether or not polynomial) defines a solution of (4.1) by $P(s) = G(s)K_D^{-1}(s)$.

We do however wish to isolate solutions of (4.1) with special properties.

Let U(s) be a nonmodular matrix such that

$$U(s)\,\bar{B}(s) = \begin{bmatrix} \bar{B}_1(s) \\ 0 \end{bmatrix} \qquad (4.4)$$

with $\bar{B}_1(s)$ square. Since $\bar{B}(s)$ has full column rank, $\bar{B}_1(s)$ is nonsingular. Replace $\bar{A},\bar{B}$ and $\bar{P}$ by $U\bar{A}$, $U\bar{B}$ and $U^{-*}\bar{P}$. Now with $\bar{P}$ any solution of (4.1) obtained as $G(s)\,K_D^{-1}(s)$ with polynomial G, define

$$M_1 = \bar{B}^*\,\bar{P}\,(\bar{A}+\bar{B}\bar{K})^{-1}\,\bar{B} = [\bar{B}_1^*\quad 0]\,\bar{P}\,(\bar{A}+\bar{B}\bar{K})^{-1}\begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \bar{B}_1^{-*} \\ 0 \end{bmatrix} M_1 \text{ -strictly} \qquad (4.5a)$$

$$\text{proper part of } M_1]\,[\bar{B}_1^{-1}\quad 0] \qquad (4.5b)$$

$$\bar{P}_1 = \bar{P} - M_2\,(\bar{A} + \bar{B}\bar{K}) \qquad (4.5c)$$

Lemma 4.2.

With the above definition, $\bar{P}_1$ satisfies (4.1), $\bar{B}^*\,\bar{P}_1\,\bar{W}_{CL}$ is strictly proper, with all poles at the determinantal zeros of $(\bar{A}\,\bar{K}_D + \bar{B}\,\bar{K}_N)$. If any other solution $\bar{P}$ of (4.1) obtainable as $\tilde{G}(s)\,K_D^{-1}(s)$ for a polynomial $\tilde{G}(s)$ is used to derive a $\tilde{P}_1$, than $\bar{B}^*\,\tilde{P}_1\,\bar{W}_{CL} = \bar{B}^*\bar{P}_1\bar{W}_{CL}$.

Proof See Appendix.

Remarks

Now let us turn to connecting $\bar{K}(s)$ and $\bar{P}(s)$.

Lemma 4.3

Under the hypotheses of Lemma 4.1, any solution $\bar{P}(s)$ of (4.1) with $\bar{B}^*(s)\bar{P}(s)\bar{W}_{CL}(s)$ strictly proper satisfies

$$\bar{K}(s)\bar{W}(s) = \bar{B}^*(s)\bar{P}(s)\bar{W}(s) \qquad (4.6)$$

Proof See Appendix.

5 CONCLUSIONS

The paper has built a bridge from the state space to a transfer function approach for optimal control system design. Once the bridge is built, it is not surprising that many of the insights and results of the state space approach to optimal design apply to the frequency domain approach. The details associated with the corresponding results for the discrete time case are not presented in this paper, but should be obtainable by similar methods.

APPENDIX

Proof of Lemma 2.1

Solve (2.1), (2.13) and (2.14) for y(s) and u(s) in terms of $v_1(s)$ and $v_2(s)$.

$$y(s) = W_{CL}(s)\,[v_1(s) + v_2(s)]$$

$$u(s) = -K(s)W_{CL}(s)v_1(s) + [I-K(s)W_{CL}(s)]v_2(s)$$

Thus,

$$Q^{1/2}(s)y(s) = Q^{1/2}(s)W_{CL}(s)v_1(s) + Q^{1/2}(s)W_{CL}(s)v_2(s)$$

$$R^{1/2}(s)u(s) = -R^{1/2}(s)K(s)W_{CL}(s)v_1(s) + R^{1/2}(s)\,[I-K(s)W_{CL}(s)]v_2(s)$$

After some algebra one can show

$$y^*(s)Q(s)y(s) + u^*(s)R(s)u(s)$$
$$= v_1^*[W_{CL}^*Q\,W_{CL} + W_{CL}^*K^*R\,K\,W_{CL}]v_1$$
$$+ v_1^*[W_{CL}^*\,Q\,W_{CL} - W_{CL}^*K^*R(I-K\,W_{CL})]v_2$$
$$+ v_2^*\,[W_{CL}^*\,Q\,W_{CL} - (I-K\,W_{CL})^*R(I-K\,W_{CL})]v_1$$
$$+ v_2^*\,[W_{CL}^*Q\,W_{CL} + (I-K\,W_{CL})^*R\,K\,W_{CL}]v_2$$

Using (2.15), this equation may be written as

$$y^*\,Q\,y + u^*Ru = v_1^*W_{CL}^*[Q+K^*RK]W_{CL}v_1$$
$$+ v_1^*\,[RK\,W_{CL}v_2] + [RK\,W_{CL}v_2]^*v_1$$
$$+ v_2^*R\,v_2$$

Thus the cost function (2.6) becomes

$$J = \int_{-\infty}^{\infty}[y^*Qy + u^*R\,u]\Big|_{s=j\omega}d\omega$$
$$= \int_{-\infty}^{\infty} v_1^*\,W_{CL}^*[Q+K^*RK]W_{CL}v_1\Big|_{s=j\omega}d\omega$$
$$+ \int_{-\infty}^{\infty}v_1^*(RK\,W_{CL}v_2]\Big|_{s=j\omega}d\omega + \int_{-\infty}^{\infty}[RK\,W_{CL}v_2]^*v_1\Big|_{s=j\omega}d\omega$$
$$+ \int_{-\infty}^{\infty} v_2^*\,R\,v_2\Big|_{s=j\omega}d\omega$$

$$\int_{-\infty}^{+\infty}v_1^*\,[RK\,W_{CL}v_2]\Big|_{s=j\omega}d\omega = 0$$

$$= \int_{-\infty}^{+\infty}[RK\,W_{CL}v_2]^*v_1\Big|_{s=j\omega}d\omega$$

This follows from Parseval's theorem and noting that $\mathbf{L}^{-1}[R^{1/2}v_1] = v_1'(t)$ is zero for $t > 0$, while

$$R^{1/2}K\,W_{CL}v_2 = R^{1/2}KW(I+KW)^{-1}v_2 = R^{1/2}[I-(I+KW)^{-1}]v_2$$

is zero for $t < 0$. (Note that the stability of $R^{1/2}$ and $(I+KW)^{-1}$ is relevant here.) Hence

$$J = \int_{-\infty}^{\infty} v_1^*\,W_{CL}^*[Q+K^*R\,K]W_{CL}v_1\Big|_{s=j\omega}d\omega$$
$$+ \int_{-\infty}^{\infty} v_2^*\,R\,v_2\Big|_{s=j\omega}d\omega$$

Now only v_2 is at our disposal to minimize J in this expression. Clearly, the minimization is accomplished by setting

$$v_2 \equiv 0$$

which means that the optimal u(s) is given by

$$u(s) = -K(s)\,y(s)$$

In terms of the + and - subscript notation introduced in Section 2,

$$u_+(s) = -K(s)\, y_+(s) - [K(s)\, y_-(s)]_+$$

The second term in this expression is due to the initial conditions on the (now dynamic) feedback control Law. Note that if K is constant, $[K\, y_-(s)]_+$ is zero.

Proof of Lemma 3.1

Suppose first that W(s) has full column rank. Equation (2.5a) implies that W and so $WR^{-1/2}$ is strictly minimum phase, and so $\bar{B}$ has constant rank in Re[s] > 0.. Hence $\bar{K}$ has no poles in Re[s] > 0. In forming the product $\bar{K}\bar{W}$, the poles of $\bar{K}$ are cancelled, and these are all in Re[s] < 0. No poles of $\bar{W}$ are cancelled unless $\bar{C}\bar{A}^{-1}$ has a cancellation, i.e. $\bar{C},\bar{A}$ have a right common divisor. By (3.4) such a divisor would necessarily have a Hurwitz determinant, and therefore there can be no unstable pole-zero cancellation in forming the product $\bar{K}\bar{W}$.

When W does not have full rank, a simple modification of the above argument shows that, as before, in forming $\bar{K}\bar{W}$, there is no unstable pole - zero cancellation. Next det $(I+\bar{K}\bar{W})$ has a Hurwitz numerator and $\bar{K}\bar{W}$ is strictly proper. This is because det $(\bar{A}+\bar{C})$ is Hurwitz, which ensures that the numerator of $I+\bar{K}\bar{W}$ has full rank in the Re(s) > 0 or that $(I+\bar{K}\bar{W})^{-1}$ is free of poles in Re[s] > 0. Also, $\bar{K}\bar{W} = \bar{C}\bar{A}^{-1}$, which is strictly proper by (3.4).

Now let us relate these results back to K and W. First, (3.2) implies (2.16a). Next, since

$$I+\bar{K}\bar{W} = I + R^{1/2} KW\, R^{-1/2} = R^{1/2}(I+KW)R^{-1/2}$$

we have

$$(I+KW)^{-1} = R^{1/2}(I+\bar{K}\bar{W})^{-1} R^{-1/2}$$

and so, [using the properties of $R^{1/2}(s)$ and $R^{-1/2}(s)$] $(I+KW)^{-1}$ has all poles in Re[s] < 0. Third, because $KW = R^{-1/2}\bar{K}\bar{W}R^{1/2}$ and $R^{1/2}(\infty)$ is finite and nonsingular since R has this property, KW has no unstable pole zero cancellation and is strictly proper because $\bar{K}\bar{W}$ has this property. Finally, because $[I+KW]^{-1}$ is stable and KW has no unstable pole-zero cancellations, $W_{CL} = W[I+KW]^{-1}$ is stable, i.e. has all its poles in Re(s) < 0.

Proof of Lemma 3.2

Z(s) is asymptotically stable from the results of the last section. Moreover for all $s = j\omega$, (3.2) implies trivially that $[I+\bar{W}^*_{OL}(s)][I+\bar{W}_{OL}(s)] > I$. This inequality can be reorganized as

$$[I+\bar{W}^*_{OL}(s)][Z(s) + Z^*(s)][I+\bar{W}_{OL}(s)] > 0 \text{ for } s=j\omega$$

from which it is clear that $Z(s) + Z^*(s) > 0$ for all $s=j\omega$. Hence Z(s) is positive real.

The zeros of $\bar{W}_{OL}(s)$ are zeros of Z(s). Because Z(s) is positive real, its $j\omega$-axis zeros must be simple and all other zeros must be in Re[s]< 0. (To see this, we note that if Z(s) is nonsingular almost everywhere, $Z^{-1}(s)$ exists and is positive real. The poles of $Z^{-1}(s)$ which are the zeros of Z(s), are all in Re[s] < 0 or are simple on the $j\omega$-axis. If Z(s) is not nonsingular almost everywhere, there exists a full row rank constant T such that $Z(s) = T'Z_1(s)T$ with Z_1 positive real and nonsingular almost everywhere. Then the earlier argument can be applied.)

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Preliminary result for proof of Lemma 4.1

Lemma A.1 Let Y,Z be polynomial matrices such that det Y is not identically zero and has all zeros in Re[s] < 0, and $Z = Z^*$. Then there exists a polynomial X with

$$X^*Y + Y^*X = Z$$

Proof Write, using a partial fraction expansion,

$$Y^{-*}ZY^{-1} = V_+ + V_- + U$$

where U is polynomial, V_+ is strictly proper with poles in Re[s] < 0 and V_- is strictly proper with poles in Re[s] > 0. Obviously $V_+ = V_-^*$ and $U = U^*$. Now

$$V_+Y = Y^{-*}Z - V_-Y - UY$$

and since the left side has poles only in Re[s] < 0, the right side has poles only in Re[s] > 0, both sides are polynomial. Let $V_+Y = N$ i.e. $V_+ = NY^{-1}$ for some polynomial N. Then

$$Y^{-*}ZY^{-1} = Y^{-*}N^* + NY^{-1} + U$$

$$Z = N^*Y + Y^*N + Y^*UY$$

$$= (N^* + \tfrac{1}{2}Y^*U)Y + Y^*(N^* + \tfrac{1}{2}UY)$$

Remark With Z=I, the above result appears in [14].

Proof of Lemma 4.1

The fact that (I+KW) is strictly minimum phase while KW has no unstable pole-zero cancellations means that the same is true of $(I+\bar{K}\bar{W})$ and $\bar{K}\bar{W}$, and so $\bar{A}(s)\bar{K}_D(s) + \bar{B}(s)\bar{K}_N(s)$ has all determinantal zeros in Re[s] < 0. This guarantees by Lemma A.1 that we can find a polynomial G(s) satisfying (4.3) and thus a P(s) satisfying (4.1).

Proof of Lemma 4.2

It is easily verified that if X is a rational matrix for which $X + X^* = 0$, then $G - X(\bar{A}\bar{K}_D+\bar{B}\bar{K}_N)$ is a (not necessarily polynomial) solution of (4.3) if and only if G is a solution of (4.3). Hence $\bar{P} - X(\bar{A} + \bar{B}\bar{K})$ is a rational solution of (4.1) if and only if $\bar{P}$ is a rational solution. In light of (4.5c), to show that $\bar{P}_1$ is a solution of (4.1), we can show that $M_2 + M_2^* = 0$.

Now (4.3) implies that

$$G[\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1} + [\bar{K}_D^*\bar{A}^* + \bar{K}_N^*\bar{B}^*]^{-1}G^*$$

$$= [\bar{K}_D^*\bar{A}^* + \bar{K}_N^*\bar{B}^*]^{-1}\bar{K}_N^*\bar{K}_N[\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1}$$

$$+ [\bar{K}_D^*\bar{A}^* + \bar{K}_N^*\bar{B}^*]^{-1}N^*N[\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1}$$

Now it is not hard to verify that $\bar{K}_N[\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1}\bar{B}$ and $N[\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1}\bar{B}$ must be strictly proper. Hence

$$[\bar{B}_1^* \;\; 0]\, G\, [\bar{A}\bar{K}_D + \bar{B}\bar{K}_N]^{-1}\begin{bmatrix}\bar{B}_1\\0\end{bmatrix}$$

$$+ [\bar{B}_1^* \;\; 0][\bar{K}_D^*\bar{A}^* + \bar{K}_N^*\bar{B}^*]^{-1}G^*\begin{bmatrix}\bar{B}_1\\0\end{bmatrix}$$

$$= M_1^*(s) + M_1(s)$$

is strictly proper. Hence if $Y(s) = M_1(s) -$

strictly proper part of $M_1(s)$, then $Y^*(s) + Y(s) = 0$. Using (4.5b), we see then that $M_2^*(s)+M_2(s) = 0$, and so $\bar{P}_1$ satisfies (4.1). Next, observe that

$$\bar{B}^*\bar{P}_1\bar{W}_{CL}(s) = \bar{B}\ \bar{P}(\bar{A}+\bar{B}\bar{K})^{-1}\bar{B} - \bar{B}^*M_2\bar{B}$$

$$= M_1 - [M_1 - \text{strictly proper part of } M_1]$$

$$= \text{strictly proper part of } M_1$$

Further, the poles of $\bar{B}^*\ \bar{P}_1\ \bar{W}_{CL}(s)$ are evidently the poles of the strictly proper part of M_1, which are, by (4.5a) a subset of the poles of the strictly proper part of $\bar{P}\ (\bar{A} + \bar{B}\bar{K}) = G\ (\bar{A}K_D + \bar{B}K_N)^{-1}$. These lie in $Re[s] < 0$ as already observed.

Next, the strictly proper part of M_1 is unique since any two solutions of (4.1) representable as $\bar{G}K_D^{-1}$ and $\tilde{G}K_D^{-1}$ with $\bar{G}$ and $\tilde{G}$ and satisfying (4.3) must differ by $X(\bar{A} + \bar{B}\bar{K})$ for a polynomial X with $X + X^* = 0$. Thus any two M_1 differ by $[\bar{B}_1^*\ 0]\ X\ [\bar{B}_1\ \ 0]'$ which is polynomial. Hence $\bar{B}^*\ \bar{P}_1\ \bar{W}_{CL}$ is unique.

Proof of Lemma 4.3

Pre- and Post-multiplication of (4.1) by $\bar{W}^*(s)$, $\bar{W}(s)$ respectively, application of the spectral factorization (3.2) written as

$$\bar{W}^*N^*N\bar{W} = \bar{W}^*\bar{K}^*\bar{K}\bar{W} + \bar{W}^*\bar{K}^* + \bar{K}\bar{W}$$ and substitution of the relationship $\bar{A}^{-1}\bar{B} = \bar{W}$ yields

$$\bar{W}^*\ (\bar{P}^*\bar{B}-\bar{K}^*)\ (I+\bar{K}\bar{W}) + (I+\bar{W}^*\bar{K}^*)\ (\bar{B}^*\bar{P}-\bar{K})\bar{W} = 0$$

Pre-multiplication by $(I+\bar{W}^*\bar{K}^*)^{-1}$ and post-multiplication by $(I+\bar{K}\bar{W})^{-1}$ then yields

$$\bar{W}^*_{CL}(s)[\bar{P}^*(s)\bar{B}(s)-\bar{K}^*(s)]$$

$$+ [\bar{B}^*(s)\bar{P}(s)-\bar{K}(s)]\bar{W}_{CL}(s) = 0$$

Regard this equation as expressing

$$Z^* + Z = 0 \text{ where } Z = \bar{B}^*\bar{P}\bar{W}_{CL} - \bar{K}\bar{W}_{CL}$$

By the Lemma hypothesis and Lemma 4.2, $\bar{B}^*\bar{P}\bar{W}_{CL}$ is strictly proper and has all poles in $Re[s] < 0$, while $\bar{K}\bar{W}_{CL}$ has the same properties by the results of Section 2. Hence $Z = 0$, whence (4.6) holds.

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