Introduction to Stochastic Control Theory—Karl Åström (New York: Academic, 1970, 299 pp., $14.50). *Reviewed by B. D. O. Anderson.*

Brian D. O. Anderson received his undergraduate education at the University of Sydney, Sydney, N.S.W., Australia, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, Calif. He is currently Professor of Electrical Engineering at the University of Newcastle, Newcastle, N.S.W., Australia. His research interests include control, communications, and network theory. He has coauthored Linear Optimal Control *(Prentice-Hall, 1971) and* Network Analysis and Synthesis *(Prentice-Hall, 1972).*

Stochastic control theory means different things to different people. To K. J. Åström, it means, at least for his book, a heavy dose of stochastic processes, particularly Gaussian processes associated with linear systems, and a study of stochastic linear regulator problems, including the separation theorem. By and large, there is a good deal more on stochastic processes than there is on control.

The presentation of the material on stochastic processes is highly successful. To begin with, the author has not been overly ambitious. Many of the readers of his text will not be familiar with measure theory. He acts on this assumption and does not attempt to provide them with a one-chapter instant measure-theory course. Again, it proves necessary to introduce and use the concept of convergence of a sequence of random variables. The author very sensibly confines nearly all his attention to mean-square convergence, highly efficient in the study of second-order processes and relatable to continuity properties of covariances.

Although the topic is largely outside the mainstream of the book, the author offers material on nonlinear stochastic differential equations. By introducing the notions of forward and backward differences, he is able to neatly indicate the relation between the Ito and Stratonovich integrals, and all this with an impressive ability to sense when a full proof would overwhelm, or when a heuristic proof would help in understanding. One application of these ideas to the linear Gaussian problem could well have been made, but was not; i.e., the problem of establishing a differential equation for the evolution of the state covariance $E[x(t)x'(t)]$ of a linear system $\dot{x} = F(t)x + G(t)u$, with $u(\cdot)$ a white-noise process. One application that is made, on the other hand, is to the problem of obtaining a partial differential equation for a loss function $V(x,t)$ of the form

$$V(x,t) = E\left[\int_t^{t_1} G(x(s),s)\, ds/x(t) = x\right]. \tag{1}$$

As far as I could see, the solution of this problem is used nowhere else in the book.

Associated with the discussion of stochastic differential equations is a good discussion of continuous-time white noise. This is well motivated by the author when he points out first that, by analogy with the more straightforward discrete-time problems, one needs to introduce a random process $v(\cdot)$ with $v(t)$ and $v(s)$ uncorrelated for all $t \neq s$. He then proves that no such process, other than the zero process, can exist that has a continuous finite variance $E[v^2(t)]$. This paves the way for postulating infinite-variance white noise. Having made clear the mathematical unpleasantness of white noise, the author takes pains to discuss physical processes that approximate white noise. Here numerical data are provided to give the reader a feeling for the approximations involved.

Once white noise is introduced, second-order Gaussian processes are covered in some detail, with emphasis on the analysis of linear systems driven by white noise. The author next introduces the frequency domain description of stationary processes and the concept of spectral factorization. In all this, there is a preference for discrete-time processes, with the theory for such processes usually preceding that for continuous-time processes. Although this probably does not reflect past emphasis, at least in electrical engineering, it offers pedagogical advantages. Discrete-time white noise is nowhere near the unpleasant animal that continuous-time white noise is.

The next topic covered in the book is termed "parametric optimization." I found the title mildly misleading. The first portion of material discussed under this heading deals with the evaluation of integrals of the form

$$\oint \frac{B(z)B(z^{-1})\, dz}{A(z)A(z^{-1})z} \tag{2}$$

and

$$\int_{-\infty}^{+\infty} \frac{B(s)B(-s)}{A(s)A(-s)}\, ds. \tag{3}$$

To those familiar with [1], the need to evaluate these integrals to solve classes of parametric optimization problems will be apparent. But Åström's text, on the other hand, offers very little motivation and no example of substance. I could not find a specific problem or example in the text in which one of these integrals represented the performance of a system, depended on a parameter in the system, and had to be minimized by selection of the parameter. On the other hand, the methods offered by Åström for evaluating the integrals are not well known, appear to be efficient, and their appearance in book form is welcome.

The remaining portion of the parametric optimization material deals with the Kalman–Bucy filtering problem. Here, the author assumes a filter with the Kalman–Bucy structure and attempts to optimize the gain. In this context, he is attempting a suboptimal and parametric optimization problem because at this point he has not established that the assumed structure is actually optimal. The plant considered is

$$x(t+1) = \Phi x(t) + \Gamma u(t) + v(t) \qquad y(t) = \Theta x(t) + e(t) \tag{4}$$

where $v(\cdot)$ and $e(\cdot)$ are uncorrelated zero-mean white-noise processes. His strategy is to assume an estimator of the form

$$\hat{x}(t+1) = \Phi\hat{x}(t) + \Gamma u(t) + K[y(t) - \Theta\hat{x}(t)] \tag{5}$$

and to choose $K(t_0), K(t_0+1), \cdots, K(t-1)$ to minimize $E\{a'[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]'a\}$ for any a. I feel that the omission of one detail in the solution of this problem is serious. Let t_0 be the initial time for (4). The author states that one chooses $K(t_0)$ to minimize the error variance at time t_0+1, then chooses $K(t_0+1)$ to minimize the error variance at time t_0+2, and so on. This overlooks the fact that the error variance at time t_0+2 depends on $K(t_0)$, and that the choice of $K(t_0)$ to minimize the error variance at time t_0+1 could drive up the error variance at time t_0+2. It does not, of course; but this needs to be pointed out.

The next topic comprises a lengthy discussion of a highly specific optimal control problem. The open-loop system considered consists of a stationary discrete-time single-input single-output plant with additive output colored noise and no input noise. The aim is to keep the mean-square plant output as small as possible by feedback control to the plant input. While the solution of this problem demonstrates the use of discrete-time spectral factorization, it offers me little else of pedagogical value. It would seem better in fact to have included material on Wiener filtering, perhaps developing the optimal control problem as an example. In my opinion, the absence of a chapter on Wiener filtering is a deficiency of the book, given the other material that the book contains.

The remainder of the book is concerned with the Kalman–Bucy filter in discrete and continuous time and a proof of the separation theorem. The continuous-time filter problem is solved by converting the problem to a dual linear-quadratic optimal control problem, and the continuous-time separation theorem is proved with the aid of an identity of the form

$$\int_{t_0}^{\infty} (u'Ru + x'Qx)\, dt = x'Px|_{t_0} + \int_{t_0}^{\infty} (u - u^*)'R(u - u^*)\, dt$$

where u^* is defined using a Riccati equation solution. This allows a very efficient and clear proof of the separation theorem. The discrete-time separation theorem also is presented most clearly.

One can criticize the absence of discussion of the limiting behavior, including stability, of the Kalman–Bucy filter. Nowhere do the concepts of controllability and observability seem to be introduced into the book, which leads to the unfortunate exclusion of a number of interesting ideas. Smoothing problems constitute another omission. Certainly though, many would not feel this omission is serious.

Having completed reading of the book, one is left with the impression that it will teach a student far more about stochastic processes than it will about control, for there is remarkably little pure control therein. The book of course remains valuable. The reviewer found very few technical errors in the book. The examples are extensive and well chosen, and students will find the annotations in the chapter bibliographies helpful. Any serious graduate student in the information and control science area would do well to read the book.

REFERENCE

[1] G. C. Newton, L. A. Gould, and J. F. Kaiser, *Analytical Design of Linear Feedback Controls*. New York: Wiley, 1957.

Optimal Control Theory: An Introduction—Donald E. Kirk (Englewood Cliffs, N. J.: Prentice-Hall, 1970, 452 pp. $13.50). *Reviewed by William S. Levine.*

William S. Levine received the S.B., S.M., and Ph.D. degrees, from the Massachusetts Institute of Technology, Cambridge, in 1962, 1965, and 1969, respectively. He is presently at the University of Maryland, College Park, where he is Assistant Professor of Electrical Engineering. He teaches graduate courses in system theory and in the theory of optimal control. His current research interests are primarily in the theory of optimal control and its applications.

The author of this book, Donald E. Kirk, has attempted to write an introduction to optimal control theory "at a level appropriate for a first- or second-year graduate course, an undergraduate honors course, or for directed self-study." Since the appropriate level for a textbook depends so strongly on the reader and his purpose, I will attempt to accurately describe the level of the book and then suggest what type of student would gain most from it.

The prerequisites for an understanding of the material in this book are quite elementary. All that is really necessary is a standard background in the calculus and familiarity with the state-space notation. In fact, knowledge of the concepts of controllability and observability is not a prerequisite. Thus, virtually any undergraduate course in linear system theory that included the state-space point of view would provide sufficient background.

The book develops the theory of optimal control in a novel order. After a pair of introductory chapters dealing with the formulation of optimal control problems, dynamic programming is introduced in Chapter 3. The discrete version of dynamic programming is introduced first and is ultimately used to find the solution of the discrete linear regulator problem. Then the standard derivation of the necessity of the Hamilton–Jacobi–Bellman equation is given. This result is applied to the continuous version of the linear regulator.

Chapter 4 presents an introduction to the calculus of variations. The basic definitions, functional, linear functional, norm, and variation, are presented very clearly and in close analogy to the related concepts of the calculus. The only results presented, however, are the Euler–Lagrange equation, the boundary conditions, the Weierstrass–Erdmann corner conditions, and the Lagrange multiplier for constrained problems.

The next and, in my opinion, weakest chapter presents the minimum principle and several of its standard applications to minimum time and minimum fuel problems. In addition, there is a brief discussion of singular problems. The major weakness of the chapter, which will be discussed in a moment, is in the statement and proof of the minimum principle.

Chapter 6 discusses a number of iterative techniques for computing optimal controls and trajectories. The algorithms discussed include steepest descent, variation of extremals, quasilinearization, and gradient projection. The book closes with a summary chapter.

The book has a number of considerable strengths. The introduction of dynamic programming at the very beginning gives the student a result that is easily understood and that lets him solve interesting problems very early. This should provide enough motivation to get him through the relatively unmotivated preliminary results in the calculus of variations. In addition, the book is easy to read and understand. A notable example of this is the discussion of the variation of extremals technique in Chapter 6. The introduction of the method is accomplished by means of a first-order example. This enables the author to illustrate the calculation of an improved initial costate on a (hypothetical) plot of the final costate versus the initial costate. The other major aspect of the method, the need to compute an approximation to the slope of the illustrated curve, is clearly stated. The net effect is a very clear explanation of the method and great ease in understanding the multidimensional version described next.

On the other hand, there are some factors that limit the usefulness of the book. These are well illustrated by the handling of the statement and proof of the minimum principle. The class of admissible controls is only specified by either boundedness or unboundedness. No mention is made of the piecewise continuity (measurability) of the admissible controls. This vagueness leads to confusion about the progressive improvement of the theoretical results. It also leads to confusion in more advanced problems. Strictly speaking, the minimum time control of the double integrator plant, discussed in the text, does not have a solution if one makes the reasonable assumption (in the absence of any comment by the author) that admissible controls are continuous. For the student who will study more advanced topics, such as differential games, stochastic and distributed-parameter optimal control, more precise statements of the hypotheses of the theorems and more rigorous proofs would greatly increase the value of this book.

In summary, if I were asked to recommend a graduate text for students training to do research in optimal control, I would suggest a more detailed, more precise, and a deeper text. On the other hand, if I were asked to recommend a text for an undergraduate course, for graduate students not specializing in research in control theory or for an industrial engineer who would like an introduction to the basic results of optimal control theory, I would recommend this book.

Stochastic Optimal Linear Estimation and Control—J. S. Meditch (New York: McGraw-Hill, 1969, 394 pp., $15.00). *Reviewed by H. W. Sorenson.*

H. W. Sorenson received the Ph.D. degree from the University of California, Los Angeles, in 1966. He has been with General Dynamics/Astronautics, San Diego, Calif., AC Electronics, El Segundo, Calif., and the Institut für Steuer-und Regeltechnik, Munich, Germany. In 1968 he joined the Faculty of the University of California, San Diego, and is currently Associate Professor of Applied Mechanics and Mechanical Engineering Sciences.