ROBUST STABILITY OF POLYNOMIALS WITH MULTILINEAR PARAMETER DEPENDENCE

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ABSTRACT

The problem is studied of testing for stability a class of real polynomials in which the coefficients depend on a number of variable parameters in a multilinear way. We show that the testing for real unstable roots can be achieved by examining the stability of a finite number of corner polynomials (obtained by setting parameters at their extreme values), while checking for unstable complex roots normally involves examining the real solutions of up to m+1 simultaneous polynomial equations, where m is the number of parameters. When m=2, this is an especially simple task.

I. INTRODUCTION

This paper is concerned with a robust stability problem. More specifically, we consider monic n-th degree polynomials \( f(s; \gamma_1, \ldots, \gamma_m) \) with real coefficients which depend in a multilinear fashion on the quantities \( \gamma_i \). The parameters \( \gamma_i \) are contained in intervals \([\gamma_i, \bar{\gamma}_i]\), and we seek a test for the stability of all \( f(s) \), where by the term stability, we mean that \( f(s) \) has all its roots in a prescribed region, e.g. \( \text{Re}[s] < 0, |s| < 1 \), etc. For the most part in this paper, we focus on the case \( \text{Re}[s] < 0 \); the ideas however with little variation will carry over to most other regions of interest. Stability inside the unit circle is easily covered for example by bilinear transformation.

To illustrate the occurrence of such problems we note that many physical systems described by linear differential equations in which parameters such as friction constants, mass, capacitance, etc. vary have associated transfer functions in which these variable parameters appear multilinearly in both numerator and denominator. Also, when a controller defined by a rational transfer function is connected, the characteristic polynomial of the closed-loop system is (apart from limited exceptions) necessarily multilinear in the parameters of the plant and controller transfer functions, see e.g. Section 9.17 of Zadeh and Desoer (1963), and Dasgupta and Anderson (1987).
In the following, two examples are given:

**Example 1:**
Let us consider the electrical circuit depicted in Figure 1:

![Fig. 1. Electrical circuit](image)

The transfer function $G(s)$ from $u_0$ to $u_c$ is given by

$$G(s) = \frac{1}{s^2LC + sRC + 1}$$

The parameters of the characteristic polynomial depend bilinearly on the physical parameters $R, L, C$ which can vary slowly, for example because of temperature variations or ageing.

**Example 2:**
Assume that $G_1(s) = B_1(s)/A_1(s)$ and $G_2(s) = B_2(s)/A_2(s)$ in the SISO control system of Figure 2, and that certain coefficients of $A_i, B_i, i = 1,2,$ depend linearly on some parameters, different for each of the four polynomials.

![Fig. 2. SISO control system](image)

Then the characteristic polynomial

$$N(s) = A_1(s)A_2(s) + B_1(s)B_2(s)$$

depends bilinearly on the coefficients of $A_1$ and $A_2$, and $B_1$ and $B_2$, respectively, and in turn bilinearly on the underlying parameters. This situation is significant in practical applications, because we often need to build control systems from different parts. The parameters of the parts can differ from the nominal values. But the stability of the closed loop should be preserved for all such parts.

Before proceeding further, we must define some terms. We work in three different spaces, which are parameter space, coefficient space and root space. The parameters $\gamma_i$ are contained in closed intervals $[\gamma_i, \bar{\gamma}_i]$. The endpoints of such intervals are denoted by $\{\gamma_i, \bar{\gamma}_i\}$. Open parameter intervals are given by $(\gamma_i, \bar{\gamma}_i)$. Corner points and corner polyno-
mials in parameter and coefficient space are defined by taking \( \gamma \in [\gamma_0, \gamma_1] \). Edges in parameter and coefficient space and edge polynomials are defined by taking \( \gamma \in [\gamma_0, \gamma_1] \) for all but one value of \( i \), say \( i_1 \), and \( \gamma_{i_1} \in [\gamma_{i_1, 1}, \gamma_{i_1, 2}] \). Notice that edges in both parameter and coefficient space are straight lines. Faces in parameter and coefficient space and face polynomials are defined by taking \( \gamma \in [\gamma_0, \gamma_1] \) for all but two values of \( i \), say \( i_1 \) and \( i_2 \), and \( \gamma_{i_1} \in [\gamma_{i_1, 1}, \gamma_{i_1, 2}] \) and \( \gamma_{i_2} \in [\gamma_{i_2, 1}, \gamma_{i_2, 2}] \).

In parameter space, faces are flat while in coefficient space, faces are two-dimensional curved surfaces, but in general not planar. Coefficient space faces are however ruled surfaces, i.e., through every point on the face there passes in general two straight lines of the surface defined by \( \gamma_{i_1} = \text{constant} \) and \( \gamma_{i_2} = \text{constant} \).

In a search for necessary and sufficient conditions for stability, the general aim is naturally to avoid testing at all possible values of the parameters, i.e., one wants theorems which establish stability for all values given that stability holds for some restricted set of values. A Kharitonov-like theorem (Kharitonov (1979)) would be one which requires testing at just corner points, i.e., \( \gamma \in \{\gamma_0, \gamma_1\} \). However, it is quickly seen that such a result is extremely unlikely; Kharitonov's theorem is valid for a region in coefficient space bounded by hyperplanes parallel to the coordinate axes, and only then for stability in the region \( \Re(s) < 0 \) (counterexamples exist for the region \( |s| < 1 \), see Hollot and Bartlett (1986)).

The next possibility is to examine stability at the corners and along the edges. Such an idea is suggested by the work of Bartlett, Hollot and Lin (1988); these authors show that if the coefficients of a polynomials depend in an affine way on a collection of parameters, each of which lies in an interval, so that in coefficient space the collection of polynomials under test is a polytope, then it suffices to check the edges for stability. More precisely, the authors prove the following:

(a) if \( s_0 \) is a real root of any polynomial in the set under test, it is a root of at least one edge polynomial;
(b) if \( s_0 \) is a complex root of any polynomial in the set under test, it is a root of at least one face polynomial;
(c) if \( s_0 \) is at the boundary of the set of roots of all face polynomials, then it is also a root of at least one edge polynomial;
(d) if \( D \) is a simply connected domain, then the roots of all polynomials lie in \( D \) if and only if the roots of all polynomials defined by all edges lie in \( D \). This is a consequence of (a), (b) and (c).

When we seek to carry over these ideas to our problem, where the coefficients depend multilinearly on the parameters, it turns out that only (a) remains valid. The following counterexample to (b) was supplied to us by C.V. Hollot. The polynomial

\[
f(s) = s^5 + (\gamma_1)s^4 + (\gamma_1 - \gamma_2 + 1)s^3 + \gamma_3 s^2 + (\gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3 + \gamma_1 \gamma_2 + \gamma_2 \gamma_3 + \gamma_1 \gamma_3) + (\gamma_3 + \gamma_2 \gamma_3)\]

with \( |\gamma_1| \leq 1 \) has the property that \( \pm j \) is a root when \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \). It is not, however, a root of any face polynomial.

What other approaches exist? In de Gasteron and Safonov (1988), appeal is made to the fact that the set of all Nyquist diagrams of all polynomials in the set has a key property. For each \( \omega \), \( f(\omega; \gamma_1, \ldots, \gamma_n) \) lies in the convex hull of the \( 2^n \) complex points obtained by setting the \( \gamma_i \) to their extreme value, a property pointed out in Zadeh and Desco (1963) with the name Mapping Theorem. This idea is exploited to tackle the robust stability problem with a type of extension of Nyquist's theorem.

These ideas have something in common with those of Saecli (1986), which consider a roughly equivalent problem, but one in which the \( \gamma_i \), in effect, are allowed to be complex. It turns out that in many ways, this simplifies the problem. Yet another possibility is to make
special assumptions on the polynomials \( f(s) \), which have the aim of making the problem equivalent to, or very like, the problem considered in Bartlett, Hollot and Lin (1988). For example, Panier, Fan and Tits (1987) postulate uncoupled perturbations in the coefficients of even and odd powers of \( f(s) \), while Djaferis and Hollot (1988) and Djaferis (1988) impose restrictions which ensure that the image for \( f(\gamma_1;\gamma_2;...;\gamma_m) \) for each \( \gamma \) and variable \( \gamma \) is a polytopic set. This allows again an extension of Nyquist's theorem to be applied. The difficulty with this type of result is that it is highly non-generic.

Rather than working up from results such as Kharitonov's theorem and the edge theorem, another approach is to work down from the very general Tarski-Seidenberg decision algebra theorem described in textbooks such as Bose (1982) and Jacobson (1964). This theorem implies that the robust stability problem we have posed can always be solved using a finite number of rational calculations (in the sense that for a given polynomial dependent on \( \gamma_1,\gamma_2;...;\gamma_m \) a yes/no answer to the robust stability question can be obtained). The number of calculations may be prohibitive, and the real interest then lies in finding shortcuts so that the number of calculations becomes acceptable.

A variant on the Tarski-Seidenberg theorem was suggested in Anderson and Scott (1977), who showed that an alternative approach for any decision algebra problem could be found which involved the construction and solutions of \( q \) polynomials equations in \( q \) unknowns, \( q \) being an integer determined by the problem statement. When this procedure is followed, much of the interest lies in ensuring that \( q \) is as small as possible. This actually will be the approach followed in two later sections of the paper, where we shall have \( q = m+1 \). Note that there exist systematic methods for solving such equations based on resultants, see Bose (1982). Also, software is increasingly becoming available, see (Watson et al., 1987).

When the \( \gamma \) correspond to physical parameters, in many cases the value of \( m \) will be quite small, say 2, 3 or 4. Under these circumstances, there is a good possibility that the computational burdens will not prove excessive.

The layout of the paper is as follows: In the next section, we shall first establish the following result:

**Theorem 2.1:** Let \( f(s; \gamma_1,...,\gamma_m) \) be an \( n \)-th degree monic polynomial, with real coefficient dependent multilinearly on the \( \gamma \) where \( \gamma \) is contained in an interval \( [\gamma_i, \bar{\gamma}_i] \), \( i = 1,...,m \). Let \( \gamma_0 \) be a real root of some such polynomial. Then \( \gamma_0 \) is also a real root of an edge polynomial.

**Proof:** The proof of this result is by induction. Let \( \gamma_0 \) be a real root of the polynomial \( f \) for some given \( \gamma_i \in [\gamma_i, \bar{\gamma}_i], i = 1,...,m \). Suppose that for \( \gamma_0, f(s; \gamma_1,...,\gamma_r, \gamma_{r+1},...\gamma_m) = 0 \) for \( \gamma_r \in (\gamma_r, \bar{\gamma}_r),...\gamma_m \in (\gamma_m, \bar{\gamma}_m) \) and to avoid trivial cases \( \gamma_{r+1} \in (\gamma_{r+1}, \bar{\gamma}_{r+1}),...\gamma_m \in (\gamma_m, \bar{\gamma}_m) \), for some \( r < m-1 \).

2. SIGNIFICANCE OF THE EDGES FOR REAL ROOTS

As mentioned before, we shall first establish the following result:

**Theorem 2.1:** Let \( f(s; \gamma_1,...,\gamma_m) \) be an \( n \)-th degree monic polynomial, with real coefficient dependent multilinearly on the \( \gamma \) where \( \gamma \) is contained in an interval \( [\gamma_i, \bar{\gamma}_i] \), \( i = 1,...,m \). Let \( \gamma_0 \) be a real root of some such polynomial. Then \( \gamma_0 \) is also a real root of an edge polynomial.

**Proof:** The proof of this result is by induction. Let \( \gamma_0 \) be a real root of the polynomial \( f \) for some given \( \gamma_i \in [\gamma_i, \bar{\gamma}_i], i = 1,...,m \). Suppose that for \( \gamma_0, f(s; \gamma_1,...,\gamma_r, \gamma_{r+1},...\gamma_m) = 0 \) for \( \gamma_r \in (\gamma_r, \bar{\gamma}_r),...\gamma_m \in (\gamma_m, \bar{\gamma}_m) \) and to avoid trivial cases \( \gamma_{r+1} \in (\gamma_{r+1}, \bar{\gamma}_{r+1}),...\gamma_m \in (\gamma_m, \bar{\gamma}_m) \), for some \( r < m-1 \).
We shall show that we can adjust either \( y_{r+1} \) to \( y_{r+1} \in \{ y_{r+1}, \bar{y}_{r+1} \} \) with \( f(s_0, \gamma_1, \ldots, \gamma_r, \gamma_{r+1}, y_{r+1}, \bar{y}_{r+1}, \bar{y}_{r+2}, \ldots, y_m) = 0 \), or \( y_{r+2} \) to \( y_{r+2} \in \{ y_{r+2}, \bar{y}_{r+2} \} \) with \( f(s_0, \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m) = 0 \).

To verify this claim, set \( \delta_{r+1} = y_{r+1} - \bar{y}_{r+1}, \delta_{r+2} = y_{r+2} - \bar{y}_{r+2} \); then we may write

\[
\begin{align*}
&f(s_0; \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m) \\
&= f(s_0; \gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m) \\
& \quad + \delta_{r+1} g_1(s_0, \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m) \\
& \quad + \delta_{r+2} g_2(s_0, \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m)
\end{align*}
\]

or in abbreviated notation

\[
f(s) = g_0(s) + \delta_{r+1} g_1(s) + \delta_{r+2} g_2(s) + \delta_{r+1} \delta_{r+2} g_3(s)\]

The \( g_i(s) \) are multilinear in the parameters on which they depend. Also, \( g_0(s_0) = 0 \). Now if \( g_1(s_0) = 0 \), set \( \delta_{r+2} = 0 \) and choose \( \delta_{r+1} \) to correspond to an extreme value for \( \delta_{r+1} = \bar{y}_{r+1} - \gamma_{r+1} \) is the upper boundary for \( \delta_{r+1} \) for example. Thus, \( \gamma_{r+1} \in \{ y_{r+1}, \bar{y}_{r+1} \} \), and also \( f(s_0; \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, \gamma_{r+2}, \bar{y}_{r+2}, \ldots, y_m) = 0 \). Similarly, if \( g_2(s_0) = 0 \), set \( \delta_{r+1} = 0 \) and \( \delta_{r+2} \) at an extreme value. If neither \( g_1(s_0) \) nor \( g_2(s_0) \) are zero, plot in the \( \delta_{r+1}, \delta_{r+2} \) plane the straight line or hyperbola defined by

\[
\begin{align*}
&\delta_{r+1} g_1(s_0) + \delta_{r+2} g_2(s_0) + \delta_{r+1} \delta_{r+2} g_3(s_0) = 0
\end{align*}
\]

(The straight line is encountered precisely when \( g_3(s_0) = 0 \).) This hyperbola necessarily intersects at least one of the four lines in the \( \delta_{r+1}, \delta_{r+2} \) plane which define the boundaries of the allowed parameter values. We choose one of the intersection points. At this intersection point, one of the associated \( y_{r+1}, y_{r+2} \) has an extreme value, say \( y_{r+2} = \bar{y}_{r+2} \). So we have obviously

\[
f(s_0; \gamma_1, \ldots, \gamma_r, y_{r+1}, \bar{y}_{r+1}, y_{r+2}, \bar{y}_{r+2}, \ldots, y_m) = 0
\]

and therefore the induction step \( r+1 \) is proved. This proves the theorem.

Obviously, the theorem states that the set of all real roots of all polynomials is given by the set of all real roots of all edge polynomials. If one is interested in knowing whether or not there are unstable real roots, it is actually unnecessary to examine all edge polynomials, and it suffices, as we now argue, to consider corner polynomials only. We are indebted to J. Ackermann for this derivation. Suppose all corner polynomials are stable. This means that \( f(s_0; \gamma_1, \ldots, \gamma_r) > 0 \) for all real non-negative \( s_0 \) and \( \gamma_i \in [\gamma_i, \bar{\gamma}_i] \) for all \( i \).

When \( s_0 \to \infty \) the monic character of \( f \) ensures that \( f(s_0) \to \infty \) and if \( f(s_0; \gamma_1, \ldots, \gamma_m) \leq 0 \) for some non-negative \( s_0 \), then by continuity, there would exist \( s_0 \geq 0 \) such that \( f(s_0; \gamma_1, \ldots, \gamma_m) = 0 \).

Now the inequality \( f(s_0; \gamma_1, \ldots, \gamma_m) > 0 \) ensures that \( f(s_0; \gamma_1, \ldots, \gamma_1, \gamma_1, \gamma_1, \ldots, \bar{\gamma}_m) > 0 \) for all \( \gamma_i \in [\gamma_i, \bar{\gamma}_i] \) since \( f \) is affine in \( \gamma_i \) alone, so that \( f(s_0; \gamma_1, \ldots, \gamma_1, \gamma_1, \gamma_1, \ldots, \bar{\gamma}_m) \) is a convex combination of the two values obtained by identifying \( \gamma_i \) with \( \gamma_i \) and \( \bar{\gamma}_i \).

It is highly probable that one or more edges does need to be tested for stability, to rule out the possibility of either real or complex roots.

Edge tests are the most straightforward; basically, root locus procedures can be used. Actually, it is only necessary to use rational calculation. Suppose that the polynomial \( f(s_0; \gamma_1, \ldots, \gamma_m) = g_0(s) + \gamma_1 g_1(s), \gamma_i \in [\gamma_i, \bar{\gamma}_i] \). Stability is achieved by requiring all Hurwitz determinants to be positive; as functions of \( \gamma_i \), these determinants are polynomials. So stability is equiva-
lent to certain polynomials in $\gamma_1$ being positive $\forall \gamma_1 \in [\gamma_1^-, \gamma_1^+]$. This can be checked by Sturm's theorem. Actually, two simplifications are possible. One can use the Lévi-Chipart form of stability conditions, and one only needs to check that all stability conditions are satisfied for one value of $\gamma_1$ and the $(n-1)$-th Hurwitz determinant is positive $\forall \gamma_1 \in [\gamma_1^-, \gamma_1^+]$. Alternatively, a result given in Ackermann and Barmish (1987) using Hurwitz matrices at the corners can be used to test the edges. Another method is given in Zeheb (1987), which requires evaluating the roots of a single polynomial.

3. SIGNIFICANCE OF THE FACES FOR COMPLEX ROOTS

In the previous section we have shown that if $s_0$ is a real root of any real polynomial, it is a real root of an edge polynomial. Now even when $f(s; \gamma_1, ..., \gamma_m)$ is linear in the $\gamma_i$, the same result, with real replaced by complex, is not true. Rather, any point on the boundary of the complex root set of all polynomials is necessarily a root of an edge polynomial (Bartlett, Hollott and Lin (1987)). It is thus natural to seek to extend this idea to the structures where $f(s)$ is multilinear in the $\gamma_i$.

In general such an extension is impossible.

Example: Consider the polynomial

$$f(s; \gamma_1, \gamma_2) = s^2 + (\gamma_1 - 1)s + (\gamma_1 \gamma_2 + 4) = s^2 + a_1(\gamma_1, \gamma_2)s + a_2(\gamma_1, \gamma_2)$$

with $\gamma_1 \in [-1, 1]$ and $\gamma_2 \in [-3, 2]$. In Figure 3, we have drawn the associated regions of parameter space and coefficient space. Points $A_1$ and $B_1$ correspond, the straight line $A_5A_6$ corresponds to the curve $B_5B_6$ (actually part of a parabola) and the two points $A_7, A_7'$ both correspond to $B_7$. Notice that each of $A_1A_2, A_2A_3, A_3A_4$ and $A_4A_1$ maps into a straight line, but these straight lines do not bound the image of the rectangle $A_1A_2A_3A_4$.

![Figure 3. Parameter space and coefficient space](image)

268
Now consider the point \( y_1 = y_2 = -1/2 \). This corresponds to a point on the curve \( B_2B_6 \), viz. \( a_1 = 1, a_2 = 17/4 \). There do not exist variations \( \Delta y_1, \Delta y_2 \) around \( y_1 = y_2 = -1/2 \) which allow perturbations \( \Delta a_1, \Delta a_2 \) in an arbitrary direction — moving "above" \( B_3B_6 \) is impossible. Consequently, since \( y_1 = y_2 = -1/2 \) corresponds to a point on the boundary in coefficient space, and we are working with second order polynomials, it also corresponds to a point on the boundary in root space. Obviously, the root is complex. It is easy to see that there is no edge polynomial with the same complex root pair, for there is no point on any one of the straight lines \( B_1B_2, B_2B_3, B_3B_4, B_4B_1 \) (which define all the edge polynomials) that corresponds to the polynomial \( s^2 + s + 17/4 \).

This example illustrates a further point, which is that the boundary in coefficient space need not correspond with the boundary in parameter space; of course, for this two-dimensional example, this is almost the same statement as that concerning the roots. But it is non-trivially different for higher degree polynomials.

In this example, the problem arises because within the region of parameter space of interest to us, the Jacobian determinant

\[
\frac{\partial (a_1, a_2)}{\partial (y_1, y_2)} = y_2 \cdot y_1
\]

can take zero values. Were this not the case, then the boundary of the parameter region would map into the boundary of the coefficient region. As we shall see below, the Jacobian determinant is of critical importance in a more general treatment.

We shall now explain how stability on faces can be checked. This is equivalent to checking stability when there are only two variable parameters. Without loss of generality, let these two parameters be \( y_1, y_2 \), and let us suppress mention of the other parameters, if any.

The idea is as follows. Suppose it has been established that all edges are stable. Suppose also that for some \( y_1, y_2 \) there exist unstable mts of \( f(s; y_1, y_2) \), then by continuity, there exists a value or values of \( y_1, y_2 \) for which \( f(s; y_1, y_2) \) has a purely imaginary root, and indeed a purely imaginary root on the boundary of the mt set. We shall show how such roots can be determined; if none exist, this means that \( f(s; y_1, y_2) \) has no roots in \( \text{Re}(s) \geq 0 \) over the entire face.

Let \( \sigma + j\omega \) be a complex root of \( f(s; y_1, y_2) \) and consider the Jacobian determinant

\[
\frac{\partial (\sigma, \omega)}{\partial (y_1, y_2)} = \frac{\partial (\sigma, \omega)}{\partial (\text{Re}, \text{Im})} \frac{\partial (\text{Re}, \text{Im})}{\partial (y_1, y_2)}
\]

\[
= \left[ \frac{\partial (\text{Re}, \text{Im})}{\partial (\sigma, \omega)} \right]^{-1} \frac{\partial (\text{Re}, \text{Im})}{\partial (y_1, y_2)}
\]

Certainly,

\[
\frac{\partial (\text{Re}, \text{Im})}{\partial (\sigma, \omega)}
\]

can never be infinite, being the 2x2 determinant of a matrix with entries polynomial in \( \sigma \) and \( \omega \). Hence,

\[
\frac{\partial (\sigma, \omega)}{\partial (y_1, y_2)} \text{ can only be zero if } \frac{\partial (\text{Re}, \text{Im})}{\partial (y_1, y_2)} \text{ is zero}.
\]
So we must recognize the possibility that boundary values in the root set of all roots of 
\( f(s;\gamma_1,\gamma_2) \), \( \gamma_1 \in [\gamma_1,\overline{\gamma_1}] \), \( \gamma_2 \in [\gamma_2,\overline{\gamma_2}] \) could only be achieved where

\[ f(s;\gamma_1,\gamma_2) = 0 \quad (3.2a) \]

and

\[ \frac{\partial(\text{Ref,Imf})}{\partial(\gamma_1,\gamma_2)} = 0 \quad (3.2b) \]

Now the root set of all polynomials \( f(s;\gamma_1,\gamma_2) \) with \( \gamma_1 \in [\gamma_1,\overline{\gamma_1}] \), \( \gamma_2 \in [\gamma_2,\overline{\gamma_2}] \) is a union of a finite number of closed connected sets \( R_1, \ldots, R_n \), and is bounded. If all edge polynomials are stable, and if \( f(s;\gamma_1,\gamma_2) \) has an unstable complex root pair for some \( \gamma_1,\gamma_2 \), it follows that the boundary of one of the sets \( R_i \) intersects the imaginary axis. Then we have proved the "only if" part of the following proposition. The "if" part is trivial.

**Proposition 3.1:** Consider \( f(s;\gamma_1,\gamma_2) \) with \( \gamma_1 \in [\gamma_1,\overline{\gamma_1}] \), \( \gamma_2 \in [\gamma_2,\overline{\gamma_2}] \), a monic \( n \)-th degree polynomial bilinear in \( \gamma_1 \) and \( \gamma_2 \). Suppose that all edge polynomials are stable. Then at least one polynomial fails to be stable if and only if for some \( s = j\omega, \omega \) real, \( (3.2) \) are satisfied.

We shall now show that the question of whether \( (3.2) \) are satisfied for some \( s = j\omega \) is an easily answered question, via a procedure which we now indicate.

Let us set

\[ \text{Ref}(j\omega) = g_0(\omega) + \gamma_1 g_1(\omega) + \gamma_2 g_2(\omega) + \gamma_1 \gamma_2 g_3(\omega) \quad (3.3a) \]

\[ \text{Imf}(j\omega) = h_0(\omega) + \gamma_1 h_1(\omega) + \gamma_2 h_2(\omega) + \gamma_1 \gamma_2 h_3(\omega) \quad (3.3b) \]

Each of the \( g_i;h_j \) takes real values for real \( \omega \). Observe then that

\[ \frac{\partial(\text{Ref,Imf})}{\partial(\gamma_1,\gamma_2)} = \det \begin{bmatrix} g_1(\omega) + \gamma_1 g_2(\omega) + \gamma_2 g_3(\omega) \\ h_1(\omega) + \gamma_1 h_2(\omega) + \gamma_2 h_3(\omega) \end{bmatrix} = (g_1 h_2 - g_2 h_1) + \gamma_1 (g_1 h_3 - g_3 h_1) + \gamma_2 (g_2 h_3 - g_3 h_2) = 0 \quad (3.4a) \]

Now \( (3.2a) \) implies

\[ g_0 + \gamma_1 g_1 + \gamma_2 g_2 + \gamma_1 \gamma_2 g_3 = 0 \quad (3.4b) \]

\[ h_0 + \gamma_1 h_1 + \gamma_2 h_2 + \gamma_1 \gamma_2 h_3 = 0 \quad (3.4c) \]

From \( (3.4b) \) and \( (3.4c) \), there follows

\[ (g_0 h_3 - h_0 g_3) + \gamma_1 (g_1 h_3 - g_3 h_1) + \gamma_2 (g_2 h_3 - g_3 h_2) = 0 \quad (3.5) \]

Together, \( (3.4a) \) and \( (3.5) \) allow \( \gamma_1,\gamma_2 \) to be expressed in terms of the \( g_i;h_j \). Their expressions can then be substituted into \( (3.4b) \) to obtain a single polynomial equations in \( \omega \). It may have no nonzero real solution. If it does, each solution determines values for \( \gamma_1,\gamma_2 \), via \( (3.4a) \) and \( (3.5) \). If these values lie inside the allowed region \([\gamma_1,\overline{\gamma_1}] \times [\gamma_2,\overline{\gamma_2}] \), then instability is proved.

Several other remarks should be made. First, in case \( f() \) is only linear rather than bilinear in \( \gamma_1,\gamma_2 \) (which is the situation considered in Bartlett, Hollot and Lin (1988),
equations (3.4) become
\begin{align}
\text{(3.6a)} & \quad g_0 h_2 - h_2 g_1 = 0 \\
\text{(3.6b)} & \quad g_0 + \gamma_1 h_1 + \gamma_2 g_2 = 0 \\
\text{(3.6c)} & \quad h_0 + \gamma_1 h_1 + \gamma_2 h_2 = 0
\end{align}

If there exists a real \( \omega \) for which (3.6a) is zero, then (3.6b) and (3.6c) can only both be satisfied in cases \( g_0 h_2 - g_2 h_0 \) is also zero at this frequency. In this case, the \( (\gamma_1, \gamma_2) \) pairs satisfying (3.6b) and (3.6c) lie on a straight line, and consequently, there exist edge values of either \( \gamma_1 \) or \( \gamma_2 \) which cause satisfaction for the same \( \omega \), i.e. there exists a pair satisfying (3.6) of one of the forms \( (\gamma_1, \gamma_2) \), \( (\gamma_1, \gamma_2) \), \( (\gamma_1, \gamma_2) \) or \( (\gamma_1, \gamma_2) \). Consequently, any root of a face polynomial on the boundary of the root set is a root of an edge polynomial. Then one never has to explicitly study face polynomials. This is the conclusion of Bartlett, Hollot and Lin (1988).

Second, decision algebra provides a tool for checking stability across a face which should not be too demanding, see Bose (1982). The Hurwitz determinants depend on two parameters \( y_1, y_2 \) and have to be checked for positivity inside a rectangle. Algorithms are available for this task, as set out in Bose (1982). These algorithms involve a finite number of rational calculations. The method we have suggested here, which introduces the need for polynomial factorization is an example of a general approach to decision algebra problems involving the setting up of a polynomial equations in \( q \) unknowns.

Third, bilinearity with respect to \( y_1, y_2 \) has not played a central role here, although it has played a helpful role. The derivation of a single equation in \( \omega \) through the elimination of \( y_1, y_2 \) from (3.3) and (3.4a) is more complicated when the dependence of \( f \) on \( y_1, y_2 \) is polynomial rather than multilinear.

Fourth, the paper of Djaferis (1988) is entirely concerned with the case when a so-called shaping condition is fulfilled, namely \( g_3 h_2 - h_3 g_2 = 0 \). Clearly, this makes \( \Re f(j\omega) \) independent of \( y_2 \). It is also easy to check that when this condition holds

\[ \frac{\partial (\Re f, \Im f)}{\partial (\gamma_1, \gamma_2)} \]

and also \( \frac{\partial (\Re f, \Im f)}{\partial (\gamma_1, \gamma_2)} \) is zero, then \( \Re f(j\omega) \) and \( \Im f(j\omega) \) are independent of \( y_2 \).

Consequently, if \( j\omega_0 \) is on the boundary of the root set, \( j\omega_0 \) remains a root on the root set boundary when \( y_2 \) varies. In particular, when \( y_2 \) is set equal to an edge value \( y_2 \), it remains true that \( j\omega_0 \) is a root. Hence all purely imaginary roots on the root set boundary are roots of edge polynomials, which means that under the condition \( g_3 h_2 - h_3 g_2 = 0 \) only edge polynomials need to be tested. Obviously, the same holds true if \( g_1 h_3 - h_1 g_3 = 0 \).

Example (J. Ackermann, 1988):

Consider \( f(y_1, y_2) = s^3 + (y_1 + y_2 + 1)s^2 + (2y_1y_2 + 6y_1 + 6y_2 + 1.25) \) with \( y_1 \in [0.3; 2.5] \) and \( y_2 \in [0; 1.7] \). It turns out (and can be established with the aid of for example the Hurwitz test) that the parameter values giving unstable \( f \) are defined by the shaded regions in Figure 4. The boundaries of the regions are given by \( 2y_1y_2 + 6(y_1 + y_2) + 1.25 = 0 \) and \( (y_1 - 1)^2 + (y_2 - 1)^2 - 0.5^2 = 0 \). These points are noted in order to allow comparison with the methods of this paper.

First, edge stability must be verified. Let us see how this can be done for one edge, say the edge \( y_2 = 0 \). The Hurwitz conditions are

\[ 0 < \det \begin{bmatrix} y_1 + 1 & 6y_1 + 1.25 \\ 1 & y_1 + 3 \end{bmatrix} = y_1^2 - 2y_1 + 1.75 \]
It is clear that these inequalities are all satisfied for $\gamma_1 \in [0.3, 2.5]$.

Next, we must look for points in the interior of the parameter region corresponding to purely imaginary roots on the boundary of the root set. These are determined from

$${\text{Re}} f(j\omega; \gamma_1, \gamma_2) = 0, \quad {\text{Im}} f(j\omega; \gamma_1, \gamma_2) = 0,$$

and

$$\frac{\partial}{{\partial}(\gamma_1, \gamma_2)} \left( {\text{Re}} f, {\text{Im}} f \right) = 0.$$ 

The relevant equations are

$$-(\gamma_1 + \gamma_2 + 1)\omega^2 + (2\gamma_1\gamma_2 + 6\gamma_1 + 6\gamma_2 + 1.25) = 0,$$

$$-\omega^2 + (\gamma_1 + \gamma_2 + 3) = 0,$$

$$2(\gamma_1 - \gamma_2) = 0.$$ 

It is readily verified, that these equations are satisfied by

$$\gamma_1 = \gamma_2 = 1 \pm \sqrt{2}/4, \quad \omega = \sqrt{2\gamma_1 + 3} = \begin{cases} 2.39 \quad &2.07 \end{cases}.$$ 

The corresponding points in parameter space are designated by $X, Y$ in Figure 4. The root set corresponding to all allowed $\gamma_1, \gamma_2$ is sketched in Figure 5, and it will be observed that the values for $\omega$ computed above define those boundary parts of the root set which lie on the imaginary axis.

In this example, it is also possible to exactly determine the root boundary. Candidates for this boundary are besides the edge polynomials also points in the interior of the parameter region $ABCE$ with
The relevant equations are

\[ f(\sigma + j\omega; \gamma_1, \gamma_2) = 0 \]

\[ \frac{\partial (\text{Re}, \text{Im})}{\partial (\gamma_1, \gamma_2)} = 0 \]

The relevant equations are

\[ 2\gamma_1\gamma_2 + (\gamma_1 + \gamma_2)(6 + \sigma + \sigma^2 - \omega^2) + (1.25 + 3\sigma + \sigma^2 + \sigma^3 - 3\sigma \omega^2) = 0 \]

\[ (\gamma_1 + \gamma_2)(1 + 2\sigma) + (3 + 2\sigma + 3\sigma^2 - \omega^2) = 0 \]

\[ 2\omega(1 + 2\sigma)(\gamma_1 - \gamma_2) = 0 \]

For the complex root boundary, two different cases are distinguished:

\[ \sigma = -0.5 \]

Then \[ \omega_0^2 = 2.75 \]

\[ \gamma_2 = -(3\gamma_1 + 1.25)/(2\gamma_1 + 3) \]

This is an isolated singular point. For variations along the given \( \gamma_1, \gamma_2 \) hyperbola the root pair \( -0.5 \pm j\omega_0 \) does not change.

\[ \gamma_1 = \gamma_2 \]

Because of the symmetry of \( f(\cdot) \) with respect to \( \gamma_1, \gamma_2 \) it is obvious that for \( \gamma_1, \gamma_2 \) from the triangle DEF the same roots result as from the triangle DEF. Therefore, the root boundaries
for \( y_1, y_2 \) from ABCE and from ABCDF are the same. For \( y_1, y_2 \) from DF a part of the root boundary is built. This is for \( y_1, y_2 \) between X and Y unstable.

4. STABILITY TESTING IN PARAMETER REGION INTERIOR

We have already described how testing of edges and faces may proceed. In an \( m \)-dimensional parameter space (\( m > 2 \)) it is necessary to look successively at 3-dimensional boundaries (all but three of the \( y_i \) take extreme values), 4-dimensional boundaries, ... the interior of the entire \( m \)-dimensional region. In each case, we seek to identify frequencies \( \omega \) such that \( j\omega \) is on the boundary of the root set of all polynomials. When looking at say 4-dimensional regions, this is done by setting up 5 simultaneous equations in 5 unknowns, viz. \( \omega \) and the four variable \( y_i \), and seeking solutions which are real in \( \omega \) and the \( y_i \), with each \( y_i \) in the prescribed interval \([y_i, y_i]\). In the absence of such solutions, it is known that the entire 4-dimensional region defines stable polynomials if the 3-dimensional regions bounding it are known to define stable polynomials.

We shall explain the idea in more detail for the case when three parameters vary. It is a generalization of the two variable parameter case considered in the previous section; the generalization to more than three variable parameters is straightforward.

Let \( \sigma + j\omega \) be a complex root of \( f(s; y_1, y_2, y_3) \) for \( y_i \in [y_i, y_i] \). Consider the effect of changing the \( y_i \) on the root. In particular, let \( \Delta y_i, i=1,2,3 \) denote very small changes in the \( y_i \), and let \( \Delta \sigma, \Delta \omega \) denote the corresponding changes in the root. Then, neglecting second order terms,

\[
\begin{bmatrix}
\Delta \sigma \\
\Delta \omega
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial \sigma}{\partial y_1} & \frac{\partial \sigma}{\partial y_2} & \frac{\partial \sigma}{\partial y_3} \\
\frac{\partial \omega}{\partial y_1} & \frac{\partial \omega}{\partial y_2} & \frac{\partial \omega}{\partial y_3}
\end{bmatrix}
\begin{bmatrix}
\Delta y_1 \\
\Delta y_2 \\
\Delta y_3
\end{bmatrix}
\]

\[=
\begin{bmatrix}
\frac{\partial \sigma}{\partial \text{Re} f} & \frac{\partial \sigma}{\partial \text{Im} f} \\
\frac{\partial \omega}{\partial \text{Re} f} & \frac{\partial \omega}{\partial \text{Im} f}
\end{bmatrix}
\begin{bmatrix}
\text{Re} f & \text{Re} f & \text{Re} f \\
\text{Im} f & \text{Im} f & \text{Im} f
\end{bmatrix}
\begin{bmatrix}
\Delta y_1 \\
\Delta y_2 \\
\Delta y_3
\end{bmatrix}
\]

\[=
\begin{bmatrix}
\frac{\partial \text{Re} f}{\partial \text{Re} f} & \frac{\partial \text{Re} f}{\partial \text{Im} f} \\
\frac{\partial \text{Im} f}{\partial \text{Re} f} & \frac{\partial \text{Im} f}{\partial \text{Im} f}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial \text{Re} f}{\partial \sigma} & \frac{\partial \text{Re} f}{\partial \omega} & \frac{\partial \text{Re} f}{\partial y_1} \\
\frac{\partial \text{Im} f}{\partial \sigma} & \frac{\partial \text{Im} f}{\partial \omega} & \frac{\partial \text{Im} f}{\partial y_1}
\end{bmatrix}
\begin{bmatrix}
\Delta y_1 \\
\Delta y_2 \\
\Delta y_3
\end{bmatrix}
\] (4.1)

Now we if we are on the boundary of the root set, there cannot exist perturbations \( \Delta y_i \) which can give arbitrary \( \Delta \sigma, \Delta \omega \). So candidates for values of \( \sigma, \omega \) and \( y_i \) yielding a point on the boundary of the root set are given by

\[
\text{rank}
\begin{bmatrix}
\frac{\partial \text{Re} f}{\partial y_1} & \frac{\partial \text{Re} f}{\partial y_2} & \frac{\partial \text{Re} f}{\partial y_3} \\
\frac{\partial \text{Im} f}{\partial y_1} & \frac{\partial \text{Im} f}{\partial y_2} & \frac{\partial \text{Im} f}{\partial y_3}
\end{bmatrix}
\leq 1
\] (4.2)
Equivalently
\[ \frac{\partial (\Re f, \Im f)}{\partial (\gamma_1, \gamma_2)} = 0 \] (4.3a)

and
\[ \frac{\partial (\Re f, \Im f)}{\partial (\gamma_1, \gamma_3)} = 0 \] (4.3b)

as well as
\[ \Re f = 0 \quad \Im f = 0 \] (4.3c)

It is enough to look for purely imaginary points on the boundary of the root set, i.e. to set \( \sigma = 0 \). Then (4.3) represent four simultaneous equations in the four unknowns \( \gamma_1, \gamma_2, \gamma_3, \omega \). In general there are a finite number of solutions. If and only if one of these solutions is real, with \( \gamma_i \in [\gamma_i, \overline{\gamma_i}] \), can there be a purely imaginary point on the boundary set.

The computation of solutions of simultaneous polynomial equations is a problem which has been studied. Older methods have depended on successive elimination of variables using resultants until a single equation in a single variable is obtained. This is solved, and then through successive back substitution, values of the other variables are obtained, see e.g. Bose (1982), and Hodge and Pedoe (1968).

Note that if \( \omega \) is the variable eliminated from \( \Re f = 0, \Im f = 0 \) and all other equations are neglected, there results a single equation which corresponds to setting a Hurwitz determinant equal to zero. The terms in this equation depend on the \( \gamma_i \). The question is then whether this determinant can be made zero for some choice of \( \gamma_i \) in the parameter region of interest or not. This is of course a natural question, and is roughly the approach exposed in Bose (1982).

**Example:**

Consider
\[ f(s) = s^3 + 1 + (s^2 + s)\gamma_2 + s\gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_3 \]

with \( \gamma_i \in [\gamma_i, \overline{\gamma_i}], i = 1, 2, 3 \).

For investigation of the root boundary, it is necessary to check all sides of the \( \gamma \)-cube, i.e. \( \gamma_i \in [\gamma_i, \overline{\gamma_i}], i = 1, 2, 3 \), and all points of the interior of the parameter region with
\[ \text{Rank } J \leq 1 \quad \Re f = 0 \quad \Im f = 0 \]

where \( J \) is the Jacobi matrix:
\[
J = \begin{bmatrix}
\sigma \gamma_2^2 + \gamma_2 \gamma_3 & \sigma^2 - \omega^2 + \sigma + \sigma \gamma_1 + \gamma_1 \gamma_3 & \gamma_1 \gamma_2 \\
\gamma_2 & 2\sigma + 1 + \gamma_1 & 0
\end{bmatrix}
\]

the relevant equations are
These equations must be simultaneously fulfilled.

For $\gamma_2 = 0$ the polynomial family degenerates to

$$f(s) = s^3 + 1$$

a simple polynomial. Rank $J$ is zero. No special investigation is necessary.

For $\gamma_1 = 0$ we obtain

$$f(s) = s^3 + 1 + (s^2 + 1)\gamma_3$$

and from rank $J = 1$ the condition

$$(\gamma_2 - 2)\gamma_3 = 1$$

results, which is a hyperbola in the $\gamma_2\gamma_3$ plane. Along this hyperbola, there are pairs $\gamma_1 = 0, \gamma_3$ which give possible boundary polynomials. But from the degeneration of the polynomial family $f(s)$ it is obvious that rank $J = 1$ follows from the degeneration of the parameter space to a straight line. Therefore, there exist internal points of the parameter space, which fulfill the necessary conditions for the root boundary. However, these points do not yield this boundary.

Next, the six sides of the parameter box must be checked. In general, this can be done by testing all boundaries of such a side and all candidates for the root boundary on this side. These are exactly the same steps as we have done before but on a lower dimensional space. In this way, we proceed for every side until the 2-dimensional faces are reached.

In our special case, the box sides are directly the 2-dimensional faces. The procedure of the last section can be used.

For $\gamma_1 =$ constant we have $\text{Re} f = 0$ and $\text{Im} f = 0$, and

$$(2\sigma + 1 + \gamma_1)\gamma_2 = 0$$

With $\gamma_1 \neq 0 \neq \gamma_2$ and $\text{Im} f = 0$, we obtain

$$2\sigma + 1 + \gamma_1 = 0$$

$$\omega^2 = 3\sigma^2 + \gamma_2(2\sigma + 1) + \gamma_1\gamma_2$$

and then

$$\sigma = \frac{1}{2}(1 + \gamma_1)$$

$$\omega^2 = 3\sigma^2$$

The critical points of the $\gamma_1$-sides yield only a point in the $s$-space and because of the continuity conditions not a significant part of the root boundary.
In the same way one may proceed for \( \gamma_2 \) and \( \gamma_3 \)-sides.

Recently, methods for solving simultaneous polynomial equations based on homotopy theory have been suggested, see Watson et al. (1987).

A number of further points should be noted. First, in this section, no special use has been made of the multilinearity, i.e. the same ideas will apply even if the dependence of the coefficients on the parameters is a general polynomial dependence.

Second, it is easy to recover various special cases. Suppose following Panier, Fan and Tiss (1987) that the coefficients of even powers of \( f(\cdot) \) depend on \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m \) and the coefficients of odd powers of \( f(\cdot) \) depend on \( \gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_m \). The condition (4.2) then becomes

\[
\begin{bmatrix}
\frac{\partial \text{Re} f}{\partial \gamma_1} & \cdots & \frac{\partial \text{Re} f}{\partial \gamma_r} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{\partial \text{Im} f}{\partial \gamma_{r+1}} & \cdots & \frac{\partial \text{Im} f}{\partial \gamma_m}
\end{bmatrix} \leq 1
\]

Suppose for example that \( \frac{\partial \text{Re} f}{\partial \gamma_i} = 0 \), \( i = 1, \ldots, r \) and \( \frac{\partial \text{Im} f}{\partial \gamma_j} = 0 \), \( j = r+1, \ldots, m-1 \). Now \( \text{Re} f \) is multilinear in \( \gamma_1, \ldots, \gamma_r \) and accordingly can take no extreme value inside the region \( \gamma \in [\gamma_1, \gamma_2] \) unless that value is also assumed on the boundary. Thus if for some \( \omega \frac{\partial \text{Re} f}{\partial \gamma_i} = 0 \)

this continues to hold when the \( \gamma_i \) take extreme values. Similarly, if \( \frac{\partial \text{Im} f}{\partial \gamma_j} = 0 \), \( j = r+1, \ldots, m-1 \) for some \( \omega \), these equations continue to hold when the \( \gamma_i \) take extreme values. Hence the rank condition if fulfilled anywhere is necessarily fulfilled when all but one \( \gamma_i \), say \( \gamma_m \), take extreme values, i.e. it is fulfilled on the edge defined by variable \( \gamma_m \). Consequently, it is only necessary to check edges for stability.

Another special case is provided by the shaping conditions of Djaferis and Hollot (1988). To fix ideas, suppose that \( f \) depends on four parameters, with

\[
f(s; \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \phi_1(s)f_1(\gamma_1, \gamma_2) + \phi_2(s)f_2(\gamma_1, \gamma_2) + \phi_3(s)f_3(\gamma_1, \gamma_3) + \phi_4(s)f_4(\gamma_1, \gamma_4)
\]

The \( \phi_i(s) \) are of course independent of \( \gamma_i \). Moreover, \( \phi_i(j\omega) = \text{Re} \phi_i(j\omega) + j\omega \text{Im} \phi_i(j\omega) \) where

\[
\text{Re} \phi_i(j\omega) = \frac{1}{2} \{ \phi_i(j\omega) + \phi_i(-j\omega) \}, \quad \text{Im} \phi_i(j\omega) = \frac{1}{2j} \{ \phi_i(j\omega) - \phi_i(-j\omega) \},
\]

and the side conditions (shaping conditions)

\[
E_{\phi_1}O_{\phi_2} - E_{\phi_2}O_{\phi_1} = 0 \quad E_{\phi_3}O_{\phi_4} - E_{\phi_4}O_{\phi_3} = 0
\]

hold identically in \( \omega \). The \( 2 \times 4 \) Jacobian matrix condition becomes

\[
\begin{bmatrix}
E_{\phi_1} \frac{\partial f_1}{\partial \gamma_1} + E_{\phi_2} \frac{\partial f_2}{\partial \gamma_1} + E_{\phi_3} \frac{\partial f_3}{\partial \gamma_2} + E_{\phi_4} \frac{\partial f_4}{\partial \gamma_2} \\
O_{\phi_1} \frac{\partial f_1}{\partial \gamma_1} + O_{\phi_2} \frac{\partial f_2}{\partial \gamma_1} + O_{\phi_3} \frac{\partial f_3}{\partial \gamma_2} + O_{\phi_4} \frac{\partial f_4}{\partial \gamma_2}
\end{bmatrix} \leq 1
\]

Now the shaping conditions ensure that the minors formed from columns 1 and 2 and from columns 3 and 4 are zero automatically. Suppose the minor formed from columns 1 and 3 is zero. By the multilinearity, column 1 is independent of \( \gamma_1 \) and column 3 is independent of \( \gamma_3 \). The special form of \( f \) ensures that column 1 is independent of \( \gamma_3 \) and
column 3 is independent of \( \gamma_1 \). Hence, if the minor formed from columns 1 and 3 is zero, it must remain so if \( \gamma_1 \) and \( \gamma_3 \) are varied to extreme values. The shaping condition ensures that the minors formed from columns 1 and 2 and columns 3 and 4 remain zero with this variation of \( \gamma_1 \) and \( \gamma_3 \). Similarly, one can argue that \( \gamma_2 \) and \( \gamma_4 \) could be varied to their extreme values. Hence if the Jacobian matrix has reduced rank somewhere, it has this property for all \( \gamma \). A consequence of this is that the image of \( f(j_0;\gamma_1,\ldots,\gamma_m) \) for fixed \( j_0 \) and \( \gamma \) lies in the convex hull of \( f(j_0;\gamma_1,\ldots,\gamma_m) \) for \( \gamma \in \{ \gamma \in \mathbb{R}^m \} \) is a set bounded by the images of the edges. In general, this is a polytope. But with the Jacobian matrix of rank 1, the image will be a line segment, and when of rank 0, it will be a point.

A third special case can be obtained by limiting the way in which the nonlinear parameter dependence arises. Specifically, assume that any one \( \gamma \) can occur bilinearly with at most one other parameter \( \gamma \), and that in the polynomial \( f(s;\gamma_1,\ldots,\gamma_m) \) the s-polynomial multiplying \( \gamma_1 \gamma \) is either even or odd. An easy calculation shows that this ensures that all 2x2 minors of the generalized Jacobian matrix are linear in the parameters. The solution of the associated simultaneous equations is made much easier in these circumstances.

5. DIFFERING AND CONVERGING NECESSARY AND SUFFICIENT CONDITIONS

We have referred earlier to the work of de Gaston and Safonov (1988), who exploited the observations of Zadeh and Desoer (1963) that the image of \( f(j_0;\gamma_1,\ldots,\gamma_m) \) for fixed \( j_0 \) and \( \gamma \in \{ \gamma \in \mathbb{R}^m \} \) lies in the convex hull of \( f(j_0;\gamma_1,\ldots,\gamma_m) \) for \( \gamma \in \{ \gamma \in \mathbb{R}^m \} \) to develop a test for stability based on Nyquist ideas.

It is possible to exploit the observation of Zadeh and Desoer (1963) in another way. Denote the corners of the allowed parameter space region by \( A_1,\ldots,A_k \) where \( k = 2^n \). Denote the corresponding points in the \( n \)-dimensional coefficient space by \( B_1,\ldots,B_k \). Now a perusal of the argument of Zadeh and Desoer (which involves scalar functions dependent multilinearly on \( m \) parameters) shows that it extends very straightforwardly to vector functions. As a result, the image in coefficient space of any point in the allowed parameter region, i.e., in the convex hull of \( A_1,\ldots,A_k \) necessarily lies in the convex hull of \( B_1,\ldots,B_k \).

Figure 6 depicts a rectangular region in parameter space and certain straight lines in its image in coefficient space. These straight lines are the images of the edges in parameter space. It is possible to construct the convex hull of \( B_1,\ldots,B_k \) by going with straight lines all possible so far unjoined pairs of points and then "filling in" the enclosed region. Thus straight lines such as \( B_4B_7, B_1B_6 \) etc. must be joined. Note that \( B_4B_7 \) is not the image of the straight line \( A_4A_7 \) (on a certain face) in parameter space. The image of \( A_4A_7 \) will in general be curved, and be within the convex hull determined by \( B_4 \) through \( B_7 \).

A necessary condition for robust stability is clearly that the edges in parameter space (or their images in coefficient space) are all stable. A sufficient condition is that the straight lines joining all possible pairs of corner points in coefficient space (i.e., those which are images of parameter space edges and those which are not) must be stable; for the Edge Theorem of Bartlett, Elliott, and Lin (1988) ensures that all points in coefficient space in the convex hull of \( B_1,\ldots,B_k \) will be stable, and so in particular those which are images of points in the defined region of parameter space.

Now if the necessary conditions for stability are fulfilled and the sufficiency ones are not, one can proceed in a similar fashion to de Gaston and Safonov (1988). That is, one partitions the original rectangular box in parameter space in two, and develops separate necessity and sufficiency conditions. More precisely, if in the example \( B_2B_7 \) proves to contain an unstable polynomial one could make a slice in parameter space parallel to
$A_1A_2A_3A_4$ or parallel to $A_1A_2A_6A_5$ thus ensuring that $A_2A_7$ go into different rectangular boxes. Then the line $B_2B_7$ will no longer enter into a sufficiency condition.

To the original necessity conditions are added four more, while a number of the original sufficiency conditions fall away to be replaced by a greater number of collectively less demanding conditions.

![Diagram](image)

Fig. 6. Parameter and coefficient space convex hull covering. a) Parameter space 3-dimensional; b) coefficient space 4-dimensional

6. CONCLUDING REMARKS

This paper has extended consideration of the robust polynomial stability problem by allowing mild forms of nonlinear dependence of the polynomial coefficients on variable parameters. It is seen very rapidly that even this mild form of dependence introduces substantial complications, so that for example the Edge Theorem applicable with affine dependence is probably no longer a valid tool. The key to examining interior points in parameter space is to consider a generalized Jacobian matrix and study the points where its rank is 1 or 0. Various special cases can be identified which allow a conclusion like that of the Edge Theorem to be applied. It would be interesting to expand this range of special cases.
REFERENCES


