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On the zero properties of tall linear systems with single-rate and multirate outputs

Brian D. O. Anderson
Australian National University
Canberra, Australia
brian.anderson@anu.edu.au

Mohsen Zamani
Australian National University
Canberra, Australia
mohsen.zamani@anu.edu.au

Giulio Bottegal
University of Padova
Padova, Italy
bottegal@dei.unipd.it

Abstract. The zero properties of tall discrete-time multirate linear systems are studied in this paper. In the literature, zero properties of multirate linear systems are defined as those of their corresponding blocked systems, which are time-invariant systems whose behavior is broadly equivalent to that of the generating multirate system. In this paper, we review some recent scattered results of the authors and their colleagues dealing with the zero properties of the blocked systems associated with multirate systems. First, we show that tall linear time-invariant unblocked systems are zero-free when the parameter matrices A,B,C,D assume generic values. Then it is argued that tall blocked systems obtained from blocking of tall linear time-invariant systems with generic parameter matrices A,B,C,D, are also zero-free. Finally, we illustrate that tall blocked systems associated with multirate systems generically have no finite nonzero zeros.

1 Introduction

Our motivation for studying the zero properties of tall transfer function matrices, transfer function matrices which have more outputs than inputs, comes from their potential application in generalized dynamic factor models. Such models arise in a number of fields e.g. econometric modeling, signal processing and systems and control, and the associated transfer functions are almost always tall. Hence the authors of this work have become interested in the zero properties of tall systems due to their application in generalized dynamic factor models, though now consider these properties of interest in their own right. Now as just mentioned, in generalized dynamic factor models, it is very common to have models with a larger number of outputs compared to their number of inputs i.e. tall models; furthermore, when they are used for econometric modeling, it is also very common to have some outputs measured monthly while some other outputs may be obtained quarterly or even annually [10], [15], [14]. Thus, the models are periodic. Moreover, in this context, the latent variables, i.e. the noiseless part of the outputs, or the part remaining after removal of contaminating additive measurement noise, are modeled by systems with unobserved white noise inputs. In a single-rate setting i.e. monthly data only, [8] has shown that model tallness generically implies that the generalized dynamic factor model is zero-free, and then the latent variables can be modeled as a singular autoregressive process whose parameters can be easily identified from covariance data using Yule-Walker equations. A corresponding demonstration is still lacking for the multirate case i.e. with both monthly and quarterly data. The results of this paper are aimed at enabling us to understand better the properties of multirate factor models, and in particular establishing that tall systems again are generically zero-free; this is done with a view to later establishing the utility of the Yule-Walker approach for identifying multirate factor models from their covariance data. Quite apart from this motivation however, the results of this paper suggest in relation to classical control design that, if one adds extra sensors to make a plant have more outputs than inputs, then controller design will be much easier due to the generic absence of plant zeros. Note though that this paper does not focus on the applications problem, but rather on the system theoretical issues involved with the zero properties of multirate systems with tall structure.

Our main goal is to establish generic conditions for when the system matrix associated with (a blocked version of) a tall multirate system will have a property corresponding to the system having no zeros, and also to establishing generic conditions for when all zeros are simple and finite. In order to reach this goal we review some of our recent results regarding the zero properties of multirate systems, and in particular, how the technique of blocking or lifting can allow them to be treated as time-invariant systems.

In the systems and control literature, the technique of blocking or lifting has been used for a long time to transfer a multirate linear system into a linear time-invariant system which is generally referred to as a blocked system (see e.g., [3], [16], [2]). The nomenclature arises because blocked systems can be obtained from stacking the input and output vectors of multirate systems within a period into new larger vectors, see [3], [16] and later in this paper. Moreover, in the literature, the zeros of multirate systems are defined as those of their corresponding blocked systems [3], [7], [5].

The zero properties of blocked linear systems have been studied to some degree in the literature; for instance, [4], [11] have explored the zero properties of blocked systems obtained from blocking of linear periodic systems (a class of systems which includes multi-rate systems). The results show that the blocked system has a finite zero if it is obtained from a time-invariant unblocked system, and the latter has a finite zero, which is a form of sufficiency condition. However, this reference does not provide a necessary condition for a blocked system obtained this way to have a finite zero. This gap has been covered in [19] and [6] where the authors have introduced some additional information about the zero properties of blocked systems obtained from blocking of time-invariant systems. References [19] and [6] have used different approaches but they have obtained largely similar results. The results in those references show that the blocked system is zero-free if and only if its associated linear time-invariant unblocked system is zero-free. In contrast to the case where the unblocked system is time-invariant, very few results indeed however deal with zeros of blocked systems where these systems have been obtained by blocking of a truly multirate system, i.e. one that is not time-invariant. We note that [18] does have some partial results.

In this paper, to achieve the main goal stated in the second paragraph, some of the recent works of the authors and their colleagues are recalled. In particular, we utilize results from [1], [19] and [18]. First in Section 2 one of the results of [1] is modified to study the zero properties of a tall unblocked linear time-invariant system under a generic setting i.e. when parameter matrices A,B,C,D assume generic values. Then in Section 3 the results of [19] are exploited to explore the zero properties of the blocked systems associated with unblocked linear time-invariant systems. Subsequently, these results are used in Section 4 to examine the zero-freeness of tall blocked systems associated with tall multirate linear systems. Finally, Section 5 provides concluding remarks and plans for future works.

It would be inappropriate to close this section without reflecting on the fact that this volume celebrates the achievements of Uwe Helmke. Two of the three authors of this paper count among their happiest professional experiences their interactions with Uwe, a person who is both very talented professionally, and possessing of an affable and generous nature. A special skill has characterized all these interactions: his ability to put himself out of the world of mathematics and into the world of engineering, and to speak the language of engineering where needed, with an infinite supply of patience in answering the engineers' questions.

2 Tall linear time-invariant unblocked systems

In this section we study the zero properties of tall linear time-invariant unblocked systems. Here, one of the results of [1] is modified to show that a tall system with generic parameter matrices A,B,C,D is zero-free i.e. its associated system matrix has full-column rank for all $z \in \mathbb{C} \cup \{\infty\}$.

Consider the following time-invariant unblocked system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^p$ and $u_t \in \mathbb{R}^m$, $p \geq m$. For this system, we assume that y_t is available at every time instant t.

In order to study the zero properties of the system (1), we need to provide a proper definition for the zeros. Here, we first recall the following definition for zeros of the unblocked system (1) from [13] and [12] (page 178).

Definition 1. The finite zeros of the transfer function $W(z) = C(zI - A)^{-1}B + D$ with minimal realization $[A,B,C,D]$ are defined to be the finite values of z for which the rank of the following system matrix falls below its normal rank

$$M(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}. \tag{2}$$

Further, $W(z)$ is said to have an infinite zero when $n + rank(D)$ is less than the normal rank of $M(z)$, or equivalently the rank of D is less than the normal rank of $W(z)$.

The following lemma inspired from results in [9] studies the zero properties of the system (1) when $p = m$ and the parameter matrices A, B, C, D accept generic values. It states that for generic parameter matrices, the rank reduction of the system matrix at any zero is 1. The lemma will help us to prove the main result of this section regarding the zero-freeness of tall linear time-invariant unblocked systems with generic parameter matrices.

Lemma 2. *The set* $\mathcal{F} = \{[A,B,C,D] | p = m, rank(D) = m, rank(M(z)) \geq n+m-1, \forall z \in \mathbb{C}\}$ *is open and dense in the set* $\{[A,B,C,D] | p = m, rank(D) = m\}$.

Proof. **Dense**: Consider the system matrix $M(z)$ and suppose that there exists a z_0 such that $rank(M(z_0)) = n+m-2$ (note that only the case where rank drops to $n+m-2$ is discussed here and generalization to $n+m-k, k \geq 2$ is straightforward). Therefore, there exist two linearly independent vectors, say x_1 and x_2, which span the kernel of $M(z_0)$. Let $x_i = [x_{i1}^\top \, x_{i2}^\top]^\top$, $i = 1,2$ with $x_{i1} \in \mathbb{R}^n$, then x_{11} and x_{21} must be linearly independent otherwise there would exist nonzero scalars a_1 and a_2 such that $a_1x_1 + a_2x_2 = [0 \; a_1x_{12}^\top + a_2x_{22}^\top]^\top$ with $D[a_1x_{12} + a_2x_{22}] = 0 \Longrightarrow a_1x_{12} + a_2x_{22} = 0$. The latter implies that x_1 and x_2 are linearly dependent which violates the initial assumption. Now it is easy to verify that $[z_0I - A + BD^{-1}C][x_{11} \; x_{21}] = 0$, which implies that $A - BD^{-1}C$ has a repeated eigenvalue. By manipulation of an entry of A by an arbitrarily small amount, we see that the kernel of the new $M(z)$ for any z will have dimension at most 1 since $A - BD^{-1}C$ will have nonrepeated eigenvalues.

Open: Set $\mathcal{F}$ being open is equivalent to its complement, call it $\mathcal{F}^C$, being closed. To obtain a contradiction, suppose $\mathcal{F}^C$ is not closed. Then there must exist a sequence $[A_m, B_m, C_m, D_m]_{m \in \mathbb{N}}$ in $\mathcal{F}^C$ with $[A_m, B_m, C_m, D_m] \to [A_0, B_0, C_0, D_0] \in \mathcal{F}$. Moreover, there exists a $z_m \in \mathbb{C}$ such that $rank(M_m(z_m)) \leq n+m-2$, where $M_m(z)$ denotes the system matrix associated with $[A_m, B_m, C_m, D_m]$. Consequently, $\sigma_1(M_m(z_m)) = \sigma_2(M_m(z_m)) = 0$ where $\sigma_i(F)$ denotes the i-th smallest singular value of F. Now $M_m(z_m) \to M_0(z_0)$ holds as $[A_m, B_m, C_m, D_m] \to [A_0, B_0, C_0, D_0]$ and $z_m \to z_0$. Hence, $\sigma_2(M_m(z_m)) \to \sigma_2(M_0(z_0))$; however, by assumption $\sigma_2(M_0(z_0)) > 0$ which contradicts the fact that $\sigma_2(M_0(z_0)) \to 0$ and the result follows. □

Theorem 3. *Consider a transfer function matrix $W(z)$ with minimal realization $[A,B,C,D]$ of dimension n in which B, C have m columns and p rows respectively with $p > m$. If the entries of A, B, C, D assume generic values, then $W(z)$ has no finite or infinite zeros.*

Proof. Observe first that the normal rank (which is the rank for almost all z) of a generic $M(z)$ i.e. system matrix $M(z)$ with generic matrices, is $n+m$: to see this, take $A = C = 0$ and D as any full column rank matrix, to get a particular $M(z)$ which for any nonzero z has rank $n+m$. Since the normal rank cannot exceed $n+m$ and this rank is attained for a particular choice of A etc, so $n+m$ must be the normal rank for generic $M(z)$. Observe also that D generically has rank m, and hence the normal rank of M equals $n + \text{rank}\, D$, which shows that generically $W(z)$ has no infinite zero. For the finite zeros, observe that any such zero must be a zero of every minor of dimension $(n+m) \times (n+m)$. Since $M(z)$ has normal rank $n+m$, there must be at least one minor of dimension $(n+m) \times (n+m)$ which is nonzero for almost all values of z. Choose A,B and the first m rows of C,D generically, and consider the associated minor. For each of the finite set of values of z for which the minor is zero, determine the associated kernel which has the dimension at most one based on the result of Lemma 2. Then a generic $(n+m)$-dimensional vector will not be orthogonal to any single one of these kernels, and since there are a finite number of such kernels, a generic $(n+m)$-dimensional vector will not be orthogonal to any of the kernels considered simultaneously. If the next, i.e $(m+1)$-th, row of $[C\ D]$ is set equal to this vector, then any vector in any of the finite set of kernels of the $(n+m)$-dimensional minors formed using the first m rows of $[C\ D]$ will not be orthogonal to the added row of $[C\ D]$, which means that the $(m+n+1)$ row matrix obtained by adjoining the new row of $[C\ D]$ must have an empty kernel for any value of z, i.e. there is no zero. Given that C,D are actually generic and may have more rows again, the result is now evident. □

3 Tall blocked linear systems

It was shown in the previous section that tall time-invariant unblocked systems are generically zero-free. In this section we study the zero properties of their associated blocked systems. The results of this section enable us to study the zero properties of blocked systems resulted from blocking of linear systems with multirate output in the next section. We note that, proofs for some of the theorems are omitted and an interested reader can refer to [19] for detailed proofs.

Now we define for a fixed but arbitrary positive number $N>1$

$$\begin{aligned} U_t &= \begin{bmatrix} u_t^\top & u_{t+1}^\top & \dots & u_{t+N-1}^\top \end{bmatrix}^\top, \\ Y_t &= \begin{bmatrix} y_t^\top & y_{t+1}^\top & \dots & y_{t+N-1}^\top \end{bmatrix}^\top, \end{aligned} \tag{3}$$

where $t=0,N,2N,\dots$.

Then the blocked system is given by

$$\begin{aligned} x_{t+N} &= A_b x_t + B_b U_t \\ Y_t &= C_b x_t + D_b U_t. \end{aligned} \tag{4}$$

The blocked system, mapping the U_t sequence to the Y_t sequence, has a time-invariant state-variable description given by

$$\begin{aligned} A_b &= A^N, \quad B_b = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}, \\ C_b &= \begin{bmatrix} C^\top & A^\top C^\top & \dots & A^{(N-1)^\top} C^\top \end{bmatrix}^\top, \\ D_b &= \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \dots & D \end{bmatrix}. \end{aligned} \tag{5}$$

An operator Z is defined such that $Zx_t = x_{t+N}$, $ZU_t = U_{t+N}$, $ZY_t = Y_{t+N}$. In the rest of this paper, the symbol Z is also used to denote a complex value. We denote the transfer function associated with (4) as $V(Z) = D_b + C_b(ZI - A_b)^{-1}C_b$ and it is worthwhile remarking that minimality of $[A,B,C]$ is equivalent to minimality of $[A_b, B_b, C_b]$.

3.1 The zero properties of blocked linear systems

Since in this section we are interested in the zero properties of the system (4), we need to first define zeros for that system. Similar to Definition 1 we have the following definition for the zeros of the system (4).

Definition 4. The finite zeros of the transfer function $V(Z) = C_b(ZI - A_b)^{-1}B_b + D_b$ with minimal realization $[A_b, B_b, C_b, D_b]$ are defined to be the finite values of Z for which the rank of the following system matrix falls below its normal rank

$$M_b(Z) = \begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix}. \tag{6}$$

Further, $V(Z)$ is said to have an infinite zero when $n + rank(D_b)$ is less than the normal rank of $M_b(Z)$, or equivalently the rank of D_b is less than the normal rank of $V(Z)$.

According to the above definition the normal rank of the system matrix $M_b(Z)$ plays an important role in the zero properties of its associated blocked system.

Lemma 5. *Suppose that $p \geq m$. Then the normal rank of $M(z)$ is $n+m$ if and only if the normal rank $M_b(Z)$ is $n + Nm$.*

Proof. One can refer to [19] for a complete proof. □

The above lemma establishes a relation between the normal rank of $M(z)$ and the normal rank of $M_b(Z)$. In the following we recall the relation between zeros of these system matrices.

Theorem 6. *Suppose the system matrix of* (1) *has full-column normal rank. Then if* (1) *has a finite zero at $z = z_0 \neq 0$, then the system* (4) *has a finite zero at $Z = Z_0 = z_0^N \neq 0$. Conversely, if the system* (4) *has a finite zero at $Z = Z_0 = z_0^N \neq 0$, then the system* (1) *has a finite zero at one or more of $z = z_0 \neq 0$ or $z = \omega z_0 \neq 0, \ldots, z = \omega^{N-1} z_0 \neq 0$, where $\omega = \exp(\frac{2\pi j}{N})$.*

Proof. The proof is omitted and an interested reader can refer to [19] for a complete proof. □

So far we have related nonzero zeros of the system (4) and those of the system (1). Now, we present theorems which establish a relation between zeros of those aforementioned systems at infinity and the origin, see [19] for the proofs, which are straightforward.

Theorem 7. *Suppose the system matrix of* (1) *has full-column normal rank. Then the system* (4) *has a zero at* $Z_0 = \infty$ *if and only if the system* (1) *has a zero at* $z_0 = \infty$.

The above theorem treats the zeros of systems (4) and (1) at the infinity. In the theorem below, we deal with zeros of the blocked system (4) and its associated unblocked system (1) at the origin.

Theorem 8. *Suppose the system matrix of* (1) *has full-column normal rank. Then the system* (4) *has a zero at* $Z_0 = 0$ *if and only if the system* (1) *has a zero at* $z_0 = 0$.

3.2 Zero-free blocked system

The results in the previous subsection established a clear connection between zeros of the blocked system (4) and zeros of the associated unblocked system (1). Furthermore, in the first section, it was shown that the system (1) with $p > m$ and a choice of generic parameter matrices $[A, B, C, D]$, is zero-free. Now, to complete our analysis for the zero properties of tall blocked systems we provide the theorem below which studies the zero properties of the blocked system (4) obtained from blocking of a linear time-invariant system with generic parameter matrices. It is worthwhile remarking that parameter matrices of the blocked system (4) cannot be generic because they are structured.

We first need the lemma below which studies the normal rank of $M_b(Z)$ when matrices A, B, C, D assume generic values.

Lemma 9. *For a generic choice of matrices* $[A, B, C, D]$ *with* $p \geq m$*, the system matrix of* (4) *has normal rank equal to* $n + Nm$.

Proof. In the generic setting and $p \geq m$, matrix D is of full column rank. So, due to the structure of D_b, see (5), one can easily conclude that D_b is of full column rank as well. Then the result easily follows. □

Theorem 10. *Consider the system* (1) *defined by the quadruple* $[A, B, C, D]$*, in which the individual matrices are generic. Then*

1. *If* $p > m$*, the system matrix of the blocked system has full column rank for all* Z.
2. *If* $p = m$*, then the system matrix of the blocked system can only have finite zeros with one-dimensional kernel.*

Proof. Suppose first that $p > m$. Using the results of Lemma 9 and Lemma 5, it can be readily seen that the system matrix of tall unblocked systems generically have full-column normal rank. Furthermore, Theorem 3 shows that tall unblocked systems are generically zero-free. If the blocked system had its system matrix with less than full column rank for a finite $Z_0 \neq 0$, then according to Theorem 6, there would be necessarily a nonzero nullvector of the system matrix of the unblocked system for $z_0 \neq 0$ equal to some $N-th$ root of Z_0, which would be a contradiction. If the blocked system had a zero at $Z_0 = \infty$, then based on Theorem 8 the D matrix of the unblocked system would be less than full column rank which would be a contradiction. Analogously, using the argument in Theorem 6, one can easily conclude that the blocked system has full column rank system matrix at $Z_0 = 0$.

Now we consider the case $p = m$; since D is generic, it has full column rank. Hence, based on the conclusion of Theorem 8, both the unblocked system and the blocked system do not have zeros at infinity. In the second part of this proof we use the conclusion of Theorem 6. Furthermore, one should note that since the matrices A, B, C and D assume generic values it can be easily understood that the quadruple $[A_b, B_b, C_b, D_b]$ is a minimal realization. Now, based on the fact that D_b is nonsingular, one can conclude that the zeros of the blocked system are the eigenvalues of $A_b - B_b D_b^{-1} C_b$. If the eigenvalues of $A_b - B_b D_b^{-1} C_b$ are distinct, then the associated eigenspace for each eigenvalue is one-dimensional; it is equivalent to saying that the associated kernel of $M_b(Z)$ evaluated at the eigenvalue has dimension one. One should note that the unblocked system has distinct zeros due to the genericity assumption. Furthermore, zeros of the unblocked system generically have distinct magnitudes except for complex conjugate pairs. It is obvious that those zeros of the unblocked system with distinct magnitudes produce distinct blocked zeros. Now, we focus on zeros of the unblocked system with the same magnitudes, i.e. complex conjugate pairs. The only case where the generic unblocked system has distinct zeros but its corresponding blocked system has non-distinct zeros happens when the $N-th$ power of the complex conjugate zeros of the unblocked system coincide. We now show by contradiction that this is generically impossible. In order to illustrate a contradiction, suppose that the unblocked system has a complex conjugate pair, say z_{01} and z_{01}^*. If they produce an identical zero for the blocked system, their $N-th$ powers must be the same. The latter condition implies that the angle between z_{01} and z_{01}^* has to be exactly $\frac{2\pi h}{N}$, where h is an integer, which contradicts the genericity assumption for the unblocked system. Hence, the zeros of the blocked system generically have distinct values and consequently the corresponding kernels of system matrix evaluated at the zeros are one-dimensional. □

4 Multirate systems

In Section 2 we consider the system (1), in which y_t exists for all t, and, as a separate matter, can be measured at every time t. The zero properties of this system for a choice of generic parameter matrices were completely discussed in Section 2. Then in Section 3 it was shown that when the system (1) and its associated blocked system (4) are tall, they are generically zero-free. Now, in this section we are also interested in the situation where y_t exists for all t, but not every entry is measured for all t. In particular, we consider a case where y_t has components that are observed at different rates. For simplicity, a case where outputs are provided at two rates which we refer to as the fast rate and the slow rate is assumed. In this section, we use the result of the previous section to specify conditions under which blocked systems associated with multirate systems are generically zero-free.

As mentioned earlier we discuss the case where y_t has components that are measured at two rates so, without loss of generality we decompose y_t as

$$y_t = \begin{bmatrix} y_t^f \\ y_t^s \end{bmatrix}$$

where $y_t^f \in \mathbb{R}^{p_1}$ is observed at all t, the fast part, and $y_t^s \in \mathbb{R}^{p_2}$ is observed at $t = 0, N, 2N, \ldots$, the slow part, also $p_1 > 0, p_2 > 0$ and $p_1 + p_2 = p$. Accordingly, we decompose C and D as

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix}, D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.$$

Thus, the multirate linear system corresponding to what is measured has the following dynamics:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \quad t = 0, 1, 2, \ldots \\ y_t^f &= C^f x_t + D^f u_t \ t = 0, 1, 2, \ldots \\ y_t^s &= C^s x_t + D^s u_t \ t = 0, N, 2N, \ldots \end{aligned} \tag{7}$$

We have actually N distinct alternative ways to block the system, depending on how fast signals are grouped with the slow signals. Even though these N different systems share some common zero properties, their zero properties are not identical in the whole complex plane (see [3], pages 173-179).

For, $\tau \in \{1, 2, \cdots, N\}$, define

$$\begin{aligned} U_t^\tau &\triangleq \begin{bmatrix} u_{t+\tau} \\ u_{t+\tau+1} \\ \vdots \\ u_{t+\tau+N-1} \end{bmatrix}, \\ Y_t^\tau &\triangleq \begin{bmatrix} y_{t+\tau}^f \\ y_{t+\tau+1}^f \\ \vdots \\ y_{t+\tau+N-1}^f \\ y_{t+N}^s \end{bmatrix}, \ t = 0, N, 2N, \ldots \\ x_t^\tau &\triangleq x_{t+\tau}. \end{aligned} \tag{8}$$

Then the blocked system $\sum_\tau$ is defined by

$$\begin{aligned} x_{t+N}^\tau &= A_\tau x_t^\tau + B_\tau U_t^\tau \\ Y_t^\tau &= C_\tau x_t^\tau + D_\tau U_t^\tau, \end{aligned} \tag{9}$$

where,

$$
\begin{aligned}
A_\tau &\triangleq A^N,\\
B_\tau &\triangleq \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix},\\
C_\tau &\triangleq \begin{bmatrix} C^{f\top} & A^\top C^{f\top} & \dots & A^{(N-1)^\top} C^{f\top} & A^{(N-\tau)^\top} C^{s\top} \end{bmatrix}^\top,\\
D_\tau &\triangleq \begin{bmatrix} D^f & 0 & \dots & 0 \\ C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ C^s A^{N-\tau-1}B & \dots & D^s & * \end{bmatrix},
\end{aligned}
\tag{10}
$$

where "*" at the very right corner denotes $\tau-1$ zero matrices of size $p_2 \times m$ and when $N-\tau-1<0$, $C^s A^{-1}B$ is replaced by D^s and rest of the terms in the last row are replaced by zero matrices of size $p_2 \times m$.

Reference [3] defines a zero of (7) at time τ as a zero of its corresponding blocked system $\sum_\tau$ [1]. Hence, in the rest of this section we focus on the zero properties of the blocked system $\sum_\tau$, $\forall \tau \in \{1,2,\dots,N\}$.

Definition 11. The finite zeros of system $\sum_\tau$ are defined to be finite values of Z for which the rank of the following system matrix falls below its normal rank

$$M_\tau(Z) = \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}.$$

Further, $V_\tau(Z) = C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau$, $\tau \in \{1,2,\dots,N\}$, is said to have an infinite zero when $n + rank(D_\tau)$, $\tau \in \{1,2,\dots,N\}$, is less than the normal rank of $M_\tau(Z)$, $\tau \in \{1,2,\dots,N\}$, or equivalently the rank of D_τ, $\tau \in \{1,2,\dots,N\}$, is less than the normal rank of $V_\tau(Z)$, $\tau \in \{1,2,\dots,N\}$.

In addition to the above definition the following results from [6] and [7] are useful to the rest of this paper.

Lemma 12. *The pair (A,B) is reachable if and only if the pairs (A_τ, B_τ), $\forall \tau \in \{1,2,\dots,N\}$ are reachable.*

The above lemma studies the reachability property of $\sum_\tau$, $\forall \tau \in \{1,2,\dots,N\}$ and the lemma below explores its transfer function.

Lemma 13. *The transfer function $V_\tau(Z)$ associated with the blocked system (9) has the following property*

$$V_{\tau+1}(Z) = \begin{bmatrix} 0 & I_{p_1(N-1)} & 0 \\ ZI_{p_1} & 0 & 0 \\ 0 & 0 & I_{p_2} \end{bmatrix} V_\tau(Z) \begin{bmatrix} 0 & Z^{-1}I_m \\ I_{m(N-1)} & 0 \end{bmatrix},$$

where $\tau \in \{1,2\dots,N\}$.

[1] Zeros of the transfer function obtained from (9) and defined following [3] are identical with those defined here, provided the quadruple $\{A_\tau, B_\tau, C_\tau, D_\tau\}$ is minimal.

The result of the above lemma is crucial for the study of the zero properties of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$, for the choice of finite nonzero zeros. The latter is the main focus for the remainder of this section. We treat the zero properties of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$, under genericity and tallness assumptions. Given that $p_1, p_2 > 0$ and tallness is defined by $Np_1 + p_2 > Nm$, it proves convenient to consider partitioning the set of p_1, p_2 defining tallness into two subsets, as follows.

1. $p_1 > m$.
2. $p_1 \le m$, $Np_1 + p_2 > Nm$.

The first case is common, perhaps even overwhelmingly common in econometric modeling but the second case is important from a theoretical point of view, and possibly in other applications. Moreover, our results are able to cover both cases.

4.1 Case $p_1 > m$

According to Definition 11, the normal rank for the system matrix of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$, plays an important role in the analysis of its zero properties; thus, we make the following observation on the normal rank of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$ using the conclusion of Lemma 9.

Remark 14. For generic choice of the matrices $[A, B, C^s, C^f, D^f, D^s]$, $p_1 \ge m$, the system matrix of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$, has normal rank of $n + Nm$.

In the situation where $p_1 > m$, obtaining a result on the absence of finite nonzero zeros is now rather trivial, since the blocked system contains a tall subsystem obtained by deleting some outputs which is provably zero-free.

Theorem 15. *For a generic choice of the matrices $[A, B, C^s, C^f, D^s, D^f]$, $p_1 > m$, the system matrix of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$, has full column rank for all finite zero Z.*

Proof. With the help of the conclusion of Theorem 10, one can easily conclude that there is a submatrix of $M_\tau(Z)$, obtained by deleting rows of $M_\tau(Z)$ associated with slow part, which is full-column rank for all finite Z. Then the conclusion of the theorem easily follows. □

4.2 Case $p_1 \le m$, $Np_1 + p_2 > Nm$

In the previous subsection the case $p_1 > m$ was treated where only considering the fast outputs alone generically leads to a zero-free blocked system, and the zero-free property is not disturbed by the presence of the further slow outputs. A different way in which the blocked system will be tall arises when $p_1 \le m$ and $Np_1 + p_2 > Nm$. The main result of this subsection is to show that $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$ with $p_1 \le m$, $Np_1 + p_2 > Nm$ is again generically zero-free. In order to study the latter case we need to review properties of the Kronecker canonical form of a matrix pencil. Since the system matrix of $\sum_\tau$, $\forall\tau \in \{1,2,\ldots,N\}$ is actually a matrix pencil, the Kronecker canonical form turns out to be a very useful tool to obtain insight into the zeros of (9) and the structure of the kernels associated with those zeros.

The main theorem on the Kronecker canonical form of the matrix pencil is obtained from [17].

Theorem 16. *[17] Consider a matrix pencil $zR+S$. Then under the equivalence defined using pre- and postmultiplication by nonsingular constant matrices $\widetilde{P}$ and $\widetilde{Q}$, there is a canonical quasidiagonal form:*

$$\widetilde{P}(zR+S)\widetilde{Q} = \mathrm{diag}[L_{\varepsilon_1},\ldots,L_{\varepsilon_r},\tilde{L}_{\eta_1},\ldots,\tilde{L}_{\eta_s},zN-I,zI-K], \tag{11}$$

where:

1. *L_μ is the $\mu\times(\mu+1)$ bidiagonal pencil*

$$\begin{bmatrix} z & -1 & 0 & \ldots & 0 & 0 \\ 0 & z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & z & -1 \end{bmatrix}. \tag{12}$$

2. *$\tilde{L}_\mu$ is the $(\mu+1)\times\mu$ transposed bidiagonal pencil*

$$\begin{bmatrix} -1 & 0 & \ldots & 0 & 0 \\ z & -1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & z & -1 \\ 0 & 0 & \ldots & 0 & z \end{bmatrix}. \tag{13}$$

3. *N is a nilpotent Jordan matrix.*

4. *K is in Jordan canonical form.*

Note the possibility that $\mu = 0$ exists. The associated L_0 is deemed to have a column but not a row and $\tilde{L}_0$ is deemed to have a row but not a column, see [17].

The following corollary can be directly derived easily from the above theorem and provides detail about the vectors in the null space of the Kronecker canonical form. Because the matrices $\tilde{P}$ and $\tilde{Q}$ are nonsingular, it is trivial to translate these properties back to an arbitrary matrix pencil, including a system matrix.

Corollary 17.

1. *For all z except for those which are eigenvalues of K, the kernel of the Kronecker canonical form has dimension equal to the number of matrices L_μ appearing in the form; likewise the co-kernel dimension is determined by the number of matrices $\tilde{L}_\mu$.*

2. *The vector $[1\ z\ z^2 \ldots z^\mu]^\top$ is the generator of the kernel of L_μ, a set of vectors $[0\ldots0\ 1\ z\ z^2 \ldots z^\mu\ 0\ldots0]^\top$ are generators for the kernel of the whole canonical form which depend continuously on z, provided that z is not an eigenvalue of K; when z is an eigenvalue of K, the vectors form a subset of a set of generators.*

3. *When z equals an eigenvalue of K, the dimension of the kernel jumps by the geometric multiplicity of that eigenvalue, the rank of the pencil drops below the normal rank by that geometric multiplicity, and there is an additional vector or vectors in the kernel apart from those defined in point 2, which are of the form $[0\,0\ldots v^\top]^\top$, where v is an eigenvector of K. Such a vector is orthogonal to all vectors in the kernel which are a linear combination of the generators listed in the previous point.*

4. *When z is an eigenvalue, say z_0 of K, the associated kernel of the matrix pencil can be generated by two types of vectors: those which are the limit of the generators defined by adding extra zeros to vectors such as $[1\,z_0\,z_0^2\ldots,z_0^\mu]^\top$ (these being the limits of the generators when $z \neq z_0$ but continuously approaches z_0), and those obtained by adjoining zeros to the eigenvector(s) of K with eigenvalue z_0, the latter set being orthogonal to the former set.*

In the rest of this subsection, we explore the zero properties of $M_\tau(Z)$, $\forall\ \tau \in \{1, 2,\ldots,N\}$. To achieve this, we first focus on the particular case of $M_1(Z)$. Later, we introduce the main result for the zero properties of $M_\tau(Z)$, $\forall\,\tau \in \{1,2,\ldots,N\}$.

First we need to introduce some parameters. To this end, we argue first that the first $n+Np_1$ rows of $M_1(Z)$ are linearly independent. For the submatrix formed by these rows is the system matrix of the blocked system obtained by blocking the fast system defined by $\{A,B,C^f,D^f\}$, and accordingly has full row normal rank, since the unblocked system is generic and square or fat under the condition $p_1 \le m$. Now define the square submatrix of $M_1(Z)$:

$$N(Z) \triangleq \begin{bmatrix} ZI - A_1 & -B_1 \\ \mathcal{C}_1 & \mathcal{D}_1 \end{bmatrix}, \tag{14}$$

such that normal rank $N(Z)$ =normal rank $M_1(Z)$, by including the first $n+Np_1$ rows of $M_1(Z)$ and followed by appropriate other rows of $M_1(Z)$ to meet the normal rank and squareness requirements. Hence there exists a permutation matrix P such that

$$PM_1(Z) = \begin{bmatrix} N(Z) \\ \mathcal{C}_2\ \ \mathcal{D}_2 \end{bmatrix} \tag{15}$$

where $\mathcal{C}_2$ and $\mathcal{D}_2$ capture those rows of C_1 and D_1 that are not included in $\mathcal{C}_1$ and $\mathcal{D}_1$, respectively.

The zero properties of $N(Z)$ are studied in the following proposition.

Proposition 18. *Let the matrix $N(Z)$ be the submatrix of $M_1(Z)$ formed via the procedure described. Then for generic values of the matrices A,B, etc. with $p_1 \le m$ and $Np_1 + p_2 > Nm$, for any finite Z_0 for which the matrix $N(Z_0)$ has less rank than its normal rank, its rank is one less than its normal rank.*

Proof. The proof is omitted; an interested reader can refer to [18] for a complete proof. □

The result of the previous proposition, although restricted to $\tau = 1$, enables us to establish the following main result applicable for any τ.

Theorem 19. *Consider the system* $\sum_\tau$, $\forall \tau \in \{1,2,\ldots,N\}$, *with* $p_1 \le m$, *and* $Np_1 + p_2 > Nm$. *Then for generic values of the defining matrices* $\{A,B,C^f,D^f,C^s,D^s\}$ *the system matrix* $M_\tau(Z)$, $\forall \tau \in \{1,2,\ldots,N\}$, *has rank equal to its normal rank for all finite nonzero values of* Z_0, *and accordingly* Σ_τ *has no finite nonzero zeros.*

Proof. We first focus on the case $\tau = 1$. Now, apart from the $p_2 - N(m-p_1)$ rows of the C^s, D^s which do not enter the matrix $N(Z)$ defined by (14), choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let $Z_a, Z_b, \ldots$ be the finite set of Z for which $N(Z)$ has less rank than its normal rank (the set may have less than n elements, but never has more), and let $w_a, w_b, \ldots$ be vectors which are in the corresponding kernels (*not co-kernels*) and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$ etc. Now, due to the facts that $M_1(Z)$ and $N(Z)$ have the same normal rank and any existing vector in the kernel of $M_1(Z)$ is in the the kernel of $N(Z)$ one can conclude that the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$ etc, coincides with the subspace in the kernel obtained from the limit of the kernel of $M_1(Z)$ as $Z \to$ zeros of $M_1(Z)$.

Now, to obtain a contradiction, we suppose that the system matrix $M_1(Z)$ is such that, for $Z_0 \neq 0$, $M_1(Z_0)$ has rank less than its normal rank, i.e. the dimension of its kernel increases. Since the kernel of $M_1(Z_0)$ is a subspace of the kernel of $N(Z_0)$, Z_0 must coincide with one of the values of $Z_a, Z_b, \ldots$ and the rank of $M_1(Z_0)$ must be only one less than its normal rank; moreover, there must exist an associated nonzero w_1 unique up to a scalar multiplier, in the kernel of $M_1(Z_0)$ which is orthogonal to the limit of the kernel of $M_1(Z)$ as $Z \to Z_0$. Then w_1 is necessarily in the kernel of $N(Z_0)$, orthogonal to the limit of the kernel of $N(Z)$ as $Z \to Z_0$ and thus w_1 in fact must coincide to within a nonzero multiplier with one of the vectors $w_a, w_b, \ldots$.

Write this w_1 as

$$w_1 = \begin{bmatrix} x_1 \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \tag{16}$$

and suppose the input sequence $u(i) = u_i$ is applied for $i = 1,2\ldots,N$ to the original system, starting in initial state x_1 at time 1. Let $y^f(1), y^f(2), \ldots$ denote the corresponding fast outputs and $y^s(N)$ the slow output at time N. Break this up into two subvectors, $y^{s1}(N), y^{s2}(N)$, where $y^{s1}(N)$ is associated with those rows of C^s, D^s

which are included in $\mathcal{C}_1$, $\mathcal{D}_1$ and $y^{s2}(N)$ is related with the remaining rows of C^s and D^s . We have

$$N(Z_0)w_1 = \begin{bmatrix} Z_0 I_n - A^N & -A^{N-1}B & -A^{N-2}B & \dots & -B \\ C^f & D^f & 0 & \dots & 0 \\ C^f A & C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ C^{s1}A^{N-1} & C^{s1}A^{N-2}B & C^{s1}A^{N-3}B & \dots & D^{s1} \end{bmatrix} w_1$$
$$= \begin{bmatrix} Z_0 x_1 - x(N+1) \\ y^f(1) \\ y^f(2) \\ \vdots \\ y^f(N) \\ y^{s1}(N) \end{bmatrix} = 0. \tag{17}$$

Now it must be true that $x_1 \neq 0$. For otherwise, we would have $N(Z)w_1 = 0$ for all Z, which would violate assumptions. Since also $Z_0 \neq 0$, there must hold $x(N+1) \neq 0$. Hence there cannot hold both $x(N) = 0$ and $u(N) = 0$. Consequently, we can always find C^{s2}, D^{s2} such that $y^{s2}(N) = C^{s2}x(N) + D^{s2}u(N) \neq 0$, i.e. the slow output value is necessarily nonzero, no matter whether $w_1 = w_a, w_b$, etc. Equivalently, the equation $[\mathcal{C}_2 \ \mathcal{D}_2]w_1 = 0$ cannot hold. Hence, if $M_1(Z)$ defines a system with a finite zero and it is nonzero, this is a nongeneric situation. Hence, $M_1(Z)$ generically has rank equal to its normal rank for all finite nonzero Z. Now, we show that the latter property holds for all $M_\tau(Z)$, $\tau \in \{1,2,\dots,N\}$. First, note that the pair (A,B) is generically reachable so, according to Lemma 12 the pair (A_τ, B_τ), $\forall \tau \in \{1,2,\dots,N\}$, is also reachable. Consider $Z_\zeta \in \mathbb{C} - \{0,\infty\}$, if Z_ζ does not coincide with the eigenvalues of A_τ then

$$rank(M_\tau(Z_\zeta)) = n + rank(V_\tau(Z_\zeta)). \tag{18}$$

Hence, using the result of Lemma 13, it is immediate that $rank(M_\tau(Z_\zeta)) = rank(M_{\tau+1}(Z_\zeta))$. If Z_ζ does coincide with an eigenvalue of A_τ then $rank(V_\tau(Z_\zeta))$ is ill-defined. However, since zeros of $M_\tau(Z)$, $\tau \in \{1,2\dots,N\}$, are invariant under state feedback and the pair (A_τ, B_τ) is reachable, one can easily find a state feedback to shift that eigenvalue [20] and then (18) is a well-defined equation and $rank(M_\tau(Z_\zeta)) = rank(M_{\tau+1}(Z_\zeta))$. Thus, we can conclude that all $M_\tau(Z)$, $\tau \in \{1,2,\dots,N\}$ generically have no finite nonzero zeros. This ends the proof. □

The above theorem studies the case of finite nonzero zeros. The cases of zeros at the origin and at infinity seem to be more complicated because the structure of the system matrices depend on τ; furthermore, the various $M_\tau(Z)$, for $\tau \in \{1,2,\dots,N\}$, may not share the same zeros at those aforementioned points. Hence, these two points need special treatments. Here, we offer the following conjecture which partly treats the case of zeros at the origin and infinity.

Conjecture 20. Consider the system $\sum_\tau$, $\forall \tau \in \{1,2,\ldots,N\}$, with $p_1 < m$ and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $[A,B,C^f,D^f,C^s,D^s]$ the system matrix $M_\tau(Z)$, $\tau \in \{1,2,\ldots,N\}$ always has zeros at either $Z = 0$ or $Z = \infty$ or at both points.

The above conjecture has been proved for a particular case where the normal rank of the system matrix $M_\tau(Z)$ is equal to the number of columns. Furthermore, it is consistent with numerical examples. Here, we provide the following example which exhibits a very simple scenario and is consistent with the conclusion of the conjecture.

Example 21. Consider a tall multi-rate system with $n = 1$, $m = 3$, $N = 2$, $p_1 = 1$, $p_2 = 5$. Let the parameter matrices for the multi-rate system be $A = a$, $B = [b_1\, b_2\, b_3]$, $C = [{c^f}^\top {C^s}^\top]^\top$, $C^s = [c_1^s\, c_2^s\, c_3^s\, c_4^s\, c_5^s]^\top$, $D = [D^f\, D^s]$, $D^f = [d_1^f\, d_2^f\, d_3^f]$ and

$$D^s = \begin{bmatrix} d_{11}^s & d_{12}^s & d_{13}^s \\ \vdots & \vdots & \vdots \\ d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

First, consider $\tau = 1$ and write the associated system matrix as

$$M_1(Z) = \begin{bmatrix} Z-a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s a & c_1^s b_1 & c_1^s b_2 & c_1^s b_3 & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s a & c_2^s b_1 & c_2^s b_2 & c_2^s b_3 & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s a & c_3^s b_1 & c_3^s b_2 & c_3^s b_3 & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s a & c_4^s b_1 & c_4^s b_2 & c_4^s b_3 & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s a & c_5^s b_1 & c_5^s b_2 & c_5^s b_3 & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

It is obvious that first two rows are linearly independent. Now, consider the rows 3 to 8; they can be written as

$$\begin{bmatrix} c^f & c^f & c^f & c^f & d_1^f & d_2^f & d_3^f \\ c_1^s & c_1^s & c_1^s & c_1^s & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s & c_2^s & c_2^s & c_2^s & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s & c_3^s & c_3^s & c_3^s & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s & c_4^s & c_4^s & c_4^s & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s & c_5^s & c_5^s & c_5^s & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix} \operatorname{diag}(a,b_1,b_2,b_3,I_3) = G\operatorname{diag}(a,b_1,b_2,b_3,I_3)$$

The matrix G has rank at most 4; hence, with generic parameter matrices the normal rank of $M(Z)$ equals 6; furthermore, it is easy to observe that the system matrix has a zero at $Z = 0$. However, for $\tau = 2$ we can write the system matrix $M_2(Z)$ as

$$M_2(Z) = \begin{bmatrix} Z-a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s & d_{11}^s & d_{12}^s & d_{13}^s & 0 & 0 & 0 \\ c_2^s & d_{21}^s & d_{22}^s & d_{23}^s & 0 & 0 & 0 \\ c_3^s & d_{31}^s & d_{32}^s & d_{33}^s & 0 & 0 & 0 \\ c_4^s & d_{41}^s & d_{42}^s & d_{43}^s & 0 & 0 & 0 \\ c_5^s & d_{51}^s & d_{52}^s & d_{53}^s & 0 & 0 & 0 \end{bmatrix}.$$

Observe that the normal rank of the system matrix is still 6 and the matrix D_2 (with its nonzero entries assuming generic values) has rank 4; hence, the only zero of the system matrix is now at infinity.

5 Conclusions and future works

The zero properties of tall discrete-time multirate linear system were addressed in this paper. The zero properties of multirate linear systems were defined as those of their corresponding blocked systems. In this paper several required results from [1], [19] and [18] were reviewed in order to prove the main results about the zero properties of the blocked systems associated with multirate systems. In particular, it was illustrated that tall unblocked linear time-invariant systems are generically zero-free. Then, the zero properties of blocked systems associated with tall unblocked linear time-invariant systems were discussed and it was presented that tall blocked systems are generically zero-free. Finally, it was shown that tall blocked systems associated with multirate systems generically have no finite nonzero zeros. However, the behavior at $Z = 0$ and $Z = \infty$, turns out to be more complicated and we provided a conjecture which specifies a situation where tall blocked systems always have a zero at $z = 0$. As part of our future work, we intended to provide a formal proof for Conjecture 20. Moreover, we intend to generalize the results of this paper in respect of the output rates. More specifically, we are interested in the general case where there are two output streams, one available every ν time instants and the other every $\overline{\nu}$ time instants, with ν and $\overline{\nu}$ coprime integers with neither equal to 1.

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