

Decentralized Control Using Time-Varying Feedback

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I. FIXED MODES AND DECENTRALIZED CONTROL

A. WHAT ARE FIXED MODES?

Consider a finite-dimensional time-invariant linear system, possibly with multiple inputs and outputs, and suppose that the system is either not completely controllable or not completely observable. Then, as is well known, no matter what feedback controller one connects to the system, so long as the controller has available to it the system inputs and outputs and nothing more, the modes which are uncontrollable or unobservable will remain as modes of the closed-loop system. If the controller

is linear, finite-dimensional and time-invariant, this means that the closed-loop characteristic polynomial will have a zero or zeros which are independent of the particular control design, and which identify the system *fixed modes*. Such fixed modes also have the interpretation of being the eigenvalues associated with the uncontrollable or unobservable part of the system matrix. If the modes are unstable, there is then no way that the system can be controlled so as to achieve any of the common design objectives – be they based on optimal control, pole positioning, or whatever. On the other hand, if there are no fixed modes, or if there are such modes which are stable, it is possible to select one of an array of design procedures to advance a particular objective.

In this chapter we study the possibility of observing and controlling the modes of a system when the controllers are constrained to being *decentralized*. (The term is explained below.) Of particular concern to us will be the exploration of the fixed mode idea, introduced above in our recollection of results applying with conventional, or *centralized*, controllers. To do this, we must first clarify our meaning of the word decentralized, and this is best done with the aid of Fig. 1, which illustrates the *two-channel* case. The system's inputs and outputs are each supposed to be divided into two sets (any one scalar input or output can, if desired, be in more than one set). The control structure is then such that the channel 1 inputs can be derived by feedback of channel 1 outputs, but can in no way depend on channel 2 inputs or outputs. The analogous statement is true for the channel 2 inputs. Of course, one can also consider three-channel, four-channel, etc., generalizations. The key ideas however remain much the same.

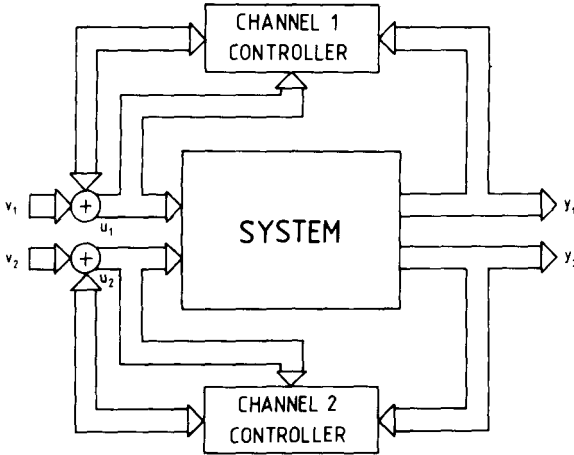


Fig. 1. Prototype system with decentralized controller.

Decentralized control structures are typically appropriate for large-scale systems. For example, most power systems normally contain more than one generator, and at each generator measurements are taken on that generator and local controls applied to it. Though in any power system, there is naturally some measure of centralized control, total centralization would be out of the question on grounds such as complexity, reliability, difficulty of update in times of expansion, and so on. Many social systems also exhibit elements of decentralized control – a company with several operating divisions for example, in which each operating division has autonomy.

Now suppose that the system being controlled is linear, finite-dimensional, and time-invariant. Suppose further that it is represented by state-variable equations which are controllable and observable. (This minimality is equivalent to there being no *centralized* fixed modes, or fixed modes of the type described in the first paragraph, if centralized controllers are used.) Next, suppose that decentralized controllers are used. Then there are some key consequences.

(1) If the controllers are restricted to being linear, and nondynamic, i.e., $u_i = K_i y_i + v_i$ for some constants K_i , it is possible for the closed-loop characteristic to have a zero s_0 which is independent of the K_i ; see [1]. An example appears in the next section.

(2) If the controllers are permitted then to be linear, time-invariant, and finite-dimensional, the new characteristic polynomial still has the same zero, s_0 , which is for obvious reasons termed a *decentralized fixed mode*.

By analogy with the centralized case, one might imagine that in these circumstances, one is encountering an unobservable or uncontrollable mode. Pursuing the analogy, one might therefore conclude that the use of nonlinear/time-varying/distributed controllers will not allow elimination of the decentralized fixed mode with characteristic frequency s_0 . As it turns out, however, the analogy is *not* safe to pursue: under circumstances set out later in the chapter, one can eliminate the mode — *by using linear, time-varying controllers*. This means that for linear, time-invariant systems which are to be controlled via a decentralized controller, the controller may have to be time-varying, if the closed loop is to be stable, or to have a prescribed degree of stability.

The first clue that such a result might be possible was provided by [2], which showed that, using a mixed closed-loop/open-loop decentralized strategy the state of a wide class of systems could be forced to zero. This result allowed an arbitrary initial state, and encompassed some systems with decentralized fixed modes (though not all). If it were true that the decentralized fixed mode had the same characteristics and properties as the centralized fixed mode, the result would not have been achievable.

We have referred above to the possibility of eliminating a decentralized fixed mode by using a time-varying controller. It perhaps should be emphasized that removing an unstable fixed mode is one thing; specifying a decentralized controller design to achieve, for example, a prescribed degree of stability is another. Once we have described how the fixed mode can be eliminated, we shall describe how controller design can be approached. However, we caution the reader that we are far from being able to answer such questions as: How much performance do we lose, or more generally how do we characterize the various trade-offs, in replacing a centralized controller by a decentralized controller?

B. DECENTRALIZED FIXED MODES - AN EXAMPLE

Consider the transfer fraction matrix

$$W(s) = \begin{bmatrix} \frac{1}{s(s+1)} & \frac{1}{s-1} \\ \frac{s-1}{s(s+1)} & \frac{1}{s+1} \end{bmatrix}. \tag{1}$$

It is not hard to check that a minimal state variable realization is defined by

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2a}$$

$$y = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} x. \tag{2b}$$

Now suppose decentralized constant feedback is applied, i.e.,

$$u_1 = k_1 y_1 + v_1 \tag{3a}$$

$$u_2 = k_2 y_2 + v_2. \tag{3b}$$

Here, k_1 and k_2 are constant gains, and v_1 and v_2 are external inputs. The feedback is decentralized since u_1 is allowed to depend only on y_1 (rather than y_1 and y_2), and u_2 is allowed to depend only on y_2 . The closed-loop system matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -k_2 & 2k_2 & k_2 \\ k_1 & k_1 & -k_1 & 0 \\ k_1 & k_1 & -1 - k_1 & 0 \\ 0 & -k_2 & 2k_2 & -1 + k_2 \end{bmatrix}.$$

The closed-loop characteristic polynomial is now the characteristic polynomial of this matrix, which can be evaluated as

$$(s - 1) \left[s^3 + s^2(2 - k_2) + s(1 - k_1 - k_2 - k_1 k_2) + k_1 - 2k_1 k_2 \right].$$

The key point is that $s = 1$ is a zero of the closed-loop characteristic polynomial, irrespective of the values assumed by k_1 , k_2 . As such, $s = 1$ is an example of a decentralized fixed mode.

C. CHARACTERIZATION OF DECENTRALIZED FIXED MODES

The first formal treatment of decentralized fixed modes appeared in [1]. However, this reference did not indicate how one could characterize fixed modes without first evaluating the characteristic polynomial of the closed-loop system and then observing somehow that it had a zero that was invariant with respect to k_1 , k_2 selections. We indicate here several characterizations.

State Variable Characterization [3]

Suppose the system is

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad y_1 = C_1' x, \quad y_2 = C_2' x \quad (4)$$

and is minimal. (The superscript prime denotes matrix transposition.) Suppose also the feedback is of the form $u_i = K_i y_i + v_i$, $i = 1, 2$. Then there is a decentralized fixed mode if and only if for some eigenvalue λ of A , one of the following conditions holds:

$$\text{rank} \begin{bmatrix} \lambda I - A & B_1 \\ C_2' & 0 \end{bmatrix} < \dim A, \quad \text{rank} \begin{bmatrix} \lambda I - A & B_2 \\ C_1' & 0 \end{bmatrix} < \dim A. \quad (5)$$

Remarks.

(1) For the example above, take $\lambda = 1$ and check that the first condition holds:

$$\begin{aligned} & \text{rank} \begin{bmatrix} \lambda I - A & B_1 \\ C_2' & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{bmatrix} = 3 < 4 = \dim A. \end{aligned} \quad (6)$$

(2) More than two channels can be considered; see [3] for details of the form taken by the extensions to (5).

(3) The conditions (5) are a subset of those necessary to ensure that a decentralized system is observable and controllable from one channel alone, after constant gain feedback has been applied around the second channel [4-6]. This idea is important for the sequel, so we develop it further here. But observe first that for the example of Section II, if $u_2 = k_2 y_2$,

then (as shown by simple algebra) there is lack of controllability from u_1 , irrespective of k_2 . Now to understand the result of [4-6] further, suppose the second condition in (5) holds for $x = s_0$, an eigenvalue of A . Since

$$\begin{bmatrix} s_0 I - A & B_2 \\ C_1' & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -K_2 C_2' & I \end{bmatrix} = \begin{bmatrix} s_0 I - A - B_2 K_2 C_2' & B_2 \\ C_1' & 0 \end{bmatrix}$$

the rank of the right-hand side is less than $\dim A$ for all K_2 . A fortiori

$$\text{rank} \begin{bmatrix} s_0 I - A - B_2 K_2 C_2' \\ C_1' \end{bmatrix} < \dim A. \quad (7)$$

This is a condition for unobservability of the pair $[A + B_2 K_2 C_2', C_1']$. Since it is perhaps not well known, we indicate how it relates to a more familiar condition. From (7) we conclude there exists $w \neq 0$ with

$$\begin{bmatrix} s_0 I - A - B_2 K_2 C_2' \\ C_1' \end{bmatrix} w = 0$$

whence

$$C_1' w = 0, \quad (A + B_2 K_2 C_2')^i w = s_0^i w$$

and so

$$C_1' (A + B_2 K_2 C_2')^i w = 0.$$

In summary, the second condition of (5) implies that if feedback $u_2 = K_2 y_2$ is applied around channel 2, the resulting one-channel system is unobservable for all K_2 . The unobservable mode is s_0 , while the unobservable state may depend on K_2 .

It is also easy to see that if the second condition of (5) holds and feedback $u_1 = K_1 y_1$ is used to create a one-channel system with input u_2 and output y_2 , then this system will be uncontrollable for all K_1 . Similar results also hold if the first condition in (5) holds.

The main result of [4-6] proves this type of result both ways. We shall note the precise formulation subsequently.

Matrix Fraction Characterization [3]¹

Suppose that $W(s) = A^{-1}(s)B(s)$ is a left matrix fraction description of $W(s)$, with $A(s)$, $B(s)$ coprime polynomial matrices. Define $A_i(s)$, $B_i(s)$ for $i = 1, 2$ by rewriting $A(s)y(s) = B(s)u(s)$ as

$$A_1(s)y_1 + A_2(s)y_2 = B_1(s)u_1 + B_2(s)u_2. \quad (8)$$

Then there is a decentralized fixed mode at s_0 , which is necessarily a zero of $\det A(s)$, if and only if one of the following conditions holds:

$$\begin{aligned} \text{rank}[A_1(s_0) \quad B_1(s_0)] &< \dim A_1(s_0) \\ \text{rank}[A_2(s_0) \quad B_2(s_0)] &< \dim A_2(s_0). \end{aligned} \quad (9)$$

Transfer Fraction Matrix Characterization [7]

Suppose that

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} W_{11}(s) & W_{12}(s) \\ W_{21}(s) & W_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}. \quad (10)$$

Then if s_0 is a simple zero of the open-loop characteristic polynomial, it is also a decentralized fixed mode if and only

¹This section can be omitted by the reader unfamiliar with matrix fraction descriptions of linear finite-dimensional systems.

if the following is true:

$$W(s) = \left[\begin{array}{c|c} \text{No entry has a pole at } s_0 & s_0 \text{ is a simple zero of characteristic polynomial of this block} \\ \hline \text{Every entry has a zero at } s_0 & \text{No entry has a pole at } s_0 \end{array} \right]. \tag{11}$$

Should s_0 be a multiple zero of the open-loop characteristic polynomial, the transfer function matrix characterization, in contrast to the state variable and matrix fraction characterizations, becomes more complicated. See [7] for details.

The example of Section I,B is immediately seen as conforming to the pattern for $W(s)$ in Eq. (11), with $s_0 = 1$.

There is an important intuitive idea behind the above pattern for $W(s)$. Let us suppose that $W(s)$ is a 2×2 matrix, and that we apply feedback in two stages. In the first stage feedback $u_2 = k_2 y_2$ is applied around channel 2 as illustrated in Fig. 2. In the second stage (not illustrated), feedback $u_1 = k_1 y_1$ will be applied around the resulting one-channel system.

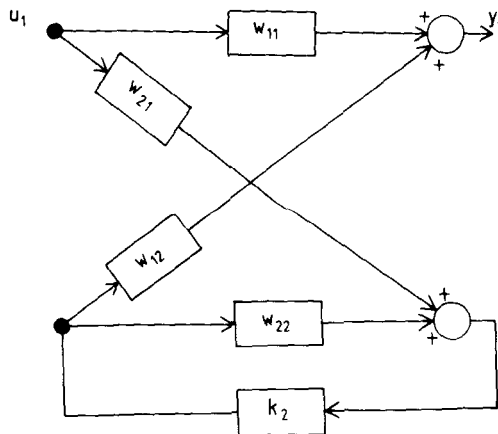


Fig. 2. System with control round channel 2.

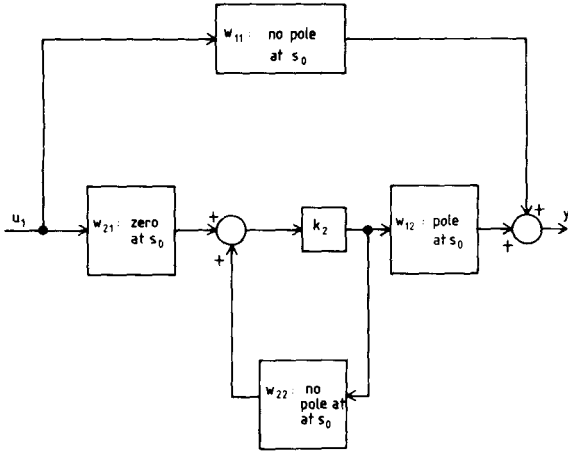


Fig. 3. Redrawing of scheme of Fig. 2.

Consider the redrawing of the Fig. 2 scheme as in Fig. 3 with $W(s)$ possessing the structure of Eq. (11). It is immediately clear that there is a pole-zero cancellation between w_{21} and w_{12} , and that this cancellation exists for all k_2 . Therefore, the one-channel system will be uncontrollable. (Of course, if w_{21} has the pole and w_{12} the zero, s_0 will be an unobservable mode for the one-channel system.) Consequently, whatever value of k_1 is selected for the feedback around the one-channel system, s_0 will remain as a zero of the closed-loop characteristic polynomial. Since s_0 is then invariant with respect to the choice of k_1 and k_2 , it is a decentralized fixed mode.

Figure 3 also illustrates, at least intuitively, several other important ideas.

- (1) If $w_{21}(s) \equiv 0$, then plainly there is no decentralized control strategy which could vary any of the poles of w_{12} which are not poles of w_{11} or w_{22} . Thus a prerequisite for eliminating a fixed mode associated with a pole of w_{12} only is that

$w_{21} \neq 0$; similarly, a prerequisite for eliminating a fixed mode associated with a pole of w_{21} is that $w_{12} \neq 0$. Such conditions are termed *connectivity conditions*.

(2) If k_2 is not a constant feedback law, but is replaced by $k_2(s)$, the transfer function of a time-invariant dynamic system, the pole-zero cancellation will still take place, and so the fixed mode will remain. For rational $k_2(s)$, this is established in [1] and for possibly irrational $k_2(s)$, in [3].

(3) If k_2 is a time-varying operator of some description, for example, a periodically switched, piecewise constant gain, then the pole-zero cancellation will *in general no longer occur*.

This is because, as is well known, a linear time-invariant integro-differential operator and a linear time-varying operator do not commute [example: With $D \equiv d/dt$,

$$Dk(t)[x(t)] = k(t)D[x(t)] + \dot{k}(t)x(t),$$

i.e., $Dk(t) \neq k(t)D$ unless $\dot{k}(t) \equiv 0$]. Such commutativity would be needed in Fig. 3 in order to juxtapose w_{21} and w_{12} . Of course, without the pole-zero cancellation, there is no fixed mode. So this argument suggests that we could eliminate the fixed mode by making k_2 time varying.

(4) Let us consider a different form of intuitive argument for the same conclusion as the remark above. With w_{12} possessing an unstable pole at s_0 , we know that to stabilize it, there would need to be a possibility for signals at the frequency s_0 to enter w_{12} . On the other hand signals with frequency s_0 cannot pass through w_{21} . Now suppose that k_2 varies sinusoidally at frequency $j\omega_0$, where $s_0 + j\omega_0$, $s_0 - j\omega_0$ are not zeros of w_{21} . Then if a signal at frequency $s_0 \pm j\omega_0$ enters w_{21} , it will leave it, and then be modulated by k_2 with the result that the

output of k_2 can contain a frequency s_0 . This is the prerequisite for eliminating the unstable pole. This intuitive argument also gives some insight into what sort of period would be appropriate: one should select $T = 2\pi/\omega_0$ so that $s_0 \pm j\omega_0$ is not close to s_0 , or, for that matter, to any other zero s_1 of w_{21} . If the time variation in k_2 is not sinusoidal, but nevertheless periodic, the conclusion is less clear, since the "modulating action" is more complex.

Note that these arguments do not show definitively that one is able to eliminate a fixed mode, let alone that one can, for example, stabilize using a decentralized controller a system with an unstable fixed mode. They simply open the door to showing that it might be possible, as was done in [8] and as explained in the remainder of this chapter.

II. ELIMINATING DECENTRALIZED
FIXED MODES WITH TIME-VARYING
CONTROLLERS

A. THE TECHNICAL LEMMAS

We consider a system in state variable form:

$$\dot{x} = Ax + B_1u_1 + B_2u_2, \quad y_1 = C_1'x, \quad y_2 = C_2'x \quad (12)$$

and begin by asking the question, *When, with feedback of the form $u_2 = K_2(t)y_2$, will the system be observable from y_1 ?* We first derive two necessary conditions.

It is clear on intuitive grounds that we need centralized observability as

$$\{A, [C_1' \ C_2']'\} \text{ is completely observable.} \quad (13)$$

Formally, if this condition were not satisfied, $\{A + B_2K_2(t)C_2', [C_1' \ C_2']'\}$ would not be observable (observability is preserved

with output feedback), and a fortiori $\{A + B_2 K_2(t) C_2', C_2\}$ would be unobservable.

The second necessary condition is

$$\text{Either } [A, C_1] \text{ is observable, or } C_1'(sI - A)^{-1} B_2 \neq 0. \quad (14)$$

If the first alternative holds, then it is trivial that $K_2(t) \equiv 0$ ensures observability from output 1. [Actually, it then follows that almost any constant control law $u_2 = K_2 y_2$ ensures observability. For the observability condition can be formally viewed as one requiring that the rank of a certain matrix take a certain value, where the entries of this matrix are entries of $C_1'(A + B_2 K_2 C_2')^i$ for $i = 0, \dots, \dim A - 1$. Regarding the entries of K_2 as variables, observability then follows if and only if at least one of a number of determinants, which evaluate as multivariable polynomials in the entries of K_2 , is nonzero. If a multivariable polynomial is nonzero for one choice of variables, it will be nonzero for almost all such choices. The selection $K_2 \equiv 0$ guarantees one of the polynomials is nonzero.]

Suppose then the first alternative in Eq. (14) fails. We first give an intuitive argument for the necessity of the second alternative. With the first alternative failing, it is evidently necessary to somehow pass information contained in the observations at the second output to the first output. This is done by passing the y_2 information to the second input, via $u_2 = K_2 y_2$, and then allowing the internal system connection from u_2 to y_1 - defined by the transfer function matrix $C_1'(sI - A)^{-1} \times B_2$ - to carry the information right through to y_1 . If, however, $C_1'(sI - A)^{-1} B_2 \equiv 0$, no information can get through via this mechanism.

Now we turn to a formal argument for the necessity of Eq. (14). With $u_2 = K_2(t)y_2$, the closed-loop transition matrix $\phi_{K_2}(t, \sigma)$ satisfies.

$$\dot{\phi}_{K_2}(t, \sigma) = A\phi_{K_2}(t, \sigma) + B_2K_2(t)C_2'\phi_{K_2}(t, \sigma)$$

whence

$$\phi_{K_2}(t, \sigma) = e^{A(t-\sigma)} + \int_{\sigma}^t e^{A(t-\tau)} B_2K_2(\tau)C_2'\phi_{K_2}(\tau, \sigma) d\tau.$$

Now if $C_1'(sI - A)^{-1}B_2 \equiv 0$, then $C_1'e^{A(t-\sigma)}B_2 \equiv 0$ for all t, σ , and we have

$$C_1'\phi_{K_2}(t, \sigma) = C_1'e^{A(t-\sigma)}.$$

The observability Grammian associated with $[A + B_2K_2(t)C_2', C_1]$ is now

$$\begin{aligned} W(s, s + L) &\triangleq \int_s^{s+L} \phi_{K_2}'(t, s)C_1'C_1'\phi_{K_2}(t, s) dt \\ &= \int_s^{s+L} [e^{A(t-s)}]_1' C_1'C_1'e^{A(t-s)} dt \end{aligned} \quad (15)$$

and is accordingly nonsingular for all $L > 0$ if and only if $[A, C_1]$ is observable.

Having now seen the two necessary conditions Eq. (13) and Eq. (14) which are prerequisites to securing observability from channel 1 via feedback around channel 2, it is appropriate to recall the precise form of the result of [4-6], which indicates when constant feedback around channel 2 will suffice.

Theorem 1. Consider the system Eq. (12). Then Eq. (13), Eq. (14), and

$$\text{rank} \begin{bmatrix} \lambda I - A & B_2 \\ C_1' & 0 \end{bmatrix} \geq \dim A \quad (16)$$

for all λ constitute necessary and sufficient conditions for the existence of a K_2 such that $[A + B_2K_2C_2', C_1]$ is observable.

We stress that Eqs. (13) and (14) have been shown to be necessary conditions for the existence of a control law with K_2 constant or time-varying, which achieve observability of $[A + B_2K_2C_2', C_1]$. However, we have not asserted that Eq. (16) is necessary for the existence of a time-varying gain, and indeed the thrust of this chapter is to show that Eqs. (13) and (14) alone are the necessary and sufficient conditions for the existence of a time-varying gain guaranteeing observability of $[A + B_2K_2(t)C_2', C_1]$. Even should Eq. (16) fail, i.e., even if there are decentralized fixed modes, there is, as we shall show, no difficulty in achieving observability.

Our first result is for plants with u_2, y_2 scalar:

Lemma 1. Assume the observability condition Eq. (13) holds and $C_1'(sI - A)^{-1}B_2 \neq 0$. Let $K_2(t) \equiv 0$ on $[s, s_1]$ and $K_2(t) \equiv K_2 \neq 0$ on $[s_1, s + L]$ for arbitrary s_1, L with $s < s_1 < s + L$. Suppose further that u_2, y_2 are scalar. Then $W(s, s + L)$, see Eq. (15), is nonsingular.

Proof. Suppose temporarily that $C_1'(sI - A)^{-1}B_2$ is scalar and that $W(s, s + L)\gamma = 0$ for some $\gamma \neq 0$, to obtain a contradiction. Using the definition of $W(\cdot, \cdot)$ in Eq. (15) (first equality only!), we see that

$$C_1'e^{A(t-s)}\gamma = 0, \quad t \in [s, s_1]$$

$$C_1'e^{(A+B_2K_2C_2')(t-s_1)}e^{As_1}\gamma = 0, \quad t \in [s_1, s + L].$$

Set $\delta = e^{As_1}\gamma$ and recognize that $\alpha'e^{At}\beta \equiv 0$ on an interval if

and only if $\alpha' A^i \beta = 0 \forall i$. Then for all i

$$C_1' A^i \delta = 0, \quad C_1' (A + B_2 K_2 C_2')^i \delta = 0.$$

Let q be the least nonnegative integer for which $C_1' A^q B_2 \neq 0$.

Note that q exists since $C_1' (sI - A)^{-1} B_2 \neq 0$. Then the second equation implies (using the first) that

$$\begin{bmatrix} C_1' A^q B_2 K_2 & 0 & \cdot & \cdot \\ * & C_1' A^q B_2 K_2 & \cdot & \cdot \\ * & * & C_1' A^q B_2 K_2 & \cdot \\ * & * & * & \cdot \\ & & & \cdot \end{bmatrix} \begin{bmatrix} C_2' \\ C_2' A \\ C_2' A^2 \\ \cdot \\ \cdot \end{bmatrix} \delta = 0$$

whence for all i

$$C_2' A^i \delta = 0. \tag{17}$$

But now $C_1' A^i \delta = 0$ and $C_2' A^i \delta = 0$, with $\delta \neq 0$. This contradicts the observability assumption (13).

Now suppose $C_1' (sI - A)^{-1} B_2$ is a vector, and let C_{1j}' denote the j th row of C_1 , and

$$W_j(s, s + L) \triangleq \int_s^{s+L} \phi_{K_2}'(t, s) C_{1j}' C_{1j}' \phi_{K_2}(t, s) dt. \tag{18}$$

Then

$$W(s, s + L) = \sum_j W_j(s, s + L) \tag{19}$$

and $W(s, s + L)\gamma = 0$ if and only if $W_j(s, s + L)\gamma = 0$, since each $W_j(s, s + L)$ is individually nonnegative definite. Arguing as above, we conclude that for all j, i

$$C_{1j}' A^i \delta = 0, \quad C_2' A^i \delta = 0 \tag{20}$$

and again, Eq. (13) is violated.

∇∇∇

In Lemma 1, u_2 and y_2 are restricted to being scalars. We now remove this restriction. The idea is as follows. Suppose that $C_1'(sI - A)^{-1}B_{2j} \neq 0$. Then we use $K_2(t)$ to switch in turn each component of y_2 to the j th component of u_2 . More specifically:

Lemma 2. With notation as above, suppose that Eq. (13) holds and that the j th column of $C_1'(sI - A)^{-1}B_2$ is not identically zero. Let e_i denote a unit vector with 1 in the i th position, and let p_2 denote the dimension of y_2 . Take

$$\begin{aligned}
 K_2(t) &\equiv 0, & t \in [s, s_1] \\
 K_2(t) &\equiv k_{12}e_j e_1', & t \in [s_1, s_2] \\
 K_2(t) &\equiv k_{22}e_j e_2', & t \in [s_2, s_3] \\
 &\vdots \\
 K_2(t) &\equiv k_{p_2 2}e_j e_{p_2}', & t \in [s_{p_2}, s + L]
 \end{aligned}
 \tag{21}$$

with $s < s_1 < s_2 < \dots < s + L$, and $k_{i2} \neq 0$ for all i . Then $W(s, s + L)$ is nonsingular.

Proof. Suppose for convenience that C_1' has only one row. In the contrary case, the argument can be extended, just as for Lemma 1. Then $W(s, s + L)\gamma = 0$ implies

$$\begin{aligned}
 C_1' \exp[A(t - s)]\gamma &= 0, & t \in [s, s_1] \\
 C_1' \exp[(A + B_{2j}k_{12}e_1' C_2')(t - s_1)] \exp(As_1)\gamma &= 0, & t \in [s_1, s_2] \\
 C_1' \exp[(A + B_{2j}k_{22}e_2' C_2')(t - s_2)] \exp[(A + B_{2j}k_{12}e_1' C_2')(s_2 - s_1)] \\
 &\times \exp(As_1)\gamma = 0, & t \in [s_2, s_3] \\
 &\vdots
 \end{aligned}
 \tag{22}$$

Arguing as in the proof of Lemma 1, the first two equations imply, for $\delta_1 = e^{As_1} \gamma$, and all i

$$C_1' A^i \delta_1 = 0, \quad e_1' C_2' A^i \delta_1 = 0.$$

Let $\delta_2 = \exp[(A + B_2 k_{12} e_1' C_2') (s_2 - s_1)] \delta_1$. Then it is easy to see that

$$C_1' A^i \delta_2 = 0, \quad e_1' C_2' A^i \delta_2 = 0. \tag{23}$$

Now take the first of these identities with Eq. (22). One obtains by the same arguments

$$e_2' C_2' A^i \delta_2 = 0.$$

Proceeding in this fashion, we construct a vector δ_{p_2} , nonzero if and only if γ is nonzero, such that, for $j = 1, 2, \dots, p_2$, and all i

$$C_1' A^i \delta_{p_2} = 0, \quad e_j' C_2' A^i \delta_{p_2} = 0. \tag{24}$$

Equivalently,

$$C_1' A^i \delta_{p_2} = 0, \quad C_2' A^i \delta_{p_2} = 0. \tag{25}$$

This provides a contradiction to an assertion that $W(s, s + L)$ is nonsingular. VVV

Let us summarize. To have $W(s + L)$ nonsingular, it is enough

- (1) to have $[A, C_1]$ observable (then one can take $K_2(t) \equiv 0$);
- (2) if $[A, C_1]$ is not observable, to have $C_1' (sI - S)^{-1} B_2 \neq 0$ and to have

$$\text{rank} \begin{bmatrix} \lambda I - A & B_2 \\ C_1' & 0 \end{bmatrix} \geq \dim A$$

for all λ (then almost all constant K_2 will suffice);

(3) if $[A, C_1]$ is not observable, and if

$$\text{rank} \begin{bmatrix} s_0 I - A & B_2 \\ C_1' & 0 \end{bmatrix} < \dim A$$

for some s_0 (which is then a decentralized fixed mode), to have $C_1'(sI - A)^{-1}B_2 \neq 0$ (then a switched K_2 of the form described in Lemmas 1 and 2 will suffice).

Let us add one final piece of intuition to points (2) and (3) above. If $[A, C_1]$ is not observable, we need to get the information seen by channel 2 through somehow to the output of channel 1. If $K_2 = 0$, then on channel 1, we are observing only what C_1 allows us to observe. But if $K_2 \neq 0$, the output information on channel 2 is fed back to influence the input of channel 2 and then, because $C_1'(sI - A)^{-1}B_2 \neq 0$, some of this shows up at the channel 1 output. That part of the state which C_1 could not observe (and which C_2 can then observe) shows up, in an admittedly mangled fashion at output 1; as it turns out, the processing can sometimes be unraveled and the state then observed.

However, it may be the case that the separate contributions to y_1 [the direct contribution associated with $C_1'(sI - A)^{-1}B_1$ and that due to feedback of y_2 and u_2 and transmission through $C_1'(sI - A)^{-1}B_2$] are not separately resolvable when K_2 is constant. Why then will time-varying K_2 allow resolution? If $K_2 = 0$, the first contribution alone is present, and can be evaluated; then when $K_2 \neq 0$, the first contribution can be, in crude terms, subtracted off, allowing evaluation of the second contribution.

This is not inconsistent with there being a fixed mode with frequency s_0 ; for one value of K_2 , one particular state is unobservable, and for a different value of K_2 , a different state

is unobservable. No one state is unobserved for all values of K_2 , so that time variation of K_2 permit observation of all states.

B. FORMAL RESULTS

Let us suppose that there is given the decentralized two-channel system $\{A, [B_1 \ B_2], [C_1 \ C_2]\}$ where $\{A, [C_1' \ C_2']'\}$ is completely observable, $C_1'(sI - A)^{-1}B_2 \neq 0$, and there is a decentralized fixed mode s_0 such that

$$\text{rank} \begin{bmatrix} s_0 I - A & B_2 \\ C_1' & 0 \end{bmatrix} < \dim A.$$

Let $K_2(\cdot)$ be periodic with period L and vary as described in Lemma 2. Let τ be an arbitrary time and consider any interval $[\tau, \tau + 2L]$. Then this contains a subinterval $[s + mL, s + (m + 1)L]$ for some integer m , and so

$$W(\tau, \tau + 2L) \geq W(s + mL, s + (m + 1)L) = W(s, s + L).$$

Thus $W(\tau, \tau + 2L)$ is bounded below away from zero uniformly for $\tau \in (-\infty, \infty)$. It is also bounded above for all τ , as is easily seen. Hence the pair $[A + B_2 K_2(t) C_2', C_1']$ is uniformly completely observable [9-11]. (The significance of this is explained further below.)

Of course, if $[A, C_1]$ is observable, or if it is not observable but there is no fixed mode of the type described, the above conclusion is still valid. However, the use of a time-varying feedback law is not essential to achieve observability.

It is not hard now to see that if $K_2(t)$ takes any ρ differing piecewise constant values where $\rho > p_2 + 1$, and is periodic, the uniform complete observability condition still holds. The observability is characterized by the failure of certain multivariable equalities in the entries of K_2 at each of its constant values. Now, there exist periodic gain selections for

which the equalities do not all hold, so that for almost all such selections they will fail to hold. (To extend from $p_2 + 1$ to $p_2 + 2$ and then $p_2 + 3, \dots$ distinct values of K_2 , regard any set of $p_2 + 1$ values as a set of $p_2 + 2$ specialized so that two values are the same.)

Now consider the controllability question. By duality, it is clear that we need to require $\{A, [B_1 \ B_2]\}$ controllable. If $[A, B_1]$ is controllable, then almost any $K_2(t)$, constant or time-varying and including $K_2(t) \equiv 0$, will ensure controllability from channel 1. If $[A, B_1]$ is not controllable, it is necessary that $C_2'(sI - A)^{-1}B_1 \neq 0$. Then if there are no fixed modes, almost any fixed K_2 will ensure that $[A + B_2K_2C_2', B_1]$ is completely controllable. If there are fixed modes, then K_2 should take at least $\dim u_2 + 1$ distinct piecewise constant values in a periodic fashion. For almost all selections of such values, $[A_2 + B_2K_2(t)C_2', B_1]$ will be uniformly completely controllable.

Notice that with K_2 nonzero and with $C_2'(sI - A)^{-1}B_1 \neq 0$, any generic input applied on channel 1 will give rise to a signal at the input of channel 2. In this way, even though $[A, B_1]$ may not be controllable, one will have some opportunity to influence all the states.

By combining the controllability and observability results we see that if $\{A, [B_1 \ B_2], [C_1 \ C_2]\}$ is controllable and observable, and if $C_2'(sI - A)^{-1}B_1 \neq 0$, $C_1'(sI - A)^{-1}B_2 \neq 0$ then the following applies:

- (1) *If there are no fixed modes, almost any constant K_2 will ensure $[A + B_2K_2C_2', B_1, C_1]$ is controllable and observable.*

(2) If there are fixed modes almost any selection of a piecewise constant, periodic $K_2(t)$ taking at least $\max(\dim u_2 + 1, \dim y_2 + 1)$ distinct values will ensure that $[A_2 + B_2 K_2(t) C_2', B_1, C_1]$ is uniformly completely controllable and observable.

If $C_2'(sI - A)^{-1}B_1 \equiv 0$, we need $[A, B_1]$ completely controllable and if $C_1'(sI - A)^{-1}B_2 \equiv 0$, we need $[A, C_1]$ completely observable for these conclusions to remain true.

What now is the significance of these conditions? If (1) holds, then conventional linear system design methods can be applied to channel 1 to position poles, minimize a linear-quadratic index, etc. If (2) holds, then linear-quadratic design methods may be applied [11] to secure an exponentially stable state estimator and an exponentially stabilizing state feedback law. The estimator and feedback law will then have periodic gains [12].

In case we work in discrete time, the same results hold, save that $[A + B_2 K_2(t) C_2', B_1, C_1]$ may have uncontrollable/unobservable modes at the origin. (These modes of course are unlikely to create any difficulties in a control problem.) Since any periodic discrete-time system can be treated as a time-invariant system (by redefining the sampling interval as one period), other techniques of linear system theory, e.g., those appropriate to pole positioning, can be applied to the single-channel design problem.

Another important observation is that even for a system with no decentralized fixed modes, it may be easier to control if time-varying feedback is used on channel 2 than if constant feedback is used. Indeed, this will certainly be the case if the given system is close to having a fixed mode, in the sense,

for example, that the structure of Eq. (11) for $W(s)$ could be achieved if poles or zeros were perturbed slightly.

Last, we comment that we have discovered examples where with $m_2 = p_2 > 1$, only two different piecewise constant values are required for $K_2(t)$, in order to make a system with a fixed mode observable and controllable from channel 1. This means that it may be possible to improve on the main result.

C. MULTICHANNEL SYSTEMS

Up to now, we have studied two-channel systems. Now we discuss p -channel systems for $p > 2$. With each channel of a p -channel system, associate a node of a p -node graph, and draw a directed arc connecting node i to node j just in the case $C_j'(sI - A)^{-1}B_i \neq 0$. A path $\{j_1 j_2 \cdots j_r\}$ from node j_1 to j_r is a set of arcs connecting j_1 to j_2 to j_3 \cdots to j_r . In the p -channel system, if there is nonzero (decentralized) feedback from output to input in each of channels j_1, j_2, \dots, j_r , and if $\{j_1 j_2 \cdots j_r\}$ is a path, it will be possible for signals inserted at the input j_1 to affect the output of all channels j_i along the path — irrespective of whether there is an arc connecting j_1 to j_i .

A system is termed strongly connected if there is a path between any two nodes. It is not hard to see that a system is strongly connected if and only if for every partition of the channels into disjoint sets A and B we have $C_A'(sI - A)^{-1}B_B \neq 0$. An alternative condition is that there is no ordering of the channels for which the system transfer function matrix is block triangular.

In [5], it is explained that every time-invariant system can be regarded as a collection of strongly coupled subsystems which have only one-way, or no, connections between them.

Moreover, many questions of decentralized control can be analyzed by considering them in relation to the individual strongly connected subsystems.

Let us now note the variation to the idea required if periodic gains are present. In the definition of connectivity for a constant gain system above, where a transfer function matrix must not be identically zero, we require that the corresponding collection of transfer function matrices be all not identically zero, where each transfer function is now associated with an interval in which the gains are constant. Thus if in the time-invariant case, there is a connectivity condition $C'(sI - A)^{-1}B \neq 0$ and if A is replaced by a periodically varying $A(t)$, switching between A_1 and A_2 , we require that $C'(sI - A_1)^{-1}B \neq 0$ and $C'(sI - A_2)^{-1}B \neq 0$.

We now assert:

Lemma 3. Consider a p -channel strongly connected system and suppose a $(p - 1)$ -channel system is formed by putting feedback of the form $u_p = -K_p y_p$ around the p th channel, with K_p periodic and piecewise constant. Then for generic K_p , the resulting $(p - 1)$ -channel system is strongly connected.

Proof. Consider any two nodes j_1, j_r of the graph associated with the $(p - 1)$ -channel system derived after introducing feedback to the original p -channel system. Before the introduction of this feedback, these two nodes, regarded as nodes of the graph of the p -channel system, define the end points of a path because the p -channel system is strongly connected. We distinguish

Case 1: The path does not include node p ;

Case 2: The path includes node p .

Let W_{j_i} denote $C_j(sI - A)^{-1}B_i$ (or the collection of such quantities), and \bar{W}_{j_i} denote the corresponding quantity resulting after feedback. Under case 1, we have $W_{j_2j_1} \neq 0, \dots, W_{j_rj_{r-1}} \neq 0$. Since for one specialized feedback, viz., $u_p \equiv 0$, we have $\bar{W}_{j_2j_1} = W_{j_2j_1} \neq 0, \dots, \bar{W}_{j_rj_{r-1}} = W_{j_rj_{r-1}} \neq 0$ it follows that for almost all feedback, i.e., generically, we must have $\bar{W}_{j_2j_1} \neq 0, \dots, \bar{W}_{j_rj_{r-1}} \neq 0$; i.e., a path connects j_1 to j_r for the $(p - 1)$ -channel system - the same path in effect as in the p -channel system.

For case 2, suppose the path is $j_1, j_2, \dots, j_k, p, j_{k+2}, \dots, j_r$. Arguing as for case 1, we know that generically $\bar{W}_{j_2j_1} \neq 0, \dots, \bar{W}_{j_kj_{k-1}} \neq 0, \bar{W}_{j_{k+3}j_{k+2}} \neq 0, \dots, \bar{W}_{j_rj_{r-1}} \neq 0$. We must show that generically, $\bar{W}_{j_{k+2}j_k} \neq 0$. If $W_{j_{k+2}j_k} \neq 0$, we can apply the case 1 argument. So assume that $W_{j_{k+2}j_k} \equiv 0$. Then

$$\bar{W}_{j_{k+2}j_k} = -W_{j_{k+2}p} K_p (I + W_{pp} K_p)^{-1} W_{pj_k}$$

as an easy calculation shows. Since $W_{j_{k+2}p} \neq 0$, and K_p is arbitrary, we have for generic K_p that $\bar{W}_{j_{k+2}j_k} \neq 0$. Consequently in the graph of the $(p - 1)$ -channel system, there is a path $j_1, \dots, j_k, j_{k+2}, \dots, j_r$ connecting nodes j_1 to j_r . This establishes the strong connectivity result.

We remark that the above result is actually true for more complicated (e.g., dynamic) feedback. However, we need only the present form.

It is not hard to verify that if a system with periodic time-varying gains is not strongly connected, it can be decomposed into a collection of strongly connected subsystems which can only have one-way connections between them and that, as for the time-invariant case, decentralized control questions must

be analyzed by considering the individual subsystems. Accordingly, to explain the main ideas of the section, we confine attention to a three-channel, strongly connected system.

Suppose we aim to use feedback on channels 2 and 3 to provide (uniform) controllability and observability at input and output 1. Temporarily consider channels 1 and 2 together as a single channel A . It is immediately clear that unless using channel A one can observe and control the system with a feedback gain around channel 3, there is no possibility of doing the same with channel 1, given feedback around channels 2 and 3.

Using the earlier ideas, we see observability and controllability from channel A can be achieved by feedback around channel 3; in case there are no fixed modes, this feedback around channel 3 can be constant, and almost any constant feedback gain suffices. If, however, there is a decentralized fixed mode associated with the channel A , channel 3 pair a constant gain will not suffice, but a piecewise constant periodic gain taking at least ρ_3 different values [$\rho_3 = 1 + \max(\dim u_3, \dim y_3)$] will suffice.

With this feedback, there now results a system with two channels, 1 and 2, which is possibly periodically time varying, and which is uniformly controllable and observable. By virtue of Lemma 3 it is, at least for generic periodic or constant gains around channel 3, strongly connected. The question arises as to whether we can now apply feedback around channel 2 to make the system uniformly controllable and observable from channel 1. The answer is yes; we shall argue simply the observability.

If this two-channel system is time invariant, the result is immediate by the earlier results. So suppose that it is described by $\{A(t), [B_1 \ B_2]\}$ where $A(t)$ is periodic and piecewise constant. Let us assume that $A(t)$ in fact takes the value \bar{A} in $[s, s + T_1)$, \tilde{A} in $[s + T_1, s + T)$. Observability means that if there exists γ for which

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} e^{\bar{A}t} \gamma = 0, \quad \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} e^{\tilde{A}t} e^{\bar{A}T_1} \gamma = 0$$

then $\gamma = 0$. [This can be checked by examining the observability Grammian over $(s, s + T)$.] Equivalently (take $\delta = e^{\bar{A}T_1} \gamma$), the equations

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} e^{\bar{A}t} \delta = 0, \quad \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} e^{\tilde{A}t} \delta = 0$$

imply $\delta = 0$. If the "frozen" systems $\{\bar{A}, [B_1 \ B_2], [C_1 \ C_2]\}$ and $\{\tilde{A}, [B_1 \ B_2], [C_1 \ C_2]\}$ were to have no fixed modes (other than any associated with lack of centralized controllability and observability), then constant feedback around channel 2 would generically produce uniform controllability and observability at channel 1. However, it is obvious from the definition of fixed modes that if the original three-channel system has fixed modes, so must each of the frozen two-channel systems. We now explain what is done in this case.

For convenience, suppose that y_2 is a scalar. We then take $u_2 = K_2(t)y_2$ where $K_2(t) = 0$, $t \in [s, s_1)$, $K_2(t) = \bar{K}_2$, $t \in [s_1, s + T_1)$, $K_2(t) = 0$, $t \in [s + T_1, s_2)$, $K_2(t) = \tilde{K}_2$, $t \in [s_2, s + T)$ with $K_2(t)$ periodic. If γ_1 is a null vector of the observability Grammian over $[s, s + T]$ of $\{A(t) + B_2 K_2(t) C_2', C_1'\}$, then

$$C_1' \exp(\bar{A}t) \gamma_1 = 0$$

$$\begin{aligned}
 C_1' \exp\left[\left(\bar{A} + B_2 \bar{K}_2 C_2'\right)t\right] \exp\left(\bar{A}s_1\right) \gamma_1 &= 0 \\
 C_1' \exp\left(\tilde{A}t\right) \exp\left[\left(A + B_2 \bar{K}_2 C_2'\right)\left(T_1 + s - s_1\right)\right] \exp\left(\bar{A}s_1\right) \gamma &= 0 \\
 C_1' \exp\left[\left(\tilde{A} + B_2 \tilde{K}_2 C_2'\right)t\right] \exp\left(\tilde{A}\left(s_2 - s - T_1\right)\right) \\
 \times \exp\left[\left(A + B_2 \bar{K}_2 C_2'\right)\left(T_1 + s - s_1\right)\right] \exp\left(As_1\right) \gamma_1 &= 0. \quad (26)
 \end{aligned}$$

Arguing as earlier, and using the fact that $C_1'(sI - \bar{A})^{-1}B_2 \neq 0$ in the light of strong connectivity, we conclude from the first two equations that

$$C_1' e^{\bar{A}t} \gamma_1 = 0, \quad C_2' e^{\bar{A}t} \gamma_1 = 0$$

for all t , and in fact if $\delta = \exp\left[\left(\bar{A} + B_2 \bar{K}_2 C_2'\right)\left(T_1 + s - s_1\right)\right] \times \exp\left(\bar{A}s_1\right) \gamma_1$, then

$$C_1' e^{\bar{A}t} \delta = 0, \quad C_2' e^{\bar{A}t} \delta = 0. \quad (27)$$

In a similar manner the last two equations in Eqs. (26) yield

$$C_1' e^{\tilde{A}t} \delta = 0, \quad C_2' e^{\tilde{A}t} \delta = 0. \quad (28)$$

However, as argued above, the observability of the two-channel system implies that in Eqs. (27) and (28) we have $\delta = 0$, and thus $\gamma = 0$; i.e., the single-channel system is observable.

The above analysis applies for scalar y_2 . The earlier techniques can be used to derive the result for vector y_2 .

The procedure for coping with a p -channel system when $p > 3$ is a straightforward extension of the procedure for a three-channel system. Assuming the p -channel system is strongly connected and meets a centralized controllability and observability condition, one successively applies feedback around channels p , $p - 1$, ..., 2. The feedback can be constant only if there are no fixed modes associated with any of the frozen systems encountered at any stage in the procedure. Otherwise, it must be

periodic and piecewise constant, taking a certain minimum number of values that is readily computable at each stage. The end result is that for generic values for all the feedback gains, the one-channel system is uniformly controllable and observable.

D. AREAS FOR FURTHER STUDY

The results presented here open up further areas for research on linear decentralized control. Let us indicate several of these.

(1) Suppose a system is to have a decentralized controller structure and the aim is to observe and control it from one channel having put feedback (periodic if need be) around the other channel. How should one systematically select the value of the feedback around the second channel and the period in the nonconstant case, for example, to achieve a good compromise between system performance for a nominal plant and robustness to plant uncertainty or variations?

(2) Presumably, if piecewise constant periodic feedback around channel 2 ensures uniform observability and controllability from channel 1, many, if not almost all, periodic feedbacks which are not piecewise constant will have the same effect. How can such feedbacks be characterized?

(3) The controller structures studied in this chapter have no dynamics in all but one channel. The controller on the remaining channel has dynamics with state variable dimension approximately equal to that of the system being controlled. How could one distribute the controller complexity more evenly across the different channels?

(4) Recent unpublished work of which the authors are aware shows that piecewise constant, periodic feedback controllers around a periodic single-input, single-output system operating in discrete time can very frequently be used to generate a deadbeat response. This raises questions as to whether such ideas could be wedded to the ideas of this chapter to show that decentralized, piecewise constant, periodic controllers could provide a deadbeat response for a very wide class of systems.

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