CHAPTER 27

SELF-LOCALIZATION OF FORMATIONS OF AUTONOMOUS AGENTS USING BEARING MEASUREMENTS

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THIS CHAPTER begins by treating a localization problem recently encountered in an operational context; that is, localizing three agents moving in a plane when the interagent distances are known, and in addition, the angle subtended at each agent by lines drawn from two landmarks at known positions is also known. It is shown as a key conclusion that there are in general a finite number greater than one of possible sets of positions for the three agents. In addition, the generalization of the result for more than three agents is presented. Examples are presented to show the applicability of the methods proposed here.

27.1 INTRODUCTION

In many multiagent applications, it is desired to know the positions of the agents. For example, in bushfire surveillance or a search-and-rescue operation, sensing the data without knowing the position of the sensing agents is not enough to accomplish the task at hand. A trivial solution to this problem is to install global positioning systems (GPSs) on each agent. However, when GPS signals are lost or corrupted [1], or when the agents operate indoors, use of GPSs for localization purposes may
be infeasible or limited [2]. Thus, it is imperative to design other methods for estimating the positions of the sensing agents when a GPS signal is not available. There are also many other scenarios when GPS signals are not available. Examples include outer space localization and localization on the surface of other planets. The vehicle itself can serve as a pair or more of landmarks, which the robots can use for absolute navigation and referencing, as long as the vehicle remains in the line of sight (LOS) of the robot.

In the literature, there are many methods that achieve the task of localizing agents via measuring distances [3, 4], time difference of arrival [5], and direction of arrival (DOA) [6].

In this chapter, we address the localization problem where self-localization is achieved cooperatively among a team of bearing and distance sensing agents (e.g., unmanned aerial vehicles [UAVs] or robots) and where the position information of the two landmarks is available, as are either interagent distances or the bearings measured to each agent of the agents in the formation.

The problem considered here is very similar to the problems considered in References 6–10. The cooperative localization task is then to put the pieces of information together, for example, interagent distances, subtended angles, and landmark positions, and localize the agents. Note here that the localization is to be achieved instantaneously; we are not envisaging collecting information from agents at a number of successive instants of time and using them to infer position at a single instant of time (the connection with Kalman filtering is explored further below). In distance-based localization literature, the nodes with known position are called anchors (beacons) [11], and since each anchor is determined exactly, they can be considered as anchors as well. Due to the nature of the problem mentioned above these anchors are collinear. However, the results obtained in this chapter are not limited to collinear anchors, and can be applied to scenarios with arbitrary arrangement of the anchors. Note also that we later extend the results to systems with more than three agents capable of measuring the subtended angle by the landmarks.

It is shown in this chapter that the solution count for the above localization problem (involving three agents) is less than 12. If there is no unique solution, then what is the point of this analysis? There are in fact several ways in which it can be relevant. First, GPS data may be intermittently available to an agent. When it is available, localization is obviously unique. When it ceases to be available, the fact that the agents are moving continuously with their initial position known means that there will be a basis for making the correct selection out of a finite number of localization possibilities at subsequent times.* Second, given that the number of possible solutions to the localization problem is finite, it may be that additional very crude data indeed (e.g., agent \( T \) is located in this general region) will be enough to disambiguate the multiple solutions. Third, if more agents are available, then the additional information will generically allow unique localization, and indeed, when measurements are noisy, will in general allow some amelioration of the distorting effects of the noisy measurements. Lastly, the method described in this chapter can

*At least until the agents move to positions corresponding to a double solution of the localization problem, after which two tracks would have to be followed assuming no disambiguating information.
be considered as batch processing of the agent locations serving to initialize and improve the updates of a Kalman-based filter that tracks the agent positions as the agents move in their environment. Kalman-based filters are also prone to errors when the agent’s motion model deviates significantly from the actual agent motion. Our batch processing method can guard against such behavior and reinitialize the filter.

There is a vast body of literature using Bayesian methods that deals with localization problems. For example, in Reference 12, a collective localization problem based on Kalman filtering is proposed, and Reference 13 uses a Gaussian sum filter to solve the initialization problem in bearing-only simultaneous localization and mapping (SLAM). Differently from these approaches, the tools for obtaining our results are drawn from two nonconventional sources. The first is the theory of rigid graphs (see, e.g., References 14–17). In recent years, its relevance to localization of sensor networks has come into prominence [11, 18]. A good deal of the theory of rigid graphs deals with the question of when localization is possible [19, 20]. The second source we draw on is the mechanical engineering literature dealing with four-bar linkage mechanisms [21, 22]. As it turns out, this literature has developed ideas for describing the loci of points that are part of a planar mechanism made up from pin joints and rigid bars that provide distance constraints between the joints. Moreover, for the purposes of this chapter, we assume that we have access to noiseless interagent distance measurements. The significance of considering the noiseless scenario is that it equips us with a knowledge on the number of solutions to the problem and methods to calculate these solutions, where later methods to deal with noisy case are built upon.

The outline of this chapter is as follows. In the next section, we formally set up the problem. In Section 27.3, we first introduce the results related to network localization in the field of rigidity and then establish the connection of the problem described in Section 27.2 with the rigidity theory. In Section 27.4, we study four-bar linkage mechanisms, and in Section 27.5, we propose a solution to the problem of interest using the mathematical methods developed to analyze these mechanisms. In Section 27.6, we consider the localization problem for formations with a larger number of agents than initially considered. In Section 27.7, we study the situation where information about an extra landmark is available. In Section 27.8, we introduce another localization scenario where the agents can measure all the subtended angles (not only at the landmarks). Concluding remarks come at the end.

### 27.2 PROBLEM SETUP

The arrangement to be considered, and the one on which the later developments are based, is depicted in Figure 27.1. Specifically, we consider $n$ mobile agents (in this case, $n = 3$), designated $T_1, T_2,$ and $T_3,$ that are to be localized. The agents detect two landmarks located at positions $L_1$ and $L_2$, which are known to the agents. The landmarks can be radars, radio frequency (RF) beacons, or some visible features if an imaging sensor is used. Each agent collects the bearing angle information to each of the landmarks. However, with no GPS information (no knowledge about the absolute heading reference), there is no absolute heading reference for each agent,
and the bearing angle information cannot be used directly for localization purposes. Nevertheless, using the angle difference (i.e., the difference between two bearings), the need for knowing the heading is removed. This angle difference is the angle subtended at each agent by the two landmarks ($\lambda_i$, $i = 1, 2, 3$) (see Fig. 27.1) and knowing its value. It is straightforward from the inscribed angle theorem to determine that each agent $i$ is located on a circle of known center, $A_i$, and radius, $R_i$, that passes through the two landmarks. Note that both values of $A_i$ and $R_i$ can be determined by $\lambda_i$ and the landmark positions. The centers, $A_i$ ($i = 1, 2, 3$), lie on the perpendicular bisector of the line joining the two landmarks. A priori information is assumed to be available that positions all agents on the same known side of the line joining the two landmarks. This can be ensured by choosing landmarks on the boundary of the agents region of operation.

We adopt the naming convention from the distance-based localization literature and call the points with the known positions anchors (beacons) [11]. Since each $A_i$ is determined exactly (from the knowledge of positions of the landmarks and the angles $\lambda_i$), they can be considered as (pseudo-)anchors as well. Due to the nature of the problem mentioned above, these anchors are collinear. However, the results obtained in this chapter are not actually limited to collinear anchors, and can be applied to scenarios with arbitrary arrangement of the anchors. First, we formalize the definition of the underlying graph of a formation.

**Definition 1 (Underlying Graph of a Formation):** A formation of point agents in the plane, $F$, can be represented by a graph $G(V, E)$, with vertex set $V$ and edge set $E$; the vertices in $V$ correspond to the agents, and an edge, in the graph exerts between two vertices $v_i, v_j \in V$ only when the distance between the corresponding agents of the formation is known. We call $G$ the underlying graph of $F$.

We can generalize the problem described above as the following problem.
Problem 1: Consider a formation $F$ with the underlying graph $\mathcal{G}(V, E)$, where $V = T \cup A$ is the set of vertices with $T = \{T_1, T_2, T_3\}$, $A = \{A_1, A_2, A_3\}$. The agents in $A$ are known as anchors, and those in $T$ as targets. Furthermore, $E = \{T_1T_2, T_1T_3, T_2T_3, A_1A_2, A_1A_3, A_2A_3, A_1T_1, A_2T_2, A_3T_3\}$ is the set of edges. Knowing the exact length of all edges in $E$, and the exact positions of the anchors,

(i) Can one localize the targets, uniquely, or to one of a finite number of sets of positions?

(ii) If so, what are the possible localization solutions?

Remark 1: In this problem there is no assumption on collinearity of $A_i, i = 1, 2, 3$. The case where $A_i$ are collinear can be treated as a special case of this problem.

In the sequel, we propose an answer to the first question posed in Problem 1. However, first we introduce some basic concepts from rigidity theory and its relationship to the localization problem.

27.3 A RIGID GRAPH THEORETICAL FRAMEWORK FOR FORMATION LOCALIZATION

In this section, we review some aspects of the problem of localizing, that is, determining the positions of agents in a formation where a number of interagent distances are known, and additionally, some absolute position data is available. We appeal to the literature on rigid graph theory and its application to sensor network localization [11, 18–20].

Definition 2 (Graph Realization Problem): Consider a formation $F$ with underlying graph $\mathcal{G}(V, E)$. The task of assigning coordinate values to each vertex of a graph, $\mathcal{G}(V, E)$, so that the Euclidean distance between any two adjacent vertices is equal to the edge length associated with the edge joining these two vertices, is the graph realization problem.

Given one solution to the graph realization problem, it is trivial that any translation, rotation, or reflection of this solution is another solution. All such solutions are congruent. Hence, it is relevant to ask whether there can be two solutions that are not congruent, and whether, disallowing translations, rotations, or reflections, the number of distinct solutions is finite or infinite. When there can be only one family of congruent solutions, one can say that the graph realization problem has a unique solution. This problem is also known as the problem of localizability in the literature (see Reference 23 for more information).

Hendrickson [19] presents necessary conditions for a graph to be uniquely realizable in $\mathbb{R}^2$, that is, with one family of congruent solutions, and the same conditions were proved later by Jackson and Jordan [20] to be necessary and sufficient.

These conditions involve two concepts, namely redundant rigidity of a graph and three-connectedness of a graph. The concept of redundant rigidity requires a prior concept of rigidity, that is, as follows.
Definition 3 (Rigid Formations): A formation $\mathcal{F}$ is called rigid if by explicitly maintaining distances between all the pairs of agents whose representative vertices are connected by an edge in $\mathcal{E}$, the distances between all other pairs of agents in $\mathcal{F}$ are consequentially held fixed as well.

The reader may refer to References 14–16 for detailed information on rigid formations and rigidity.

Definition 4 (Redundantly Rigid and Minimally Rigid Formations): A redundant rigid formation is one that remains rigid when any single edge constraint is removed. By way of contrast, a minimally rigid formation is one which ceases to be rigid when any single edge constraint is removed.

For the sake of simplicity, the underlying graph of a formation is called rigid, redundantly rigid, and minimally rigid if the formation is, respectively, rigid, redundantly rigid, and minimally rigid.

The notion $\mathcal{G}$ of a three-connected graph is standard (see, e.g., Reference 24). Such a graph has the property that between any two vertices, three nonintersecting paths can be found.

Jackson and Jordan’s result [20] is as follows.

Theorem 1: Consider a two dimensional formation $\mathcal{F}$ with underlying graph $\mathcal{G}(V, E)$. Then the graph realization problem is uniquely solvable for generic values of the formation edge lengths (interagent distances) if and only if $\mathcal{G}$ is redundantly rigid and three-connected.

From this result, we can have a definition for globally rigid graphs.

Definition 5 (Globally Rigid Graphs): A graph $\mathcal{G}$ with the two properties in Theorem 1, that is, redundant rigidity and three-connectedness, is termed globally rigid.

For a formation that is rigid but not globally rigid, one of at least of two ambiguities known as flip ambiguity or discontinuous flex ambiguity can occur [19]; these ambiguities are depicted in Figure 27.2. The reader may refer to References 16 and

![Figure 27.2](image-url)

Figure 27.2 Illustration of (a) flip ambiguity: vertex $D$ can be flipped over the edge $(A, B)$ to a symmetric position $D'$ and the distance constraints remain the same; (b) discontinuous flex ambiguity: temporarily removing the edge $(C, D)$, the edge triple $(D, A), (D, E), (E, B)$ can be flexed to obtain positions $E'$ and $D'$, such that the edge length $(C, D)$ equals the edge length $(C, D')$, and then all the distance constraints are the same.
and references therein for further information on these ambiguities. Examples of nonrigid, rigid, and globally rigid graphs are presented in Figure 27.3.

Now we provide an answer the first question posed in Problem 1. An example of the formation $F$ described in Problem 1 is depicted in Figure 27.4. A crucial concept pertinent to answering the first question posed in Problem 1 is a minimally rigid formation. Two ways are presented in the following paragraphs to see this fact.

Figure 27.3 Rigid and nonrigid formations. The formation represented in panel (a) is not rigid. It can be deformed by a smooth motion without affecting the distance between the agents connected by edges, as shown in panel (b). The formations represented in panels (c) and (d) are rigid, as they cannot be deformed by any such move. In addition, the formation represented in panel (c) is minimally rigid because the removal of any edge would render it nonrigid. That of panel (d) is not minimally rigid; any edge may be removed without losing rigidity.

Figure 27.4 Graph representation of Problem 1.
Laman’s Theorem [14]: Laman’s theorem provides a combinatorial way to check rigidity, and indeed minimal rigidity. It requires the idea of an induced subgraph of a graph $G = (V, E)$. Let $V'$ be a subset of $V$. Then the subgraph of $G$ induced by $V'$ is the graph $G' = (V', E')$, where $E'$ includes all those edges of $E$ that are incident on a vertex pair in $V'$.

Theorem 2 (Laman’s Theorem [14]): A graph $G = (V, E)$ in $\mathbb{R}^2$ of $V$ vertices and $E$ edges is rigid if and only if there exists a subgraph $G' = (V', E')$ with $2V - 3$ edges such that for any subset $V''$ of $V$, the induced subgraph $G'' = (V'', E'')$ of $G'$ obeys $E'' \leq 2V'' - 3$. It is minimally rigid if $E = 2V - 3$.

It is easy to check for the graph of Figure 27.4 that $E = 2|V| - 3$; one takes $G' = G$ and can verify the counting condition for all induced subgraphs.

Combination of Rigid Formations [17]: Another way of demonstrating minimal rigidity of a formation is by showing that it is a certain type of combination of two minimally rigid formations. The key theorem is as follows.

Theorem 3 ([17]): Let $F_1$ and $F_2$ be two rigid formations, and consider a formation $F$ formed by connecting these two formations with three edges, each edge incident on one vertex of $F_1$ and one of $F_2$, with no two edges incident on the same vertex. Then $F$ is rigid. Moreover, if $F_1$ and $F_2$ are minimally rigid, so is $F$.

The setting described in Theorem 3 is depicted in Figure 27.5. Observe that a triangle is obviously minimally rigid. The theorem then applies identifying $F_1$ with the triangle formed by $A_1, A_2$, and $A_3$ and $F_2$ with the triangle formed by $T_1, T_2$, and $T_3$. In the light of the minimal rigidity of the formation of Figure 27.4, there will be noncongruent formations meeting the distance constraints. Then, even though the positions of $A_1, A_2$, and $A_3$ are fixed, the positions of $T_1, T_2$, and $T_3$ will not be uniquely determinable.

27.4 FOUR-BAR LINKAGE MECHANISMS

Four-bar linkage mechanisms perform a wide variety of motions with a few simple parts. Furthermore, due to their ease of design calculations, they have gained much popularity in mechanical machine design. An example of such a mechanism is presented in Figure 27.6.
In a four-bar linkages mechanism, there is a fixed link that is called the frame. There are two side links that can revolve around the ends of the frame, and the remaining (fourth) link is called the coupler. A four-bar linkage mechanism is characterized by the length of each of its links, and the configuration of its coupler, that is, open or crossed configuration [25]. Furthermore, we have the following concepts in four-bar linkage mechanisms [26]:

- A side link that can fully revolve around (360°) the end point of the frame is called a crank.
- Any link that cannot fully revolve is called a rocker.

Although we do not use the following theorem extensively in this chapter, we present it as background information that will assist in interpreting the subsequent examples.

**Theorem 4 (Grashof’s Theorem [21]):** A four-bar mechanism has at least one crank link if and only if

\[ s + l \leq p + q, \]  

and all three mobile links are rockers if

\[ s + l > p + q. \]  

Here, \( l \) and \( s \) are the lengths of the longest link and the shortest link, respectively, and \( p \) and \( q \) are the lengths of the other two links. For example, in Figure 27.6, \( s \) and \( l \) are \( A_2T_2 \) and \( A_1A_2 \), respectively. The inequality (Eq. 27.1) is known as Grashof’s criterion.

In the mechanism depicted in Figure 27.6, \( T_3 \), which is a fixed point on a rigid body attached to the coupler link \( T_1T_2 \), is termed the coupler point. The curve that this coupler point moves on in both open and cross configurations is called the coupler curve. Generally, coupler curves are closed curves, but for some special mechanisms we will have open coupler curves. These correspond to situations where the area enclosed by the coupler curve shrinks to zero.

A coupler curve \( K_c \) may comprise either a single part or a bipartite curve. (A bipartite curve is one with two branches, like a hyperbola.) In the case where \( K_c \) is bipartite, we denote the branches as \( K_{c_1} \) and \( K_{c_2} \), where \( K_{c_1} \) is the curve obtained...
by the coupler point in open configuration and $K_{C_2}$ is constructed by the curve in cross configuration.

For a given four-bar linkage mechanism, as in Figure 27.6, the equation of the coupler curve (bipartite or single part), $K_C$, where the center of Cartesian coordinates system is placed on $A_1$, and $A_1$ and $A_2$ are placed on the x-axis, is [21]

$$r_2^2 ((x-k)^2 + y^2)(x^2 + y^2 + r_3^2 - R_1^2)^2$$
$$- 2r_3r_4((x^2 + y^2 - kx) \cos \eta_3 + k \sin \eta_3)((x^2 + y^2 + r_3^2 - R_1^2)^2$$
$$+ ((x-k)^2 + y^2 + r_3^2 - R_1^2) + r_3^3(x^2 + y^2)((x-k)^2 + y^2 + r_3^2 - R_1^2)^2 +$$
$$- 4r_3^2r_4^2((x^2 + y^2 - kx) \sin \eta_3 - k \cos \eta_3)^2 = 0$$

(27.3)

where $k$ is the length of the frame link, $A_1A_2$, $\eta_3 = \angle T_1T_3T_2$, $r_6$ is the distance between agents $T_i$ and $T_n$, and $R_i = d(2 \sin \lambda_i)$.

In addition, another coupler curve, $K_C'$, can be obtained from the reflection of $K_C$, when $A_1A_2$ is the image axis. In general, the equation describing $K_C'$ can be obtained by substituting $-y$ for $y$ in Equation 27.3. In the case of a bipartite $K_C$, we denote the branches of $K_C'$ as $K'_C$ and $K_C''$. As a result, the locus of the coupler point is made up of two polynomial curves, each with degree of six. More specifically, we have the following result from Reference 21 for bipartite coupler curves:

**Proposition 1:** Bipartite coupler curves occur when and only when Grashof's criterion holds; all other cases yield coupler curves that always consist of a single part.

### 27.5 A LOCALIZATION ALGORITHM BASED ON FOUR-BAR LINKAGE MECHANISMS

Let us revisit the problem described in Section 27.2. It has been assumed that the agents form a triangular formation, where $T_i$ is the $i$th agent, with $\mathbf{x}_i = [x_i, y_i]^T \in \mathbb{R}^2$ as its coordinates. The distance between two agents $T_i$ and $T_j$ is known and equal to $r_{ij}$ (or $r_{ji}$). For a given agent, $T_i$, and two landmarks with known positions, $L_1$ and $L_2$, the locus for the position of $T_i$ when $\angle L_iT_iL_2 = \lambda_i$ is a part of a circle denoted by $C(a_i, R_i)$, where $R_i = d/2 \sin (\lambda_i)$ is the radius of the circle and $a_i$ is the center. Furthermore, assuming that the origin of the global coordinates frame is the middle of $L_1L_2$, $d = L_1L_2$ and the x-axis coincides with the perpendicular bisector of $L_1L_2$, the position of $A_i$ is considered to be $a_i = [x_i, y_i]^T = [d/2 \tan \lambda_i, 0]^T$. Note that $T_i$, $L_1$, and $L_2$ lie on the same circle described above.

In Figure 27.1b, each mobile platform, $T_i$, $i = 1, 2, 3$, and the associated circles are depicted. In this case, the centers of the circles, $A_i$, serve as the virtual anchors since we know their exact positions in the plane. Hence, the agents, $T_i$ $(i \in \{1, 2, 3\})$, in the formation and these virtual anchors, $A_j$ $(j \in \{1, 2, 3\})$, form a graph that satisfies
the conditions presented in Problem 1. Here, first, we make the relationship between
the localization problem described in Problem 1 and the concept of four-bar linkage
mechanism clear, and then using this concept we present an upper bound for the
number of localization possibilities.

Note that the graph representation of Problem 1 in Figure 27.4 can be further
iterated to obtain the virtual four-bar linkage mechanism representation in Figure
27.6 by representing the distance constraints on $|A_1A_2|$, $|A_1T_1|$, $|A_2T_2|$, $|T_1T_2|$ as bars, and
decoupling the distance constraint on $A_3T_3$ from this representation. Because of this
decoupled distance constraint, $T_3$ is on the circle $C(a_3, R_1)$ with $A_3$ as its center and
$R_1$ as its radius. Hence, the possible solutions for the localization problem can be
obtained from the calculation of intersections of the circle, $C(a_3, R_3)$ and the two
coupler curves $K_C$ and $K'_C$ defined in Section 27.4.

In Reference 22, the number of intersections of coupler curves and a circle is
computed using concepts of circularity and order of the curves. The result, according
to Reference 22, in our context is as follows:

**Lemma 1**: The circle, $C(a_3, R_3)$ and the coupler curve described by Equation 27.3
has at least two and at most six real points of intersection.

Using Lemma 1, we establish that the maximum number of localization solutions is
12 in the following theorem:

**Theorem 5**: The maximum number of (real) localization solutions for Problem 1 is
12. For generic values of distances and angles, the minimum number of localization
solutions is four.

**Proof**: Equation 27.3 corresponds to the coupler curves for the four-bar linkages
mechanism depicted in Figure 27.6 corresponding to Problem 1 [21]. Since
there are two (single part or bipartite) coupler curves (the second one is the image
of the first one when the frame link is the image axis) and for each coupler curve
based on Lemma 1, there are a maximum of six and a minimum of two possible
solutions, we have at most 12 possible solutions, and at least four localization
solutions.

Returning to the problem presented in Section 1, based on the procedure introduced
earlier in this section for constructing a four-bar mechanism by deleting edge $A_3T_3$,
we can have the linkage mechanism depicted by solid lines in Figure 27.1b. In
addition, here for the coupler curve equation, we have, $k = (d/2 \tan \lambda_2) - (d/
[2 \tan \lambda_1])$, $R_1 = d/(2 \sin \lambda_1)$, and $R_2 = d/(2 \sin \lambda_2)$. From Theorem 5, we can have up
to 12 localization solutions. We are now ready to provide a localization algorithm
to address Problem 1(ii). Algorithm 27.1 can be used to find up to twelve localization
solutions (up to six pairs of mirror solutions with respect to the line connecting
$A_1$ to $A_2$).

**Algorithm 27.1 Three Agent and Two Landmark Localization**

Find $m \leq 6$ real intersection points of Equation 27.3 and $(x - x_i)^2 + (y - y_i)^2 - R_2^2$, $x_{3i} = [x_{3i}, y_{3i}]^T$, $i = 1, \ldots, m$.

for $i = 1$ to $m$ do
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Solve the system of equation for $x_{\mathcal{T}}$, $y_{\mathcal{T}}$, $x_{\mathcal{T}}$, and $y_{T}$:

\[
(x_{T1} - x_{T})^2 + (y_{T1} - y_{T})^2 - r_{13}^2 = 0 \\
(x_{T2} - x_{T})^2 + (y_{T2} - y_{T})^2 - r_{23}^2 = 0 \\
(x_{T} - x_{T1})^2 + (y_{T} - y_{T1})^2 - R_1^2 = 0 \\
(x_{T} - x_{T2})^2 + (y_{T} - y_{T2})^2 - R_2^2 = 0
\]

\[s_i \leftarrow [x_{T1}, y_{T1}, x_{T2}, y_{T2}, x_{T}, y_{T}] \quad s'_i \leftarrow [x_{T1} - y_{T1}, x_{T2} - y_{T2}, x_{T}, y_{T}]
\]

end for

Return $S$ and $S'$ as the sets of $2m$ localization solutions where $s_i$ and $s'_i$ are their rows respectively.

Example 27.1 Simulations with Three Agents and Two Landmarks

The angle and distance values used in each simulation scenario are presented in Table 27.1. In addition, it is worth mentioning that after running several simulations, a case with 12 localization solutions was never encountered.

The important characteristics of each simulation result and the number of localization solutions in each scenario are presented in Table 27.2. The legends used in the simulation results are described in Table 27.3. In Figures 27.7 and 27.8, two scenarios where, respectively, four and eight localization solutions exist are presented. A nongeneric case is presented in Figure 27.9a, where there are repeated localization solutions. A bad geometry is identified in Figure 27.9b, where an infinite number of localization solutions exists. This infinite localization ambiguity occurs if the three sensors and the two landmarks lie on a common circle. Invalid localization solutions occur in the scenario that is depicted in Figure 27.10a. Another nongeneric case arises when the angles subtended at any two of the agents by the landmarks are equal, and this is depicted in Figure 27.10b. Note that for all the scenarios the landmarks $L_1$ and $L_2$ are placed at $[0,1]^T$ and $[0,-1]^T$, respectively.

<table>
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<th>Scenario</th>
<th>$\lambda_1$ (rad)</th>
<th>$\lambda_2$ (rad)</th>
<th>$\lambda_3$ (rad)</th>
<th>$\mathcal{T}\mathcal{T}_1$ (m)</th>
<th>$\mathcal{T}\mathcal{T}_2$ (m)</th>
<th>$\mathcal{T}_1\mathcal{T}_2$ (m)</th>
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<td>0.6283</td>
<td>0.5236</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>0.1674</td>
<td>3.1623</td>
<td>5.099</td>
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<td>0.2487</td>
<td>2.0859</td>
<td>2.7552</td>
<td>4.6188</td>
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<td>0.5236</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Figure 27.10b</td>
<td>0.7854</td>
<td>0.7854</td>
<td>0.5236</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
27.5 A LOCALIZATION ALGORITHM BASED ON FOUR-BAR LINKAGE MECHANISMS

Figure 27.7 The possible localization solutions for scenario 1.

Figure 27.8 The possible localization solutions for scenario 2.
Figure 27.9  Simulation results. (a) Repeated localization solutions (scenario 3 as described in Table 27.2). (b) Infinite number of localization solutions for scenario 4 as described in Table 27.2. The locus of $T_3$ coincides with one of the branches of the coupler curve.
Figure 27.10 Simulation results. (a) Occurrence of invalid localization solutions (scenario 5 as described in Table 27.2). (b) Localization solutions when two subtended angles $\lambda_1$ and $\lambda_2$ are equal (scenario 6 as described in Table 27.2)
CHAPTER 27 SELF-LOCALIZATION OF UAV FORMATIONS

27.6 LOCALIZATION OF LARGER FORMATIONS

The methodology presented in the previous sections can be extended to localization of certain formations having more than three agents, as will be elaborated in what follows. The following theorem gives the maximum possible number of solutions for formations with globally rigid underlying graphs:

**Theorem 6:** Consider a formation \( \mathcal{F} \) with the underlying globally rigid graph \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \), and the three agents, \( T_1, T_2, \) and \( T_3 \), in the formation that form a triangle. Assuming that these three agents are the only agents capable of measuring the angle subtended at them by the two landmarks \( L_1 \) and \( L_2 \), with known positions, then there are at most 12 possible localization solutions for the formation.

**Proof:** Theorem 5 states that the upper bound for the number of localization solutions of a triangular formation using the value of the angles subtended at each agent by two landmarks is 12. On the other hand, in Reference 20, it has been shown that the necessary and sufficient condition for unique localization of a formation is that the associated graph is globally rigid and there are three nodes with exactly known positions. As a result, for each possible localization of three agents, there is a localization solution for the whole formation, so there are up to 12 possible localization solutions for the formation.

Now, assume that we have another agent, \( T_4 \), that can measure its distance from agents \( T_1, T_2, \) and \( T_3 \) and the angle subtended at itself by the two landmarks, \( \lambda_4 \). Knowing this angle, we can characterize another anchor node, \( A_4 \) with known position, similar to \( A_1, A_2, \) and \( A_3 \), for the time being assuming that \( A_i \) are not collinear. We can calculate the distance between \( T_4 \) and \( A_4 \), \( i = 1, \ldots, 4 \), form a complete (and therefore globally rigid) graph and we already have implicitly assumed that \( T_1, T_2, T_3, \) and \( T_4 \) also form a complete graph. We know from Reference 27 that by connecting a globally rigid graph to another one using four edges, the resulting graph

<table>
<thead>
<tr>
<th>Scenario</th>
<th>No. of Distinct Solutions</th>
<th>Characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Fig. 27.7)</td>
<td>4</td>
<td>Generic</td>
</tr>
<tr>
<td>2 (Fig. 27.8)</td>
<td>8</td>
<td>Generic</td>
</tr>
<tr>
<td>3 (Fig. 27.9a)</td>
<td>2</td>
<td>Nongeneric/repeated solutions</td>
</tr>
<tr>
<td>4 (Fig. 27.9b)</td>
<td>Infinite</td>
<td>Nongeneric/infinite ambiguity</td>
</tr>
<tr>
<td>5 (Fig. 27.10a)</td>
<td>2</td>
<td>Generic/invalid solutions</td>
</tr>
<tr>
<td>6 (Fig. 27.10b)</td>
<td>2</td>
<td>Nongeneric/two equal angles</td>
</tr>
</tbody>
</table>

**TABLE 27.3. The Legends Used in Simulation Results**

<table>
<thead>
<tr>
<th>Agent</th>
<th>Small solid circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Landmark</td>
<td>Solid diamond</td>
</tr>
<tr>
<td>Formation</td>
<td>Solid triangles</td>
</tr>
<tr>
<td>Agent locus</td>
<td>Dashed circles</td>
</tr>
<tr>
<td>Coupler curve</td>
<td>Dotted curves</td>
</tr>
</tbody>
</table>

**27.6 LOCALIZATION OF LARGER FORMATIONS**
is globally rigid as well, and as a result, for generic positions of the agents there is a unique realization. Thus, the formation described above has a unique localization solution for generic positions of agents and landmarks, and it is the addition of an agent capable of measuring the angle subtended at itself by the two landmarks that disambiguates the multiple localization solutions. However, the important point here is that the anchor nodes $A_1, A_2, A_3,$ and $A_4$ are collinear, which violates genericity. This collinearity results in having mirror localization solutions at different sides of the line that the anchors are placed on.

We conclude this section with the following remark, which is an extension of Theorem 5:

**Remark 2:** Consider a formation $F$ with the underlying globally rigid graph $G(V,E)$, and the four agents, $T_1, T_2, T_3,$ and $T_4$ in the formation that form a complete graph. Assuming that these four agents are the only agents capable of measuring the angle subtended at them by the two landmarks $L_1$ and $L_2$, with known positions, it can be shown that there are at most two possible localization solutions for the formation.

**Example 27.2**

Consider four agents $T_i, i = 1, \ldots, 4,$ at unknown positions with $\lambda_1 = 1.4407$, $\lambda_2 = 1.3051$, $\lambda_3 = 0.9193$, $\lambda_4 = 0.8635$, $T_1T_2 = 0.4310$, $T_1T_3 = 0.6236$, $T_1T_4 = 0.9402$, $T_2T_3 = 0.9481$, $T_2T_4 = 0.7957$, $T_3T_4 = 0.9306$, $L_1 = (0,1)^T$ and $L_2 = [0, -1]^T$. The two possible positions for the agents are depicted in Figure 27.11. A simple way to calculate these solutions is to use Algorithm 27.1 to find a set of possible solutions for the positions of $T_1, T_2,$ and $T_3$. Calculate a candidate solution for the position of $T_4$ using the interagent distances for each of these solutions sets, and discard all the solutions that are not consistent with the angle measurements associated with $T_4$.

![Figure 27.11](image-url)
27.7 LOCALIZATION WITH EXTRA LANDMARKS

In this section, we consider the cases where bearing measurements from another landmark are available. Suppose another landmark, $L_3$, is positioned at a known position in the plane, and further imagine that agent $T_1$ can measure the subtended angle by landmarks $L_1$ and $L_2$, $L_1$ and $L_3$, and $L_2$ and $L_3$. These three angle measurements result in having three circles with the common point of intersection exactly at $T_1$ for generic positions for $T_1$. Hence, having another landmark generically disambiguates the multiple localization solutions.

There are certain geometries for which a unique solution for the agent cannot be obtained, for instance, if the agent is located on the circumcircle of the triangle $\Delta L_1L_2L_3$ one cannot localize it. In what follows, we address the problem of localization of a formation of three agents measuring the subtended angle at them by three landmarks in the presence of noise.

When there are three agents and two landmarks, there is no special technique for dealing with the noise. One simply proceeds using noisy measurements in lieu of noiseless measurements. However, when the noiseless equations are overdetermined, as for example with three landmarks, a modified method is required to handle the equations, temporarily assuming there is no measurement noise. As described before for each agent and any selection of two landmarks, we can define a circle. Call the center and the radius of the circle defined by $T_i, L_1$ and $L_2$, $a_i = [x_i, y_i]^T$ and $R_i$, respectively; term the center and the radius of the circle defined by $T_i, L_1$ and $L_3$, $a_i' = [x_i', y_i']^T$ and $R_i'$, and term the center and the radius of the circle defined by $T_i, L_2$ and $L_3$, $a_i'' = [x_i'', y_i'']^T$ and $R_i''$. All the equations that govern the system can be written as

\[
\begin{align*}
(x_T - x_i)^2 + (y_T - y_i)^2 - R_i^2 &= 0, \quad i \in \{1, 2, 3\} \\
(x_T - x_i')^2 + (y_T - y_i')^2 - R_i'^2 &= 0, \quad i \in \{1, 2, 3\} \\
(x_T - x_i'')^2 + (y_T - y_i'')^2 - R_i''^2 &= 0, \quad i \in \{1, 2, 3\} \\
(x_T - x_j)^2 + (y_T - y_j)^2 - R_j^2 &= 0, \quad i, j \in \{1, 2, 3\}
\end{align*}
\]  

(27.4)

where $x_T = [x_T, y_T]^T$ is the position of agent $T$. It is obvious that Equation 27.4 has a unique set of solutions for the positions of the $T_i$. This solution is a root of the following polynomial.

\[
P = \sum_{i=1}^{3} \left( (x_T - x_i)^2 + (y_T - y_i)^2 - R_i^2 \right)^2 \\
+ \sum_{i=1}^{3} \left( (x_T - x_i')^2 + (y_T - y_i')^2 - R_i'^2 \right)^2 \\
+ \sum_{i=1}^{3} \left( (x_T - x_i'')^2 + (y_T - y_i'')^2 - R_i''^2 \right)^2 \\
+ \sum_{i,j \in \{1,2,3\}} \left( (x_T - x_j)^2 + (y_T - y_j)^2 - R_j^2 \right)^2
\]  

(27.5)
Furthermore, it is easy to show that this solution is the global minimizer of Equation 27.5 as well. So the solution \( \hat{x}\hat{T} = [\hat{x}_{T1}, \hat{x}_{T2}, \hat{x}_{T3}]^T \) is obtained by

\[
\hat{x}_T = \arg\min \ P.
\] (27.6)

Now assume that the measurement is noisy, hence Equation 27.4 does not necessarily have a solution; however, in the light of the introduction of Equation 27.5, one can solve the minimization equation to obtain an estimate for the solution. There are readily available software packages that enable us to minimize such cost functions, for example, Reference 28. In the next section, we introduce some numerical simulations that illustrate the applicability of the methods introduced in this chapter to some possible scenarios.

**Example 27.3 Simulations with Three Agents and Three Landmarks**

Here we consider that three landmarks are placed at \([-1,0]^T, [1,0]^T, \text{and } [0,0]^T\]. The exact position of agents 1, 2, and 3 are, respectively, \([3,6]^T, [5,8]^T, \text{and } [2,3]^T\) (the positions are in meters). We consider that interagent distance measurements are corrupted by a Gaussian noise with a variance equal to \(0.25 \text{ m}^2\), and the angle measurements are corrupted by a Gaussian noise with a variance equal to \(0.0005 \text{ rad}^2\). Running the simulation for 20 times and solving Equation 27.6, we obtain \( \bar{x}_1 = [2.9492, 5.9528]^T, \bar{x}_2 = [4.9306, 7.9594]^T, \text{ and } \bar{x}_3 = [1.9290, 2.9609]^T \) as average values for the positions of \(T_1, T_2, \text{ and } T_3\), respectively. As is clear, the average estimates are very close to the actual values. Furthermore, the variances of all the solutions for the \(x\) coordinates of \(T_1, T_2, \text{ and } T_3\) are 0.0267, 0.0466, and 0.0443, and the variances of all the solutions for the \(y\) coordinates of \(T_1, T_2, \text{ and } T_3\) are 0.0155, 0.0095, and 0.0097, respectively. We have used Gloptipoly 3 [28] to solve the optimization problem generated from this scenario.

## 27.8 AVAILABILITY OF MORE ANGLE MEASUREMENTS FOR THREE AGENTS

So far, in this chapter, each agent is assumed to be able to detect the subtended angle by the landmarks at them only and not to be able to measure relative bearings to other agents or to one agent and one landmark. This scenario is more common if the agents are equipped with dedicated RF angle of arrival devices that measure bearings to transmitting beacons or detect other signals transmitted by other RF emitters of opportunity such as broadcast TV. Another scenario is that some or all agents have optical on-board sensors with a narrow field of view. In this case, the agents may not necessarily have other agents within their sensor field of view. However, use of, for example, a 360° camera sensor gives rise to a different scenario where the agents in the formation are capable of measuring not only the angle subtended by the landmarks at them but also the angle subtended at them by any pair of the agents or the landmarks.

In this section, we briefly consider this latter scenario, in which (as before) there are three agents and two landmarks. For agent \(T_i\), we denote the angle
subtended by $L_4$ and agent $T_j$, $\mu_{j,k}^i$, and the angle subtended by $T_i$ and $T_k$ by $\eta_i$. Measurements of these subtended angles as well as $\lambda_i$, as before, make it possible to localize up to two possible positions. We formally show this in the following theorem.

**Theorem 7**: Knowing all the angles $\lambda_i$, $i = 1,2,3$, $\eta_i$, $i = 1,2,3$, and $\mu_{j,k}^i$, $k = 1,2$, $i,j = 1,2,3$ and the positions of landmarks $L_1$ and $L_2$, one can find two solutions for the position of the agents $T_i$, $i = 1,2,3$, which are reflections through the line connecting the landmarks.

**Proof**: Consider a polygon with vertices defined by $T_i$, $i = 1,2,3$, and $L_{j',j} = 1,2$. By all the angles $\lambda_i$, $i = 1,2,3$, $\eta_i$, $i = 1,2,3$, and $\mu_{j,k}^i$, $k = 1,2$, $i,j = 1,2,3$, we can characterize all similar polygons with the same angles. Furthermore, knowing the distance between $L_1$ and $L_2$ fixes the scale of the polygon, knowing the position of either $L_1$ or $L_2$ fixes the two-dimensional translation, and knowing the position of the other landmark fixes the orientation of the polygon; and hence, the solution set is composed of up to two polygons that can be constructed via symmetry from each other, where $L_1L_2$ is the mirror axis.

**Remark 3**: Since we have already fixed on which side of $L_1L_2$ the formation is located, the only possibility is the polygon on the a priori known side of $L_1L_2$.

We conclude this section by the following example.

**Example 27.4**

Consider three agents $T_i, i = 1, \ldots, 3$, at unknown positions with $\lambda_1 = 0.2075, \lambda_2 = 0.1419, \lambda_3 = 0.0821$, $\eta_1 = 0.1419, \eta_2 = 2.8890$, and $\eta_3 = 0.1107$, $\mu_{1,2}^1 = 2.8023, \mu_{2,3}^1 = 3.0342, \mu_{3,1}^1 = 2.5948, \mu_{2,3}^2 = 2.7367, \mu_{3,1}^2 = 0.1419, \mu_{1,2}^3 = 3.0309, \mu_{2,3}^3 = 0.2838, \mu_{3,1}^3 = 3.1104, \mu_{1,2}^3 = 0.0476, \mu_{2,3}^3 = 0.0631, \mu_{3,1}^3 = 0.1297, \mu_{1,2}^3 = 0.0190, L_1 = (0,1)^T$, and $L_2 = [0,-1]^T$.

Using elementary geometric arguments one can calculate all the angles of the pentagon formed by $T_1, T_2, T_3, L_1$, and $L_2$. Using these angles along with the exact positions of $L_1$ and $L_2$, one can calculate two mirrored solutions with $L_1L_2$ as the reflection axis for the target positions. The two possible sets of positions for the agents are $T_1 = [\pm2,4]^T, T_2 = [\pm6,7]^T$, and $T_2 = [\pm10,12]^T$.

**27.9 CONCLUSIONS**

In this chapter, we have demonstrated that there are up to 12 possible localization solutions to the problem of localization of a formation composed of three agents, collecting bearing measurements to two landmarks at known positions and measuring their interagent distances. Furthermore, we extend this result to a more general case where a larger formation is to be localized using the same information as before. In addition, the effect of having extra landmarks on the number of the solutions to the problem is studied as well, and in this case a method to improve the accuracy of the localization solution is proposed. Some simulation results are presented to show the applicability of the methods. In the end, we briefly consider another closely
related localization problem, where the agents can measure all the subtended angles at them, and show the uniqueness of the localization for this case.

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