CHAPTER 24

POLYNOMIAL-BASED METHODS FOR LOCALIZATION IN MULTIAGENT SYSTEMS

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IN THIS chapter we review a series of results obtained in the field of localization that are based on polynomial optimization. First, we provide a review of a set of polynomial function optimization tools, including sum of squares (SOS). Then we present several applications of these tools in various sensor network localization tasks. As the first application, we propose a method based on SOS relaxation for node localization using noisy measurements and describe the solution through semidefinite programming (SDP). Later, we apply this method to address the problems of target localization in the presence of noise and relative reference frame determination based on range and bearing measurements. Some simulation and experiment results are provided to show the applicability of the methods proposed here.

24.1 INTRODUCTION

The problem of target localization has gained much attention recently. This problem involves fusion of different measurements, for example, angle of arrival, distance, and time difference of arrival, obtained from different sensing nodes with known global positions, called anchors, in a network to estimate the position of the target. Solving the localization problem in the case where there is no noise is heavily studied

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in the network localization literature, see for example, References 1 and 2. However, in almost all engineering applications, the assumption of having noiseless measurements is not realistic. And while the analysis of the problem of localization in the presence of measurement noise is still in its infancy, recently, in addition to trying to formally define and understand this problem (see, e.g., Reference 2), many algorithms have been proposed to tackle this problem. For example, in Reference 3, a linear algorithm to address this problem is proposed. Later in Reference 4, a nonlinear method using the *Cayley–Menger* determinant is proposed, and its performance is compared with that of Reference 3, which is shown to have better results. Another paper that has considered the application of Cayley–Menger determinant to solve localization problems is Reference 5, where an in-depth error analysis for the case of having three anchors is presented. In addition to the abovementioned methods, one can name other methods based on convex optimization [6–11], SOS relaxation [13], graph connectivity[14, 15], and multidimensional scaling [16].

To make the problem of noisy target localization clearer, consider the following illustration of a range-based localization scenario in the presence of noise. Consider a set of nodes scattered in an N-dimensional space, $N \in \{2, 3\}$. Consider $N_a > N$ anchor nodes, labeled 1, 2, … , N_a. Assume that each (anchor) node $I \in \{1, \dots, N_a\}$ is at a generic position $\mathbf{a}_i \in \mathbb{R}^N$; $\mathbf{a}_i = [x_i, y_i]^T$ for $N = 2$ and $\mathbf{a}_i = [x_i, y_i, z_i]^T$ for $N = 3$. Here, by having generic positions, we mean that for $N = 2$, no triple of anchor nodes are colinear, and for $N = 3$, no quadruple of anchor nodes are coplanar. Consider a nonanchor sensor node, node 0, which is placed at an unknown position $\mathbf{x} = [x, y]^T$, which can measure its distance from anchor nodes 1, 2, … , N_a, that is, it measures $d_i^* = \|\mathbf{a}_i - \mathbf{x}\|$, for each $i \in \{1, \dots, N_a\}$. Note that in this paper we denote the actual distance $\|\mathbf{a}_i - \mathbf{x}\|$ by d_i^* and the corresponding measurement by d_i. The problem of interest is finding an accurate estimate $\hat{\mathbf{x}}$ for $\mathbf{x}$ using the measurements d_i.

If the measurements are precise ($d_i = d_i^*$) the solution is trivial: $\hat{\mathbf{x}} = \mathbf{x}$ is the unique point of intersection of the N_a circles, C_i:

$$C_i : \|a_i - x\|^2 - d_i^2 = 0, \quad i \in \{1, \cdots, N_a\}. \tag{24.1}$$

The setting (for the case where the measurements are noisy) is illustrated in Figure 24.1. However, unfortunately, it is never the case that the measurements agree with the actual distances. Furthermore, we know that when the measurements are noisy, the circles may not have a common point of intersection, so one needs another method to solve the problem. All the methods mentioned earlier in a sense try to solve this problem.

In this chapter we will report several results related to sensor networks where optimization tools based on SOS, SDP formulation and relaxation, and algebraic geometry have been applied. First, we revisit the problem of localization of nodes in sensor networks and propose a solution using the tools from algebraic geometry and specifically SOS relaxation: this relaxation requires solving an SDP problem. The SOS approach has been already implemented to solve the problem of localization in sensor networks where range measurements are available; see References 12 and 17. However, here we propose a method for localization of the nodes using not only range measurements but other types of measurements as well, and in particular

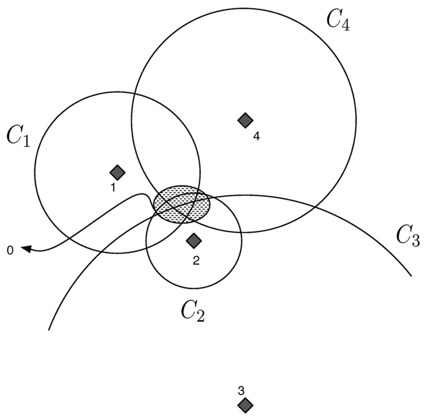


Figure 24.1 Distance based localization problem.

we formulate the problem of localization using noisy range difference measurements. Additionally we intend to introduce methodologies about how to combine different measurement types obtained from heterogeneous measurement devices to achieve the task of localization.

Later, we propose a solution to the robot pose determination problem, that is, determining the relative pose, or relative position and orientation, of a pair of robots that move on a plane, using the same methodology in two-dimensional and three-dimensional space, when the agents have access to distance or bearing measurements from each other. This problem is very important in many practical applications, such as target tracking [18] or sensor fusion [19].

24.2 POLYNOMIAL FUNCTION OPTIMIZATION

Before proceeding we present some mathematical notations and definitions that will be used in this chapter; the reader may refer to Reference 20 for further information. We use $\mathbb{R}[\boldsymbol{z}]$, where $\boldsymbol{z} = [z_1, \ldots, z_n]$, to denote the ring of all polynomials in n indeterminants (variables), $z_1, z_2, \ldots, z_n$, with real coefficients. The set $I \subset \mathbb{R}[\boldsymbol{z}]$ is an *ideal* if $q.h \in I$ for any $q \in I$ and $h \in \mathbb{R}[\boldsymbol{z}]$, where $q.h$ denotes the multiplication of polynomial q by polynomial h. Given polynomials $g_1, \ldots, g_r$ we write $(g_1, \ldots g_r)$ to represent the set of all polynomials that are polynomial linear combinations of $g_1, \ldots, g_r$.

Definition 1**:** Let $f_1, \ldots, f_m$ be polynomials in $\mathbb{R}[z_1, \ldots, z_n]$. Let the set V be

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in \mathbb{C} : f_i(a_1, \ldots, a_n) = 0, \forall 1 \le i \le m\}. \quad (24.2)$$

We call $V(f_1, \ldots, f_m)$ the *variety* defined by $f_1, \ldots, f_m$. Then, the set of polynomials that vanish in a given variety, that is,

$$I(V) = \{f \in \mathbb{R}[z_1, \ldots, z_n] : f(a_1, \ldots, a_n) = 0 \ \forall (a_1, \ldots, a_n) \in V\}, \quad (24.3)$$

is an ideal, called the *ideal of V*. Furthermore, the subset $V_{\mathbb{R}}(f_1, \ldots, f_m)$,

$$V_{\mathbb{R}}(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in \mathbb{R} : f_i(a_1, \ldots, a_n) = 0, \forall a \le i \le m\}. \tag{24.4}$$

is called the *real variety* defined by $f_1, \ldots, f_m$. By Hilbert's Basis Theorem every ideal $I \subset \mathbb{R}[z]$ is finitely generated. In other words, there always exists a finite set $\{f_1, \ldots, f_m\} \subset \mathbb{R}[z]$ such that for every $f \in I$, we can find $g_i \in \mathbb{R}[z]$ that satisfies $f = \sum_{i=1}^{m} g_i f_i$.

***Definition 2*:** Let $I \subset \mathbb{R}[z]$ be an ideal. The radical of I, denoted $\sqrt{I}$, is the set

$$\{f \mid f^k \in I \quad \textit{for some integer } k \ge 1\}. \tag{24.5}$$

It is clear that $I \subset \sqrt{I}$, and it can be shown that $\sqrt{I}$ is a polynomial ideal as well. We call an ideal I radical if $I = \sqrt{I}$.

***Definition 3*:** Consider a polynomial function f over the ring $\mathbb{R}[z]$. We call the ideal $I_{\nabla}(f) = \left\langle \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right\rangle$ the gradient ideal of f.

***Problem 1*:** Consider a polynomial function $F(z)$ of degree $2d$, over the polynomial ring $\mathbb{R}[z]$. Find F^* and $\mathbf{z}^* = [z_1^*, \ldots, z_n^*]$ such that $F^* = F(z^*) = \min F$.

An obvious method to find the solution to Problem 1 is to find all the critical points of F. The usual methods to solve such a system of nonlinear equations use Newton's method, and there is no guarantee that the solutions of the system will be found completely. However, there are other tools to find the the global minimum and minimizer of polynomial functions. We describe two of these methods briefly in what follows.

24.2.1 Polynomial Continuation (Homotopy) Methods

One exact method to find the global minimum of a polynomial optimization problem is to apply differentiation and then solve the resulting system of polynomial equations using the continuation (or homotopy) method. This method is suitable for finding isolated roots, and it basically defines a trivial polynomial system $Q(z)$ and a real variable, t, to form a family of curves by solving

$$0 = H(\mathbf{z}, t) = (1-t)Q(\mathbf{z}) + tP(\mathbf{z}).$$

Several properties need to be satisfied by $H(z)$ [21] to ensure that all solutions of $P(z) = 0$ are reached at $t = 1$ by continually deforming the trivial solutions of $Q(z) = 0$ obtained at $t = 0$. An upper bound of the number of curves (or paths) that need to be continually traced from $t = 0$ to $t = 1$ is given by the Bézout number, which is the product of the system polynomial degrees. This upper bound is often too large for the actual number of solutions that need to be considered. Numerous research papers have been published that provide much lower bound estimates and construct smaller polynomial systems, $Q(z)$. If the system is sparse [22], then polyhedral homotopies which use the Newton polytopes of $P(z)$ to construct the continuation paths are more efficient. The mixed volume of the Newton polytopes is usually much smaller than the Bézout number. Homotopy methods are often regarded as slow (because they require tracing an unacceptably large number of paths), but proponents of these methods always argue that they are well suited to parallel computing.

24.2.2 SOS and SDP Approaches

Another method to find the global minimum of a polynomial function, whose application is the focus of this chapter, is to use SOS relation. A polynomial function $F(z)$ of degree $2d$ over the polynomial ring $\mathbb{R}[z]$ is an SOS if one can write

$$F(z) = \sum_{i=1}^{q} Q_i^2(z) \tag{24.6}$$

where $q \in \mathbb{Z}^+$ and $Q_i(z)$ are the polynomials over $\mathbb{R}[z]$. Denote the global minimizer and global minimum of a polynomial function $F(z)$, respectively, by z^* and $\gamma = F(z^*)$. z^* can be calculated solving the following optimization problem:

$$\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && F(z) - \gamma \geq 0. \end{aligned} \tag{24.7}$$

One can relax Equation 24.7 and write it as

$$\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && F(z) - \gamma \quad \text{is SOS.} \end{aligned} \tag{24.8}$$

Remark 1**:** The relaxed problem is often computationally much easier to solve, and may yield the same solution. However, in general Equations 24.7 and 24.8 are not identical, since there are positive polynomials that are not in the SOS form. For more information see Reference 22.

We know that any SOS polynomial $F(z)$ of degree $2d$, with z as n-tuple of variables, can be written as $F(z) = \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{Z}$, where $\boldsymbol{Z}$ is a vector of all monomials of degree up to d obtained from the variables in z with the first entry equal to one, and $\boldsymbol{Q}$ is a positive semidefinite matrix obtained by solving a set of linear matrix inequalities (LMIs) [23]. So one can reformulate Equation 24.8 as,

$$\begin{aligned} &\text{maximize} && \gamma \\ &\text{subject to} && \boldsymbol{Q} - \hat{\boldsymbol{E}}\gamma \geq 0, \end{aligned} \tag{24.9}$$

where $\hat{\boldsymbol{E}}$ is a matrix with $E_{11} = 1$ and the rest of the entries are zero. The problem stated in Equation 24.9 is an SDP problem and can be solved by SDP techniques [22]. By solving the dual problem of the SDP problem stated in Equation 24.9, one can obtain the minimizer of F as well, using the procedure in Reference 24.

Remark 2**:** For polynomials with two variables and degree of 4, Equations 24.7 and 24.8 are equivalent ([23]).

In addition, we consider the following constrained optimization problem with polynomial F, g_i, and h_j:

$$\begin{aligned} &\text{minimize} && F(z) \\ &\text{subject to} && g_i(z) \geq 0 \quad i = 1, \dots, M \\ &&& h_j(z) = 0 \quad j = 1, \dots, N. \end{aligned} \tag{24.10}$$

Assume there exists a set of SOS $\sigma_i(z)$, and a set of polynomials $\lambda_j(z)$ such that

$$\begin{aligned} F(z) - \gamma = \sigma_0(z) + \sum_j \lambda_j(z) h_j(z) + \sum_i \sigma_i(z) g_i(z) \\ + \sum_i \sigma_{i_1,i_2}(z) g_{i_1}(z) g_{i_2}(z) + \sigma_M \prod_{i=1}^{M} g_i(z). \end{aligned} \tag{24.11}$$

Then, γ is a lower bound for Equation 24.10 [23]. So by maximizing γ as before, one can get a lower bound that gets tighter as the degree of Equation 24.11 increases. There are well-known SDP-based solutions to the aforementioned problem. For more information, one may refer to Reference 25 and references therein. We use this method to optimize the cost functions introduced in this chapter. There are a few implementations of this method as software packages that are readily available. The main two of such packages are `GloptiPoly 3` [26] and `SOSTOOLS` [27]. In what follows, we provide an example for polynomial optimization using `GloptiPoly 3` from Reference 26.

Example 24.1 Global Minimization of the Two-Dimensional Six-Hump Camel Back Function

Consider the following minimization problem:

$$\min_{\mathbf{z} \in \mathbb{R}^2} F(z)\ 4z_1^2 + z_1 z_2 - 4z_2^2 - 2.1 z_1^4 + 4 z_2^4 + \frac{1}{3} z_1^6.$$

This problem has six local minima, two of them being global minima. The problem can be solved via the following procedure in `Matlab`:

```
>> mpol z1 z2
>> F = 4*z1^2+z1*z2-4*z2^2-2.1*z1^4+4*z2^4+z1^6/3;
>> P = msdp(min(F));
>> [status,obj] = msol(P)
>> status
status =
1
>> obj
obj =
-1.0316
>> z = double([z1 z2]);
z(:,:, 1) =
0.0898 -0.7127
z(:,:, 2) =
-0.0898 0.7127
```

where `status=1` means that the problem is solved and `obj=-1.0316` is the global minimum. The two solutions for z correspond to two global minimizers of the function.

24.3 NOISY TARGET LOCALIZATION

In this section, application of the method in Section 22.2 to target localization using (1) distance measurements and (2) range difference measurements is presented. The formal problem definitions for these two application cases are given below, respectively, in Problem 2 and Problem 3.

***Problem 2*:** (Range-Based Localization). Consider N_a anchor nodes in $\mathbb{R}^N$ ($N \in \{2, 3\}$) at the known positions $\mathbf{a}_i$, $i \in \{1, \ldots, N_a\}$, and a node 0 at the unknown position $\mathbf{x}$. Let the noisy measurement d_i of the distance of node 0 to each node i *for* $i \in \{1, \ldots, N_a\}$ be available to node 0. The task of node 0 is to produce the least-square estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ using the noisy distance measurements $d_1, \ldots, d_n$.

Problems similar to Problem 2 have been already well-considered in the signal processing context, for example, References 3, 7, and 28. A solution to this problem can be obtained by finding the point $\hat{\mathbf{x}}$ which solves the following minimization problem:

$$\textit{minimize} \quad J_r(\mathbf{x}) = \sum_{i=1}^{n} \left(\|\mathbf{x} - \mathbf{a}_i\|^2 - d_i^2 \right)^2. \tag{24.12}$$

In general $J(\mathbf{x})$ is not convex (or concave), so ordinary convex optimization methods will not yield the desired result. In what follows some examples on target localization using distance measurements in two and three-dimensional space are presented.

Example 24.2 Localization of One Node with Distance Measurements to Three Anchors (Calculating the Variety)

We use the same setting as in Reference 4: Consider three anchor nodes at $\mathbf{a}_1 = [0, 0]^T$, $\mathbf{a}_2 = [7, 43]^T$, and $\mathbf{a}_3 = [0, 47]^T$, and a sensor node zero using distance measurements from these anchors to localize itself. The actual distances that are not available are $d_1^* = 34.392$, $d_2^* = 44.1106$, and $d_3^* = 41.2608$, while the noisy distance measurements by sensor 0 are $d_1 = 35$, $d_2 = 42$, and $d_3 = 43$. Hence,

$$J(\mathbf{x}) = \left(x^2 + y^2 - 1225\right)^2 + \left((x-43)^2 + (y-7)^2 - 1764\right)^2 + \left((x-47)^2 + y^2 - 1849\right)^2, \tag{24.13}$$

and solve,

$$\begin{aligned} \frac{\partial J}{\partial x} &= 4\left(x^2 + y^2 - 1225\right)x + 2\left((x-43)^2 + (y-7)^2 - 1764\right)(2x - 86) \\ &\quad + 2\left((x-47)^2 + y^2 - 1849\right)(2x - 94) \\ &= 0. \end{aligned} \tag{24.14}$$

$$\begin{aligned}\frac{\partial J}{\partial y} &= 4\left(x^2+y^2-1225\right)y+2\left((x-43)^2+(y-7)^2-1764\right)(2y-14)\\ &\quad+4\left((x-47)^2+y^2-1849\right)y\\ &=0.\end{aligned} \tag{24.15}$$

The real solutions to this system of equations construct the set,

$$V_{\mathbb{R}} = \left\{[3.43, 0.85]^{\mathrm{T}}, [13.90, 31.79]^{\mathrm{T}}, [18.22, -29.24]^{\mathrm{T}}\right\},$$

and by inspection $\mathbf{x}$ is found to be $[18.22,-29.24]^{\mathrm{T}}$ (this point is the global minimizer of the function.). Comparing with the actual position of sensor node 0, $\mathbf{x} = [17.9719,-29.3227]^{\mathrm{T}}$, it is seen that the estimate is considerably accurate. The result obtained here is the same as the one calculated in Reference 4. However, here, the steps for calculating the solution are less than those used in Reference 4. Furthermore, smaller number of floating point operations are used here, which increases the robustness of the method to numerical perturbations. See Chapter_24_Example_2.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

Example 24.3 Localization of One Node with Distance Measurements to Four Anchors in $\mathbb{R}^3$

In this example the three-dimensional case is considered. Consider four anchors at positions $\mathbf{a}_1 = [10, 13, 14]^{\mathrm{T}}$, $\mathbf{a}_2 = [5, 20, 40]^{\mathrm{T}}$, $\mathbf{a}_3 = [12, 15,-10]^{\mathrm{T}}$, and $\mathbf{a}_4 = [0, 5, 32]^{\mathrm{T}}$. Furthermore, sensor node 0 senses its distance to these anchors to be $d_1 = 22.4388$, $d_2 = 46.0740$, $d_3 = 22.8327$, and $d_4 = 33.3214$. Here,

$$V = \left\{[19.73, 4.92, -5.27]^{\mathrm{T}}, [29.95, 20.31, -1.32]^{\mathrm{T}}[-1.29, -0.79, -0.38]^{\mathrm{T}}\right\},$$

and $\hat{\mathbf{x}} = [-1.29, -0.79, -0.38]^{\mathrm{T}}$ is the global minimum of J. Our estimate is very close to the actual position of 0, $[0, 0, 0]^{\mathrm{T}}$. See Chapter_24_Example_3.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

Example 24.4 Localization of One Node with Distance Measurements to Three, Four, and Five Anchors (SOS)

In this example we compare the methods from References 3 and 4, and the one introduced here. Consider five anchors, 1, … , 5, are positioned at $\mathbf{a}_1 = [10, 31]^{\mathrm{T}}$, $\mathbf{a}_2 = [12, 45]^{\mathrm{T}}$, $\mathbf{a}_3 = [-9, 10]^{\mathrm{T}}$, $\mathbf{a}_4 = [30,-3]^{\mathrm{T}}$, and $\mathbf{a}_5 = [-7, 53]^{\mathrm{T}}$. The measured distances by the sensor node 0, at the unknown position $[0, 0]^{\mathrm{T}}$, to each of the anchors corrupted with a zero mean Gaussian noise with variance of $1\,\mathrm{m}^2$ are $\mathrm{d}_1 = 32.1404$, $\mathrm{d}_2 = 44.9069$, $\mathrm{d}_3 = 13.5789$, $\mathrm{d}_4 = 30.4373$, and $\mathrm{d}_5 = 52.3138$. The actual position of node 0 is $\mathbf{x} = [0, 0]^{\mathrm{T}}$. The estimated values for each of the methods using 3, 4, and 5 distance measurements are accessible from Tables 24.1 and 24.2. It can be noted that the linear method performs significantly worse than the nonlinear methods. In particular, for the case where three measurements are available, the improvement

TABLE 24.1. The Estimated Position of Node 0. The Method from Reference 3] Is Labeled "Linear," the Method from Reference 4] Is Labeled "C-M," and the Method Introduced in This Paper Is Labeled "Geometric"

No. Anchors	Linear	C-M	Geometric
3	$\begin{bmatrix} -4.8493 \\ 5.1347 \end{bmatrix}$	$\begin{bmatrix} 0.8825 \\ 1.0590 \end{bmatrix}$	$\begin{bmatrix} 0.8825 \\ 1.0590 \end{bmatrix}$
4	$\begin{bmatrix} 1.8971 \\ -0.0505 \end{bmatrix}$	N/A	$\begin{bmatrix} -0.0150 \\ 1.2858 \end{bmatrix}$
5	$\begin{bmatrix} -0.6306 \\ -0.1361 \end{bmatrix}$	N/A	$\begin{bmatrix} 0.2211 \\ 0.0710 \end{bmatrix}$

TABLE 24.2. The Error between the Estimated Position of Node 0 and Its Actual Position

No. Anchors	Linear	C-M	Geometric
3	7.0626	1.3785	1.3785
4	1.8977	N/A	1.2859
5	0.6452	N/A	0.2323

obtained by using the geometric method is very significant, this is due to the sensitivity of linear equations to perturbation in the coefficient. Moreover, for more than 4 measurements, the method based on Cayley-Menger determinant gets increasingly complicated and cannot be applied directly. See Chapter_24_Example_4.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes

Example 24.5 Localization of One Node with Distance Measurements to Five Anchors

In this example we compare the result obtained by applying our approach to the example introduced in Reference 11. Consider $N_a = 5$ anchors in $\mathbb{R}^2$ at $\mathbf{a}_1 = [4, 6]^T$, $\mathbf{a}_2 = [0,-10]^T$, $\mathbf{a}_3 = [5,-3]^T$, $\mathbf{a}_4 = [1,-4]^T$, and $\mathbf{a}_5 = [3,-3]^T$. The distance measurements to the sensor at the unknown position $[0, 0]^T$ corrupted by a zero-mean Gaussian noise with variance equal to 0.1 are $d_1 = 8.0051$, $d_2 = 13.0112$, $d_3 = 10.1138$, $d_4 = 7.7924$, and $d_5 = 8.0210$. The result obtained from applying range-based least square (R-LS) and squared-range-based least square (SR-LS) from Reference 7 for $\hat{\mathbf{x}}$ *are* $[-1.9907, 3.0474]^T$ and $[-2.018, 2.9585]^T$. Applying the method from this study we obtain the same value as SR-LS. It can be seen that the optimization method proposed in Reference 7 and the one based on SOS relaxation results in the same solution. See Chapter_24_Example_5.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

Now, we introduce another problem whose formulation was presented in Reference 7 in what immediately follows.

***Problem 3*:** (Range-Difference Based Cooperative Localization). Consider N_a anchor nodes in $\mathbb{R}^N$ ($N \in \{2, 3\}$) at the known positions $\mathbf{a}_i$, $i \in \{1, \ldots, N_a\}$, another node $N_a + 1$ at the origin, and node 0 at the unknown position $\mathbf{x}$. Let δ_i, the noisy measurement of $\delta_i^* = d_i^* - \|\mathbf{x}\|$ for $i \in \{1, \ldots, N_a\}$, be available (to node 0). The task (of node 0) is to produce the estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ using the noisy range difference measurements $\delta_1, \ldots, \delta_n$.

To solve the problem one is interested in, solve the following minimization problem:

$$\text{minimize } J_d(p) = \sum_{i=1}^{n} \left(\delta_i^2 - \|\mathbf{a}_i\|^2 + 2\delta_i \|\mathbf{x}\| + 2\mathbf{a}_i^{\mathsf{T}}\mathbf{x}\right)^2. \tag{24.16}$$

Denoting $\|\mathbf{x}\|$ by D, and considering that $D^2 - \|\mathbf{x}\|^2 = 0$, Equation 24.16 can be rewritten as

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{n} \left(\delta_i^2 - \|\mathbf{a}_i\|^2 + 2\delta_i D + 2\mathbf{a}_i^{\mathsf{T}}\mathbf{x}\right)^2, \\ &\text{subject to} \quad D^2 - \|\mathbf{x}\|^2 = 0,\ D \geq 0. \end{aligned} \tag{24.17}$$

By setting, $z = \mathbf{x}$, $F(z) = J_r(\mathbf{x})$ and using Equation 24.9, we can find the exact solution to Equation 24.12. For Equation 24.17, $F(z) = J_d(\mathbf{x})$, inequality constraints do not exist, and the only equality constraint is $h(\mathbf{x}) = \|\mathbf{x}\|^2 - D^2 = 0$, and we can find the solution to it by using the extended version of the methods introduced in the previous section. Now we present an example for this problem.

Example 24.6 Localization of One Node with Range-Difference Measurements to Five Anchors

In this example we consider another example presented in Reference 7. Consider five anchors and an extra anchor at the origin. The range difference measurements corrupted by a zero-mean Gaussian noise with the variance of 0.2 are, $\delta_1 = 11.8829$, $\delta_2 = 0.1803$, $\delta_3 = 4.6399$, $\delta_4 = 11.2402$, and $\delta_5 = 10.8183$. The result for $\mathbf{x}$ is $\hat{\mathbf{x}} = [-4.9798, 10.2786]^{\mathsf{T}}$, which agrees with the one obtained in Reference 7. See Chapter_24_Example_6.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

24.4 RELATIVE REFERENCE FRAME DETERMINATION

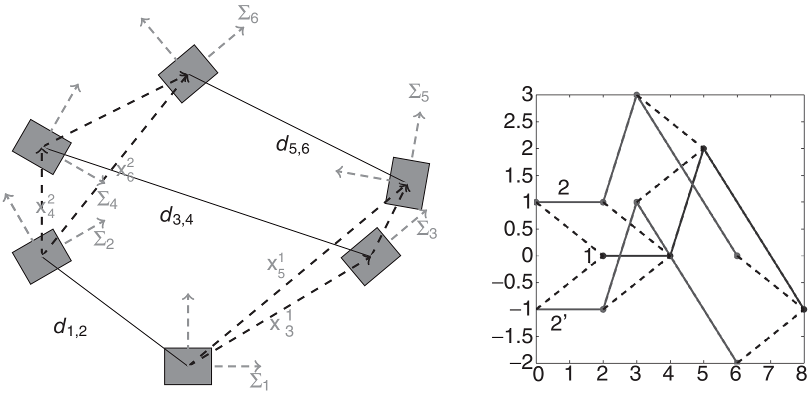
In this section we consider the problem of determining the rotation and translation associating the relative reference frames of a pair of robots. In the first subsection we consider the case where the robots can measure their distance from each other, and in the next one we consider the case where only relative angle measurements are available to one of the robots.

24.4.1 Relative Reference Frame Determination with Distance Measurements

In Reference 29, the problem of determining the relative reference frames of a pair of robots that move on a plane while measuring distance to each other is studied. Later, in Reference 30, the authors introduced the same problem in three-dimensional space. In what comes next, we state this problem.

***Problem 4*:** (Distance-Based Relative Reference (Pose) Determination). Consider two agents (robots) A_1 and A_2 in $\mathbb{R}^N$, $N = 2$, 3 whose initial reference frames are indicated by Σ_1 and Σ_2, respectively. The two agents move through a sequence of n unknown different positions with a reference frame associated with each position, $\Sigma_1, \Sigma_3, \ldots . \Sigma_{2n-1}$ for A_1 and $\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n}$ for A_2, where $n \in \mathbb{N}$. Their inter-agent distance, $d_{i,i+1}$ is measured at each of these positions, where $i \in \{1, 3, \ldots, 2n-1\}$. In addition, each agent is capable of estimating its current reference frame orientation and displacement with respect to its initial reference frame using dead-reckoning (odometry). In other words, A_1 and A_2 estimate the position vectors $\mathbf{x}_3^1, \ldots, \mathbf{x}_{2n-1}^1$ and $\mathbf{x}_4^2, \ldots, \mathbf{x}_{2n}^2$, respectively, where $\mathbf{x}_j^i$ is the position of the origin of Σ_j in Σ_i. Additionally, they know the rotation matrices that relate the orientations of their initial reference frame and all the others in their own sequence. Here we consider the rotation matrix $\mathbf{R}_j^i$ to be the rotation matrix relating Σ_i to Σ_j. The task is to find $\mathbf{x}_2^1$ and $\mathbf{R}_2^1$ using this information.

First, we consider the case $N = 2$ and assume that the origins of the reference frames of the agents at each time are the vertices of a graph, and if the distance between any pair of the origins is known, there is an edge connecting them together (see Figure 24.2a). We call the resulting graph $G_{a\text{–}a}$. Without loss of generality, we select Σ_1 as our reference frame for solving the problem. As a result of this selection, the origins of the reference frames $\Sigma_1, \Sigma_3, \ldots, \Sigma_{2n-1}$ can be calculated, and they can be considered as anchor points for the formation with $G_{a\text{–}a}$ as its underlying graph



(a) The Setting Considered in Problem 4 (b) A Non-Generic Case Where 2 Solutions Exist for the Pose Determination Problem

Figure 24.2 Pose-determination problem and a nongeneric scenario.

(in addition, the rotation matrix relating each of them to Σ_1 can be calculated as well.). The goal here is to find the positions of the origins of $\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n}$ in Σ_1. We further can compute $\mathbf{R}_2^1$ using any equation of the following form:

$$\mathbf{x}_{2i}^1 = \mathbf{x}_2^1 + \mathbf{R}_2^1 \mathbf{x}_{2i}^2, \quad i \in \{1, \ldots, n\}. \tag{24.18}$$

It is obvious that the origins of the reference frames $\Sigma_1, \ldots, \Sigma_{2n-1}$ form a complete graph, as do the origins of $\Sigma_2, \ldots, \Sigma_{2n}$. For $n \geq 4$, these two complete graphs are connected to each other with $n \geq 4$ edges. (Note that these edges do not share any vertex with each other.) From Reference 31 we know that the resulting graph is a globally rigid graph. Hence, the resulting formation when the origins of $\Sigma_1, \ldots, \Sigma_{2n-1}$ are considered anchors has a unique localization solution. As a result, due to the fact that there is a unique rotation relating Σ_1 to Σ_2, the agent-to-agent relative pose can be uniquely determined. We note that in Reference 29 the authors have suggested a maximum of four solutions for Problem 4 when $N = 2$ and $n = 4$ and with probability one, only one of the solutions can be valid. Now, we formally present the following result.

***Proposition 1*:** For $N = 2$ and $n \geq 4$, Problem 4 has a unique solution (generically).

One nongeneric scenario is when both agents move on a straight line [29]. Another nongeneric scenario is when the agents have the same displacement between any two consecutive measurements (see Fig. 24.2b). In these cases, there are two distinct solutions for Problem 4 and $N = 2$. Now we consider the three-dimensional case.

For the case $N = 3$, the origins of the reference frames $\Sigma_1, \ldots, \Sigma_{2n-1}$ form a complete graph, as do the origins of $\Sigma_2, \ldots, \Sigma_{2n}$. For $n \geq 4$, these two complete graphs are connected with n edges. (Note that these edges do not share any vertex with each other.) From Reference 31 we know that the resulting graph is a globally rigid graph if (and only if) $n \geq 7$. Hence, we have the following proposition.

***Proposition 2*:** For $N = 3$ and $n \geq 7$, Problem 4 has a unique solution (generically).

A nongeneric scenario is when the movement of each of agents is confined to a single plane.

24.4.2 Relative Reference Frame Determination with Relative Angle Measurements

In this section we introduce a problem similar to Problem 4 replacing distance measurements with angle measurements. First, we formally define the problem of interest.

***Problem 5*:** (Bearing-Based Relative Reference [Pose] Determination). Consider two agents (robots) A_1 and A_2 in $\mathbb{R}^N$, $N = 2, 3$ whose initial reference frames are indicated by Σ_1 and Σ_2, respectively. The two agents move through a sequence of n unknown different positions with a reference frame associated with each position, $\Sigma_1, \Sigma_3, \ldots, \Sigma_{2n-1}$ for A_1 and $\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n}$ for A_2, where n is a positive integer. Agent A_1 measures a unit vector corresponding to the bearing of agent A_2, $\boldsymbol{\theta}_{2i}^{2i-1}$, at each of these positions, where $i \in \{1, 2, \ldots, n\}$. Specifically,

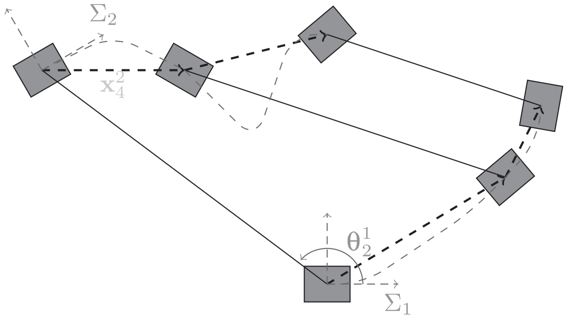


Figure 24.3 The setup of problem 5 with $n = 3$ measurements.

$\boldsymbol{\theta}_{2i}^{2i-1} = \left[\bar{\theta}_{2i-1,2i}, \bar{\theta}'_{2i-1,2i}, \bar{\theta}'_{2i-1,2i}\right]^{\mathsf{T}}$ is the unit vector associated with the line connecting $\mathbf{x}_{2i-1}^1$ and $\mathbf{x}_{2i}^1$ and in the direction toward $\mathbf{x}_{2i}^1$ in case $N = 3$, with obvious variation when $N = 2$. In addition, each agent is capable of estimating its current reference frame orientation and displacement with respect to its initial reference frame using dead-reckoning (odometry). In other words A_1 and A_2 estimate the position vectors $\mathbf{x}_3^1, \ldots, \mathbf{x}_{2n-1}^1$ and $\mathbf{x}_4^2, \ldots, \mathbf{x}_{2n}^2$, respectively, where $\mathbf{x}_j^i$ is the position of the origin of Σ_j in Σ_i. Additionally, they know the rotation matrices that relate the orientations of their initial reference frame and all the others in their own sequence. Here we consider the rotation matrix $\mathbf{R}_j^i$ to be the rotation matrix relating Σ_i to Σ_j. The task is to find $\mathbf{x}_2^1$ and $\mathbf{R}_2^1$ using this information.

We have the following result for three measurements.

***Lemma 1*:** For $N = 2, 3$ and $n = 3$, Problem 5 has at most eight solutions (Fig. 24.3).

***Proof*:** Each of the measurements $\boldsymbol{\theta}_{2i}^{2i-1}$, $i = 1, 2, 3$, corresponds to a line in Σ_1 such that $\exists t_{2i} \in \mathbb{R}$:

$$t_{2i}\boldsymbol{\theta}_{2i}^{2i-1} + \mathbf{x}_{2i-1}^1 = \mathbf{x}_{2i}^1, \tag{24.19}$$

where t_{2i} are unknown. The problem of interest is to find t_{2i}. Furthermore, $\mathbf{x}_2^1$, $\mathbf{x}_4^1$, and $\mathbf{x}_6^1$ should satisfy the following set of equations:

$$\left\|\mathbf{x}_4^2\right\|^2 = \left\|\mathbf{x}_2^1 - \mathbf{x}_4^1\right\|^2, \tag{24.20}$$

$$\left\|\mathbf{x}_6^2\right\|^2 = \left\|\mathbf{x}_2^1 - \mathbf{x}_6^1\right\|^2, \tag{24.21}$$

$$\left\|\mathbf{x}_4^2 - \mathbf{x}_6^2\right\|^2 = \left\|\mathbf{x}_4^1 - \mathbf{x}_6^1\right\|^2. \tag{24.22}$$

Using Equation 24.19, we obtain from this equation set the following three equations, each of degree 2 and with three unknowns, viz., t_2, t_4, t_6:

$$\left\|\mathbf{x}_4^2\right\|^2 = \left\|t_2\boldsymbol{\theta}_2^1 - t_4\boldsymbol{\theta}_4^3 - \mathbf{x}_3^1\right\|^2, \tag{24.23}$$

$$\left\|\mathbf{x}_6^2\right\|^2 = \left\|t_2\boldsymbol{\theta}_2^1 - t_6\boldsymbol{\theta}_6^5 - \mathbf{x}_5^1\right\|^2, \tag{24.24}$$

$$\left\|\mathbf{x}_4^2 - \mathbf{x}_6^2\right\|^2 = \left\|t_4\boldsymbol{\theta}_4^3 + \mathbf{x}_3^1 - t_6\boldsymbol{\theta}_6^5 - \mathbf{x}_5^1\right\|^2. \tag{24.25}$$

From Reference 32 we know that a system of m polynomial equations in m unknowns have at most $\prod_{j=1}^{m} q_j$ real solutions where q_j is the degree of the jth equation. Hence, the system of equations composed of Equations 24.23–24.25 has at most eight real solutions. Now, since we already know that one real solution already exists (the solution that corresponds to the actual scenario), the system has at least one real solution as well.

For more than three measurements, we have the following result.

***Proposition 3*:** For $N = 2$, 3 and $n \geq 4$, Problem 5 generically has a unique solution.

Proof: It is enough to show that Problem 5 has a unique solution for $n = 4$. For $n = 4$ we have

$$\left\|\mathbf{x}_4^2\right\|^2 = \left\|t_2\boldsymbol{\theta}_2^1 - t_4\boldsymbol{\theta}_4^3 - \mathbf{x}_3^1\right\|^2, \tag{24.26}$$

$$\left\|\mathbf{x}_6^2\right\|^2 = \left\|t_2\boldsymbol{\theta}_2^1 - t_6\boldsymbol{\theta}_6^5 - \mathbf{x}_5^1\right\|^2, \tag{24.27}$$

$$\left\|\mathbf{x}_8^2\right\|^2 = \left\|t_2\boldsymbol{\theta}_2^1 - t_8\boldsymbol{\theta}_8^7 - \mathbf{x}_7^1\right\|^2, \tag{24.28}$$

$$\left\|\mathbf{x}_4^2 - \mathbf{x}_6^2\right\|^2 = \left\|t_4\boldsymbol{\theta}_4^3 + \mathbf{x}_3^1 - t_6\boldsymbol{\theta}_6^5 - \mathbf{x}_7^1\right\|^2, \tag{24.29}$$

$$\left\|\mathbf{x}_6^2 - \mathbf{x}_8^2\right\|^2 = \left\|t_6\boldsymbol{\theta}_6^5 + \mathbf{x}_5^1 - t_8\boldsymbol{\theta}_8^7 - \mathbf{x}_7^1\right\|^2, \tag{24.30}$$

$$\left\|\mathbf{x}_4^2 - \mathbf{x}_8^2\right\|^2 = \left\|t_4\boldsymbol{\theta}_4^3 + \mathbf{x}_3^1 - t_8\boldsymbol{\theta}_8^7 - \mathbf{x}_7^1\right\|^2. \tag{24.31}$$

Calculating t_4,t_6,t_8 from Equations 24.26–24.28 in terms of t_2 and substituting them in Equations 24.29–24.31 and squaring the equations twice to get rid of the square roots, we obtain 3 degree of 8 equations in t_2. Using similar argument as in section IV of Reference 29, it can be shown that such system of polynomial equations has generically a unique solution. Following the same procedure for t_4, t_6, and t_8 we can find unique values for them as well, hence the problem has only one solution for $n = 4$. The result for $n > 4$ follows immediately.

24.4.3 Noisy Relative Reference Frame Determination

We already mentioned that Problem 4 for $N = 2$ has a unique solution, where there are four or more measurements. Here we consider the case where we have $n \geq 4$ measurements. The following set of equations governs the system for calculating $\mathbf{x}_2^1$ when we have range measurements available.

$$\begin{aligned} \left\|\mathbf{x}_{2i-1}^1 - \mathbf{x}_{2i}^1\right\|^2 &= d_{2i-1,2i}^2, \quad i \in \{1, \ldots, n\} \\ \left\|\mathbf{x}_{2j}^1 - \mathbf{x}_{2i}^1\right\|^2 &= \left\|x_{2j}^2\right\|^2, \quad i, j \in \{1, \ldots, n\} \end{aligned} \tag{24.32}$$

The solution to Equation 24.32, $\hat{\mathbf{x}} = \left[\hat{\mathbf{x}}_2^{1\mathrm{T}}, \ldots, \hat{\mathbf{x}}_{2n}^{1\mathrm{T}}\right]$, is a root of the following polynomial as well:

$$P' = \sum_{i=1}^{n} \left(\left\| \mathbf{x}_{2i-1}^{1} - \mathbf{x}_{2i}^{1} \right\|^{2} - d_{2i-1,2i}^{2} \right)^{2} + \sum_{i,j \in \{1,\ldots,n\}} \left(\left\| \mathbf{x}_{2j}^{1} - \mathbf{x}_{2i}^{1} \right\|^{2} - \left\| \mathbf{x}_{2j}^{2} - \mathbf{x}_{2i}^{2} \right\|^{2} \right)^{2}. \quad (24.33)$$

And similarly to before, it is easy to show that this solution is the global minimizer of Equation 24.33 as well. So the solution, $\hat{\mathbf{x}}$ is obtained by

$$\hat{\mathbf{x}} = \arg\min P'. \quad (24.34)$$

Again, in the presence of noise, Equation 24.32 does not have a solution, and the best estimate can be obtained by solving Equation 24.34.

To calculate $\mathbf{R}_2^1 = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$, we proceed as follows. Each vector equation of type Equation 24.18 consists of two scalar equations linear in $\cos\phi$ and $\sin\phi$, of the form

$$\Phi_{i1}(\phi) = 0, \Phi_{i2}(\phi) = 0 \quad (24.35)$$

for $i \in \{4, \ldots, 2n\}$. Furthermore, consider the following cost function:

$$J_\phi(\phi) = \sum_{i \in \{4,\ldots,2n\}} \left(\Phi_{i1}^2(\phi) + \Phi_{i2}^2(\phi) \right). \quad (24.36)$$

The best estimate for ϕ, $\hat{\phi}$ is obtained by

$$\hat{\phi} = \arg\min J_\phi. \quad (24.37)$$

Moreover, substituting $\sin\phi$ and $\cos\phi$ with x_s and x_c, we can rewrite Equation 24.37 as

$$\begin{gathered} [\hat{x}_s, \hat{x}_c]^{\mathrm{T}} = \arg\min J_\phi \\ \text{subject to } x_s^2 + x_c^2 = 1, \end{gathered} \quad (24.38)$$

where $\sin\hat{\phi} = \hat{x}_s$ and $\cos\hat{\phi} = \hat{x}_c$.

Furthermore, in order to answer Problem 4 having $n \geq 7$ measurements, we construct the following cost function:

$$P_3' = \sum_{i=1}^{n} \left(\left\| \mathbf{x}_{2i-1}^{1} - \mathbf{x}_{2i}^{1} \right\|^{2} - d_{2i-1,2i}^{2} \right)^{2} + \sum_{i,j \in \{1,\ldots,n\}} \left(\left\| \mathbf{x}_{2j}^{1} - \mathbf{x}_{2i}^{1} \right\|^{2} - \left\| \mathbf{x}_{2j}^{2} - \mathbf{x}_{2i}^{2} \right\|^{2} \right)^{2} \quad (24.39)$$

The global minimizer of P_3', $\hat{\mathbf{x}} = \left[\hat{\mathbf{x}}_2^{1\mathrm{T}}, \ldots, \hat{\mathbf{x}}_{2n}^{1\mathrm{T}} \right]$ gives us the solution to the first part of Problem 4 for $N = 2$. To calculate the rotation matrix $\mathbf{R}_2^1$, first we note that this rotation matrix can be written as found in Reference 33:

$$\begin{bmatrix} s^2 + t^2 - u^2 - v^2 & 2(tu - sv) & 2(tv + su) \\ 2(tu + sv) & s^2 - t^2 + u^2 - v^2 & 2(uv - st) \\ 2(tv - su) & 2(uv + st) & s^2 - t^2 - u^2 + v^2 \end{bmatrix}, \quad (24.40)$$

where

$$s^2 + t^2 + u^2 + v^2 = 1, \quad (24.41)$$

and s, t, u, $v \in \mathbb{R}$. Then each of the equations of the form

$$\mathbf{x}_{2i}^{1} = \mathbf{x}_{2}^{1} + \mathbf{R}_{2}^{1}\mathbf{x}_{2i}^{2}, \quad i \in \{1, \cdots, n\} \tag{24.42}$$

results in three scalar equations $\Phi_{ij} = 0$, $j = 1, 2, 3$, in indeterminants s, t, u, and v. We construct the objective function,

$$J_{\Phi}(s, t, u, v) = \sum_{i=1}^{n\geq 7} \sum_{j=1}^{3} \Phi_{ij}^{2}. \tag{24.43}$$

Then, the global minimizer of J_{Φ} subject to the constraint $s^2 + t^2 + u^2 + v^2 = 1$ is the estimate for $\mathbf{R}_2^1$. One can use the method based on SOS that was introduced earlier to find the global minimizers of J_p and J_{Φ} subject to $s^2 + t^2 + u^2 + v^2 = 1$.

For pose determination with angle measurements, as before, we construct the following cost function for $n \geq 4$ angle measurements:

$$P'_{AOA} = \sum_{i,j\in\{1,\ldots,n\}} \left(\left\| \mathbf{x}_{2j}^{1} - \mathbf{x}_{2i}^{1} \right\|^{2} - \left\| \mathbf{x}_{2j}^{2} \right\|^{2} \right)^{2}. \tag{24.44}$$

Or equivalently,

$$P'_{AOA} = \sum_{i,j\in\{1,\ldots,n\}} \left(\left\| t_{2i}\boldsymbol{\theta}_{2i}^{2i-1} + \mathbf{x}_{2i-1}^{1} - t_{2j}\boldsymbol{\theta}_{2j}^{2j-1} - \mathbf{x}_{2j-1}^{1} \right\|^{2} - \left\| \mathbf{x}_{2j}^{2} \right\|^{2} \right)^{2}. \tag{24.45}$$

For $\hat{\mathbf{x}}_2^1$ we have

$$\hat{\mathbf{x}}_{2}^{1} = \hat{t}_{2i}\boldsymbol{\theta}_{2i}^{2i-1} + \mathbf{x}_{1}^{1}. \tag{24.46}$$

To calculate the rotation matrix, we construct and minimize the same cost functions as we used for the range measurement case.

We minimize the polynomial cost functions introduced here and find their global minimizers using SOS relaxation. We conclude this section by presenting some examples for reference-frame determination using noisy distance and bearing measurements.

Example 24.7 Pose Determination with Four Distance Measurements

In the first scenario that we consider $\mathbf{x}_3^1 = [2.0407, -0.9862]^{\mathsf{T}}$, $\mathbf{x}_5^1 = [3.0233, 1.9054]^{\mathsf{T}}$, $\mathbf{x}_7^1 = [4.0186, 0.9239]^{\mathsf{T}}$, $d_{12} = 7.1626$, $d_{34} = 5.9512$, $d_{56} = 6.6932$, $d_{78} = 5.0052$, $\mathbf{x}_4^2 = [1.0555, -1.9487]^{\mathsf{T}}$, $\mathbf{x}_6^2 = [5.0235, 0.9997]^{\mathsf{T}}$, and $\mathbf{x}_8^2 = [7.0963, -3.1266]^{\mathsf{T}}$. The distance measurements are noisy, and the noise is considered to be a random Gaussian variable with zero mean and variance equal to $0.1\,\text{m}^2$. Solving the optimization problem corresponding to it we obtain, $\hat{\mathbf{x}}_2^1 = [1.2479, 7.0525]^{\mathsf{T}}$, and $\hat{\phi} = 0.0312$ rad. Comparing with the real values; $\mathbf{x}_2^1 = [1, 7]^{\mathsf{T}}$, and $\phi = 0$, we observe that estimates are very close to the real values. See Chapter_24_Example_7.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

Example 24.8 The Effect of Different Noise Levels on Pose-Determination Accuracy

In the second scenario, we aim to study the effect of different levels of noise in distance measurements and odometry readings on the solution. First we set the distance measurement variance equal to $0.1\,\mathrm{m}^2$ and set the noise variance on odometry readings $0.0001\,\mathrm{m}^2$. After repeating the procedure for 100 times, the average of the position estimate error magnitude, $\|\hat{\mathbf{x}}_2^1 - \mathbf{x}_2^1\|$, is 0.2665 m, and the average of the angle estimate error, $|\hat{\phi} - \phi|$ is 0.0311 rad. Then we set the distance measurement variance equal to $0.0001\,\mathrm{m}^2$ and set the noise on odometry readings $0.1\,\mathrm{m}^2$. After repeating the procedure for 100 times, the average of the position estimate error magnitude, $\|\hat{\mathbf{x}}_2^1 - \mathbf{x}_2^1\|$, is 0.4078 m, and the average of the angle estimate error, $|\hat{\phi} - \phi|$ is 0.0385 rad. While the average errors of the angle estimate in the two cases are close, the magnitude of the error of the position estimate is somewhat larger in the second scenario with larger odometry error, suggesting it to be more problematic. A reason for this phenomenon might be the larger number of odometry measurements used in the construction of the cost functions compared to the number of distance measurements. The code to test this scenario is very similar to the one presented in the MATLAB codes that can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

Example 24.9 Pose Determination with Four Bearing Measurements

In this scenario we test frame determination using four angle measurements. We consider $\mathbf{x}_3^1 = [2.3, 5]^\mathsf{T}$, $\mathbf{x}_5^1 = [4, 2]^\mathsf{T}$, $\mathbf{x}_7^1 = [9\ 0]^\mathsf{T}$, $\boldsymbol{\theta}_2^1 = [0.9806, 0.1961]^\mathsf{T}$, $\boldsymbol{\theta}_4^3 = [0.2873, 0.9578]^\mathsf{T}$, $\boldsymbol{\theta}_6^5 = [0.1961, 0.9806]^\mathsf{T}$, $\boldsymbol{\theta}_8^7 = [0.9487, -0.3162]^\mathsf{T}$, $\mathbf{x}_4^2 = [-3, 3]^\mathsf{T}$, $\mathbf{x}_6^2 = [0, 6]^\mathsf{T}$, and $\mathbf{x}_8^2 = [-2, 1]^\mathsf{T}$. The angle measurements are noisy, and the noise is considered to be a random Gaussian variable with zero mean and variance equal to 0.01 rad^2. Solving the optimization problem corresponding to it we obtain, $\hat{\mathbf{x}}_2^1 = [5.0585, 1.0747]^\mathsf{T}$, and $\hat{\phi} = 0.0254$ rad. Comparing with the real values, $\mathbf{x}_2^1 = [5, 1]^\mathsf{T}$, and $\phi = 0$, we observe that estimates are very close to the real values. See Chapter_24_Example_9.m for more details. MATLAB codes can be found online at ftp://ftp.wiley.com/public/sci_tech_med/matlab_codes.

24.4.4 Algorithmic Comparison with Some Existing Methods

The formulated localization problem as in Reference 4 with N_a anchors will now be described. Let ε_i be the error in the estimated squared distances between sensor 0 and anchor i. We want to minimize the sum of squared errors

$$J_{CM} = \varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2, \tag{24.47}$$

subject to $N_a - 2$ equality constraints

$$f_i(\varepsilon_1, \varepsilon_2, \varepsilon_i) = 0, \quad i = 3, 4, \ldots, n, \tag{24.48}$$

where f_i are obtained from writing different Cayley–Menger determinants [4] for different sets of anchor nodes. Using the Lagrange multiplier method, this minimization is equivalent to minimizing

$$H(\varepsilon_1, \ldots, \varepsilon_n, \lambda_1, \ldots, \lambda_{n-2}) = \sum_{i=1}^{n} \varepsilon_i^2 + \sum_{i=1}^{n-2} \lambda_i f_{i+2}(\varepsilon_1, \varepsilon_2, \varepsilon_{i+2}). \qquad (24.49)$$

One way of solving the above minimization problem is to differentiate Equation 24.49 with respect to variables ε_i and λ_i, and to find the zeros of these differentials. In this process we have $2n - 2$ equations, each with degree 2, that need to be solved using a nonlinear root finding method (see Example 24.4). Hence, since in our proposed method adding anchors has no effect on the number of equations and variables, we always have a system of polynomial equations comprising two equations and two unknowns.

For three anchors, the results obtained using the method introduced here and the one in Reference 4 are the same and are clearly better than the result obtained from linear algorithm of Reference 3.

The result obtained by applying the SR-LS method from Reference 7 in Example 24.5 and the method introduced here give the same numerical result. However, in solving SR-LS, positive definiteness of a certain matrix is assumed while here no assumption is required.

For more than three anchors, the method introduced in Reference 4 does not result in a solution using the `fsolve` routine in MATLAB, which uses a quasi-Newton algorithm to solve a set of nonlinear equations. The method introduced in this paper, however, yields a result that is more accurate compared with the one obtained by the linear method in Reference 3.

In the case of localization using range difference measurements, the biggest difference between the method introduced here and the one in Reference 7 is the number of steps to reach the final result. In this paper we only need to solve one minimization problem, while to find the result using the technique from Reference 7, one needs to go through a four-stage algorithm.

Comments on the Complexity of SOS Methods: With the addition of more unknowns (hence introduction of more variables and consequently higher dimension LMIs) the complexity of the SOS-based solution increases rapidly, but it still remains reasonable for up to 10 agents (20 variables). However, if one is interested in solving problems with a large number of variables, one should take advantage of the sparsity in the cost function to reduce the size of the underlying LMIs (see [12]). For more information on the size of LMIs to be solved, the reader may refer to Reference 34.

To reduce the numerical sensitivity of the methods, one may want to decrease the condition number of the matrices involved in solving the optimization problems; one way to do this is to normalize the coefficients of the cost functions. The choice of the polynomial δ in Equation 24.11 does not follow any rigid guidelines, but as a rule of thumb, the higher the order one chooses for δ, the better the bound one obtains on the optimal solution but at the expense of a considerable increase of the problem size (i.e., increase in matrix dimensions).

24.4.5 Colinear Anchors

In this section we discuss the situation (in $\mathbb{R}^2$) where the anchors are colinear. Consider N_a colinear anchors labeled 1, 2 to N_a at positions $\mathbf{a}_i \in \mathbb{R}^2$, where $i \in \{1, \ldots, N_a\}$. Furthermore, sensor node 0 can measure its distance d_i from the anchor i. The geometric representation of a problem with five colinear anchors in $\mathbb{R}^2$ is depicted in Figure 24.4. In the case where the distances are exact, there are two points of intersection for the circles; however, in the presence of noise, the circles will not necessarily intersect at any common intersection point. Defining $J(p)$ as before and solving the related optimization problem, we can compute an estimate for the position of node 0, and since the anchors are colinear, the other possible position for node 0 is the mirror of this point where the line connecting the anchors is the mirror axis.

Using the method here one can have estimates for the two possible positions of node 0.

Example 24.10 Localization with Colinear Anchors

Consider the problem of localization as depicted in Figure 24.4. The anchors are placed at $\mathbf{a}_1 = [10, 31]^T$, $\mathbf{a}_2 = [17, 45]^T$, $\mathbf{a}_3 = [-17, -23]^T$, $\mathbf{a}_4 = [-10, -9]^T$, and $\mathbf{a}_5 = [19, 49]^T$. Again, it is considered that distance measurements are being taken by sensor node 0 in the presence of a zero-mean Gaussian noise with variance of $1\,\text{m}^2$. The measured distances from the sensor to the anchors are $d_1 = 32.8674$, $d_2 = 46.7679$, $d_3 = 29.3150$, $d_4 = 15.0772$, and $d_5 = 51.8630$. The actual position of sensor node 0 is at $[0, 0]^T$. The estimates of the node 0 position (the ones closer to the actual position) using 3, 4, and 5 distance measurements are presented in Table 24.3. The code to generate this scenario is similar to that of Example 24.4.

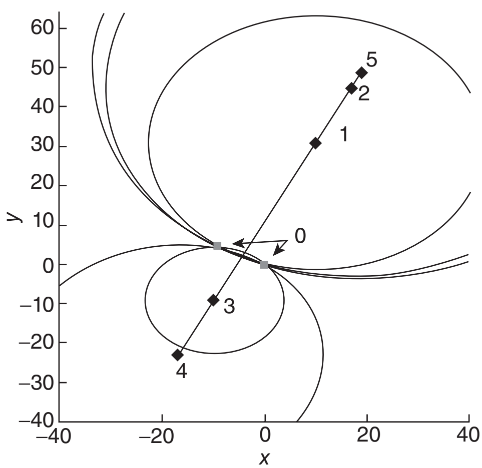


Figure 24.4 The localization problem with colinear anchors [13].

TABLE 24.3. The Error between the Estimated Position of Node 0 and Its Actual Position

No. Anchors	3	4	5
$\hat{\mathbf{x}}$	$\begin{bmatrix} -3.8399 \\ 2.8925 \end{bmatrix}$	$\begin{bmatrix} -4.0187 \\ 3.0142 \end{bmatrix}$	$\begin{bmatrix} -3.8906 \\ 2.8935 \end{bmatrix}$

24.5 AN EXTENSION OF THE SOS APPROACH

There are situations where the equivalence stated in Remark 2 does not necessarily hold, for example, in the three dimensional case, which means the solution obtained by SOS relaxation is not necessarily the global optimum. However, according to Reference 25, one can construct a finite sequence of values that converges to the global minimum of a polynomial function if the gradient ideal of that polynomial function is radical. Next, we establish that this condition holds for almost all the polynomials of degree d in $\mathbb{R}[z]$.

***Lemma 2*:** For almost all polynomials f of degree d in the ring $\mathbb{R}[z]$, the gradient ideal $I_\nabla(f)$ is radical and the gradient variety $V_\nabla(f)$ is a finite subset of $\mathbb{C}^n$.

Before proceeding to prove the lemma, we present the following theorem.

***Theorem 1*:** ([35]) Let I be a zero dimensional ideal in $\mathbb{C}[z]$, and let $A = \mathbb{C}[z]/I$. Then $\dim_{\mathbb{C}}(A)$ is greater than or equal to the number of points in $V(I)$. Moreover, equality occurs if and only if I is a radical ideal.

This means that in the case where I is not radical, there are multiplicities at each point in $V(I)$ so that the sum of the multiplicities is equal to $\dim_{\mathbb{C}}(A)$.

***Proof of Lemma 2*:** Consider the vector space S of all polynomials f in n variables of degree d. By Theorem 1, the condition for a polynomial f to have either infinitely many critical points or to have a critical point with multiplicity ≥2, or equivalently have a nonradical gradient ideal, is a closed condition. In order to prove the lemma, we simply need to show that the complementary open set in S is nonempty. For that purpose it suffices to show that there exists one polynomial that has a finite number of critical points each with multiplicity 1. One such polynomial is

$$f(z) = z_1^d + z_2^d + \cdots + z_n^d - dz_1 - dz_2 - \cdots - dz_n$$

It has a finite number of critical points, and its gradient ideal $I_\nabla(f) = \langle z_1^{d-1} - 1, z_2^{d-1} - 1, \ldots, z_n^{d-1} - 1\rangle$ is radical because it has $n(d - 1)$ distinct complex roots. Thus, generically, the gradient ideal of any f is radical.

Having established Lemma 2, we present the following optimization problem formulation that Reference 25 used to design an algorithm to construct a sequence of values converging to the global minimum:

$$\begin{aligned} &\text{maximize} \quad \gamma \\ &\text{subject to} \quad F(z) - \gamma - \sum_{j=1}^{n} \phi_j(z) \frac{\partial F}{\partial z_j} \text{ is SOS'} \end{aligned} \tag{24.50}$$

where $\phi_j(z) \in \mathbb{R}[z]$ is a polynomial of maximum degree $2N - d + 1$. Call $F^*_{\nabla,N}$ the optimal value obtained by solving Equation 24.50. $F^*_{\nabla,N}$ is a lower bound for F^*, which gets tighter as N increases. Furthermore Reference 25 shows that if I_∇ is radical, there exists an integer N such that $F^*_{\nabla,N} = F^*$.

However, while for generic cost functions one can solve the optimization problem with the abovementioned method, the situations that we introduced throughout this chapter cannot be considered as generic, since the coefficients of the polynomial cost functions are related to each other through algebraic relations, which are the direct consequence of the measurements. To clarify, consider a polynomial cost function constructed from $\bar{m}$ measurements $\mu_1, \ldots \mu_m$ with coefficients $c_1, \ldots c_n$. These coefficients can be written as the functions of the measurements, that is, $c_i = f_i\ (\mu_1, \ldots \mu_m)$ for some function $f_i(\cdot)$. This in turn results in the existence of an algebraic map g_i such that $c_i = g_i\left(\left\{c_j\right\}_{j=1, j\neq i}^{\bar{n}}\right)$. Nonetheless, this does not eliminate the possibility of the ideals of these cost functions being radical. Hence, the problem of showing that the ideals of the cost functions introduced here are radical and the applicability of the method introduced in this section to minimize these cost functions remains as an open problem.

24.6 CONCLUSIONS

In this chapter we have briefly introduced the idea of localization and pose determination in the presence of noisy measurements using polynomial optimization. We have introduced two methods for optimizing polynomials and particularly implemented SOS relaxations to solve some common problems in localization and pose determination. Some examples were provided further to show the applicability of the methods. In the end, we provided a condition for having exact global optimums for polynomial functions using SOS relaxation and established that for generic polynomials, this condition is satisfied.

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