MULTIVARIABLE DESIGN PROBLEM REDUCTION TO SCALAR DESIGN PROBLEMS

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ABSTRACT

The paper studies the problem of determining for a square multivariable plant, a precompensator, with possibly a postcompensator, such that the compensated plant transfer function matrix is diagonal or triagonal. The compensators are restricted to being stable and proper, their determinants have no right half plane zeros, and in some cases the compensators have proper inverses. If desired, compensator zeros and poles can be concentrated in a more restricted region than $Re[s] < 0$.

1. INTRODUCTION

A common theme among the UK workers in multivariable system design has been that of devising a technique to permit the use of scalar classical control system design methods on a multivariable system. See e.g. [1-4]. Typically, some form of plant precompensator is sought such that the transfer function matrix of the plant and precompensator combined is diagonal, triangular, or dominant on the $j\omega$ axis.

A number of different approaches have been advanced for obtaining such precompensators; in this paper we advance further approaches. Their applicability is yet to be validated on examples, so that we make no claim to comprehensiveness at this point. Perhaps the most negative feature of the approaches is the lack of control over the McMillan degree of the compensators, while the most positive feature would seem to be the theoretical comprehensiveness.

In this paper, we shall restrict attention to obtaining diagonal or triangular structures for the compensated plant, sometimes in the case of the diagonal structure with a post compensator as well as a precompensator. (One may then seek to move the postcompensator around the loop to a possibly less objectionable position). Some results on obtaining dominant structures have also been obtained but are in too much of a preliminary form to report at this stage.

The main tools for obtaining the compensators are the Smith and Hermite forms of a matrix [5], but not of a matrix of polynomials as is usual, but of a matrix of rational transfer functions with restrictions of properness and stability. The necessary algebraic background is summarized in the next section. In Section 3, we discuss the restrictions which it is desirable to impose on compensators, and in Sections 4 and 5 we discuss the construc-

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tion of compensators for, respectively, diagonalizing and triangularizing. In Section 6, we argue that by broadening slightly the class of compensators used for triangularizing, we may be able to diagonalize more easily than via the procedure of Section 4. Section 7 contains some concluding remarks.

2. ALGEBRAIC PRELIMINARIES

In linear system theory, two matrix algebra constructions have found significant application, the constructions leading to the Smith form and the Hermite form of a polynomial matrix [5]. The Smith form construction has been extended to provide a construction -- the Smith McMillan form-for rational matrices, see e.g. [6].

In linear algebra, the Smith and Hermite form constructions are not restricted to polynomial matrices, but to matrices with entries drawn from a principal ideal domain, [5,7]. The definition of a principal ideal domain (p.i.d.) is not required here, but the fact that certain classes of objects are p.i.d.'s is important. Non trivial examples of p.i.d.'s include:

The class of all real polynomials.
The class of all proper real rational transfer functions.
The class of all stable real rational transfer functions.
(A real rational transfer function is stable if with coprime numerator and denominator, all zeros of the denominator have negative real parts).
The class $\mathcal{S}$ of all stable, proper, real rational transfer functions.
The class of all real rational transfer functions with denominator zeros confined to any region S of the complex plane containing at least one real point, and containing a complex point only if also containing its complex conjugate.
The class $\mathcal{S}_S$ of all proper, real rational transfer functions with denominator zeros in S.

The first fact is well known. For proof that the other classes define p.i.d.'s, see [8,9]. For a system theory application of the fourth p.i.d., see [10].

To recall the Smith and Hermite form, we need the notion of a unimodular matrix over a p.i.d., [5]: this is a square matrix with entries drawn from the p.i.d. and with determinant a unit in the p.i.d., i.e., a member of the p.i.d. with inverse also in the p.i.d. Equivalently, the unimodular matrix must have an inverse whose entries are also in the p.i.d. Thus, over the p.i.d. of real polynomials, the unimodular matrices are those with constant determinant; over $\mathcal{S}$ they are matrices with determinant an all pass function nonzero at ∞, i.e., the determinant has all zeros and poles in $\mathrm{Re}[s] < 0$, and an equal number of each.

Certain p.i.d.'s are also Euclidean rings [5,7], implying roughly that one element may be divided by another, leaving a remainder of lower "degree". It is clear that the class of real polynomialsis a Euclidean ring; so is the class $\mathcal{S}$, although this is by no means obvious. An important result is that unimodular matrices over a p.i.d. which is a Euclidean ring are expressible as a product of a finite number of elementary transformation matrices, i.e., matrices inducing one of the three following operations when postmultiplying an arbitrary matrix:

(a) interchange of two columns; (b) multiplication of all elements of one column by a unit; (c) addition to one column of k times another column, for any k in the p.i.d.

The Smith form is defined as follows [5,7]: Let M be an nxn matrix over an arbitrary p.i.d. Then there exist (nonunique) unimodular matrices U_1 and U_2 such that $U_1MU_2 = D$ (the Smith form) with D diagonal and its diagonal entries uniquely defined up to multiplication by a unit via the requirement that d_{ii} divides $d_{i+1,i+1}$ for $i = 1,\ldots,n$. [A version for nonsquare M is also available]. The Hermite form is as follows [5]: Let M be an nxn matrix over an arbitrary p.i.d. Then there exists a unimodular U such that $MU = H$ (the Hermite form) where H is lower triangular, the diagonal entries are uniquely defined up to multiplication by a unit, and the off diagonal elements in any row (when the p.i.d. is Euclidean) all have lower "degree" than the diagonal element in the same row, with a generalization of this true in the non-Euclidean ring case.

Algorithms for computing Smith and Hermite forms can be found in [5].

3. CLASSES OF COMPENSATORS

In this section, we consider the nature of compensators for an nxn rational plant transfer function matrix $G(s)$, always assumed to be strictly proper. Usually, $G(s)$ will be assumed stable, and frequently to have $|G(s)|$ with no zeros in $Re[s] \geq 0$. What sort of compensator can we use? One very appealing class of compensators are those which are stable, have no right half plane determinantal zeros (which make closure of loops with high gain more difficult) and which are nonsingular at $s = \infty$ (causing less overall phase lag than a compensator with transfer function matrix approaching zero as $s \to \infty$). With $U(s)$ the transfer function matrix of such a compensator, we see that $U(s)$ is unimodular over $\mathcal{P}$, the class of stable proper transfer function matrices. Let $\mathcal{U}^{nxn}$ denote the set of nxn unimodular matrices over $\mathcal{P}$.

A second type of compensator, somewhat less appealing but still attractive, is one which is stable and has no right half plane determinantal zeros. Its transfer function may be singular at $s = \infty$. We shall denote the transfer function matrix of such a compensator by $V(s)$ and the set of all such transfer functions over $\mathcal{P}$ by $\mathcal{V}^{nxn}$. Such transfer function matrices are often termed minimum phase.

A third type of compensator is a diagonal matrix. Let $\mathfrak{D}^{nxn}$ denote the set of nxn diagonal proper transfer function matrices; though $D(s) \in \mathfrak{D}^{nxn}$ is not then stable, usually we shall impose a stability constraint.

4. TRANSFER FUNCTION MATRIX DIAGONALIZATION

The following theorem is the key to the use of compensators for transfer function matrix diagonalization. The theorem is a particularization of the Smith form decomposition.

Theorem 1. Let $G(s) \in \mathcal{P}^{nxn}$, the class of nxn stable, proper, real rational transfer function matrices. Then there exist $U_1(s), U_2(s) \in \mathcal{U}^{nxn}$ such that $U_1GU_2 = D$ with $D \in \mathfrak{D}^{nxn} \cap \mathcal{P}^{nxn}$.

As the first step in design, suppose that in Fig. 1 below a diagonal feedback compensator $M(s)$ is chosen to ensure that the n decoupled single loops perform satisfactorily.

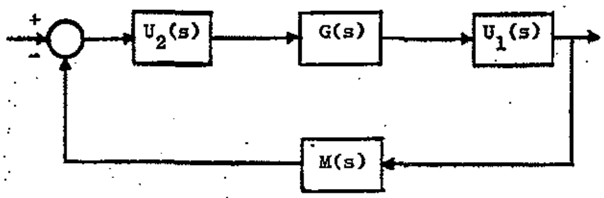
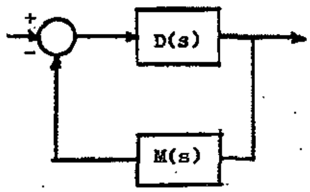


Figure 1



Then we may rearrange as shown in Figure 2 if desired, to obtain a unity feedback system with a single series precompensator and with the same nice loop gain properties as the arrangement of Figure 1.

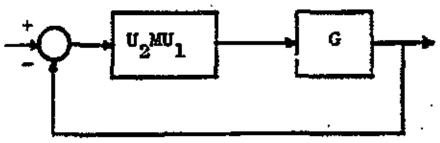


Figure 2

Provided that the design step of Figure 1 has no unstable pole-zero cancellation between M and D, the arrangement of Figure 2 will contain no unstable pole zero cancellations between U_2MU_1 and G.

Several remarks should be made.

1. In case $|G(s)|$ has no zeros in $Re[s] \geq 0$, $D(s)$ evidently has no zeros in $Re[s] \geq 0$ and this simplifies the design task.

2. The requirement that d_{ii} divide $d_{i+1,i+1}$ means that if $d_{ii}(s) = e_i(s)/\Psi_i(s)$, $d_{i+1,i+1} = e_{i+1}(s)/\Psi_{i+1}(s)$, then $[e_{i+1}\Psi_i/e_i\Psi_{i+1}]$ is stable and proper. This requirement is not exploited in the design, which means that there may be other diagonal D than that of the Smith form available as U_1GU_2 for $U_1, U_2 \in \mathcal{U}^{n\times n}$.

3. The standard Smith-McMillan construction of linear system theory can be modified to enable us to cope with unstable proper $G(s)$. Let $\Psi(s)$ be the unstable part of the least common denominator of entries of $G(s)$. Then $[\Psi(s)/\Psi(-s)]G \in \mathcal{P}^{n\times n}$, and from the Smith decomposition $U_1(s)[\Psi(s)/\Psi(-s)]G(s)U_2(s) = D(s)$, we then have $U_1(s)G(s)U_2(s) = [\Psi(-s)/\Psi(s)]D(s)$. The design task now involves stabilizing one or more unstable scalar plants.

4. There is initially no control exerted over the McMillan degree (usual) meaning) of $U_1(s)$ and $U_2(s)$ by the algorithms. Consequently, controllers may well prove too complex, and clearly some examples are required to evaluate the feasibility of designing this way.

5. There is a danger that phase shift introduced by U_1, U_2 could be impractically great. One partial remedy for this problem is to work with a different principal ideal domain $\mathcal{P}_S$ -- that of proper real rational transfer functions with poles confined to a more restricted region than simply $\mathrm{Re}[s] \geq 0$. The zeros and poles of $\det U_1$, $\det U_2$ will then be confined to the same region, and if the region is small enough, difficulties associated with phase shift in their realization may be avoided. The difficulties may, however, show up in attempting to design $M(s)$.

5. PRECOMPENSATION WITH INVERTIBLE COMPENSATORS

By an invertible compensator, we mean here one with transfer function matrix $U(s) \in \mathcal{U}^{n\times n}$. In this section, we shall note how triangularization and sometimes diagonalization, can be achieved with such a compensator.

The Hermite form described in Section 2 gives an immediate triangularization result.

Theorem 2. With $G(s) \in \mathcal{P}^{n\times n}$, there exists $U(s) \in \mathcal{U}^{n\times n}$ such that $G(s)U(s) = H(s)$ is lower triangular; moreover, $h_{ii}(s)$ may be taken to be of the form $d_i(s)/\pi^{m_i}(s)$ where $\pi(s) = (s+a)$ for some $a > 0$ and $d_i(s)$ has all zeros in $\mathrm{Re}[s] \geq 0$; also, $h_{ij}(s)$ may be taken to be of the form $d_{ij}(s)/\pi^{r_{ij}}(s)$ where $d_{ij}(s)$ and $\pi(s)$ are coprime, and $r_{ij} < m_i$.

The detailed structure of the Hermite form is established in [9].

Again, we make a number of observations.

1. In [1], Rosenbrock has a very similar result, though without explicitly labelling the construction an Hermite form. There is, however, one important difference; Rosenbrock's construction is over the p.i.d. of stable, not necessarily proper, real rational transfer functions. As a consequence, $U(s)$ and $U^{-1}(s)$ are not guaranteed to be proper.

2. In case $|G(s)|$ has no right half plane zeros, all diagonal entries of $H(s)$ can be taken to be minimum phase and after multiplication by a unit in the p.i.d. $\mathcal{P}$ can be written as $1/\pi^{r_i}(s)$.

3. In case $G(s)$ is unstable, we premultiply $G(s)$ by a diagonal matrix $E(s) = \mathrm{diag}\,\dfrac{e_i(s)}{e_i(-s)}$, where $e_i(s)$ is the least common multiple of the unstable part of the denominators of i-th row elements of $G(s)$. Thus $E(s)G(s) \in \mathcal{P}^{n\times n}$. We find $U(s)$ and then have $GU = E^{-1}H$. What significance this has is not clear.

4. In case $G(s)$ is strictly proper and stable, and $|G(s)|$ has no right half plane zeros and behaves as $O(s^{-n})$ as $s \to \infty$, $H(s)$ can be taken to be diagonal. This can be seen either from the detailed Hermite form structure, or by noting that $(s+1)G(s)$ is proper, has determinant with no right half plane zeros and approaching a constant as $s \to \infty$,

i.e., $(s+1)G(s)$ is unimodular over $\mathcal{P}^{n\times n}$. Thus $U(s) = (s+1)^{-1}G^{-1}(s)$ diagonalizes $G(s)$. (If one drops the properness restriction on the p.i.d., one obtains Rosenbrock's result [1], to the effect that if G is stable and $|G(s)|$ has no right half plane poles, there exists a $U(s)$ such that $H(s)$ is diagonal. In this case, $G^{-1}(s)$ in fact works as a $U(s)$! We return to this point in the next section.)

5. Once again, by working with a p.i.d. $\mathcal{P}_S$ with restricted S, one can perhaps ease the task of realizing $U(s)$.

6. If one requires $H(s)$ to be triangular but otherwise not to have the Hermite form structure, this allows additional freedom in $U(s)$. However, it is not clear how this freedom might be explored.

6. TRANSFER FUNCTION MATRIX PRECOMPENSATION WITH MINIMUM PHASE COMPENSATORS

In the last section, we considered what could be achieved using compensators which were stable, had no determinantal zeros in $\mathrm{Re}[s] \geq 0$, and were nonsingular at $s = \infty$. Here, we consider the effect of dropping the last assumption. The reason for doing this is to extend the class of plant transfer function matrices which can be diagonalized, as opposed to triangularized.

The following result appears to be of little use other than as a tool for proving Theorem 3 below.

Lemma 1. Let $G(s) \in \mathcal{P}^{n\times n}$. Then there exists a $V(s) \in \mathcal{V}^{n\times n}$, the class of stable proper transfer function matrices with all zeros of $|V(s)|$ in $\mathrm{Re}[s] < 0$, such that $G(s)V(s) = T(s)$ is lower triangular, and satisfying the following constraints. Let $t_{ij}(s) = p_{ij}(s)/q_{ij}(s)$, with p_{ij} and q_{ij} coprime for $i > j$ and let $t_{ii}(s) = p_{ii}(s)/q_{ii}(s)$ with $p_{ii}(s)$ possessing zeros in $\mathrm{Re}[s] \geq 0$, without loss of generality. Then $\deg[p_{ij}] < \deg[p_{ii}]$ for $j \neq i$.

Proof. Let $U(s) \in \mathcal{U}^{n\times n}$ be such that $GU = H$, the Hermite form of G over $\mathcal{P}^{n\times n}$. Let $f_i(s)$ be the least common denominator of the i-th row of H and let $F = \mathrm{diag}[f_i(s)]$. Then $FGU = FH$ is polynomial. Let $W(s)$ be a unimodular matrix over the p.i.d. of polynomials such that $FHW = K$ is the associated Hermite form over the p.i.d. of polynomials. Then $k_{ii}(s)$ has all zeros in $\mathrm{Re}[s] \geq 0$, and $\deg[k_{ij}] < \deg[k_{ii}]$ for $j \neq i$.

Let $X(s)$ be any diagonal polynomial matrix such that WX^{-1} is proper and stable. Let $V(s) = UWX^{-1}$. Note that $V(s) \in \mathcal{V}^{n\times n}$. Moreover,

$$G(s)V(s) = [\mathrm{diag}\, f_i^{-1}]\, K(s)\, [\mathrm{diag}\, x_j^{-1}] = T(s)$$

whence it is clear that $T(s)$ has the required properties.

The application of this lemma is to plants which are themselves minimum phase.

Theorem 3. Let $G(s) \in \mathcal{V}^{n\times n}$. Then there exists $V(s) \in \mathcal{V}^{n\times n}$ such that $G(s)V(s) = D(s)$ is diagonal, with $D(s) \in \mathcal{V}^{n\times n}$.

Proof. Form $T(s)$ as above. Because $|G(s)V(s)|$ has no zeros in

$\mathrm{Re}[s] \geq 0$, $p_{ii}(s)$ is constant for all s . Thus $p_{ij}(s) = 0$ for all $i \neq j$, i.e., $T(s)$ is diagonal.

Many of the same comments made at the end of the last two sections can be made here.

7. FINAL REMARKS

In this section, we confine ourselves almost entirely to making remarks additional to those made before.

In the preceding sections, we have not discussed the question of plants with nonsquare transfer function matrices. While, of course, the main interest is in the square case, nonsquare situations may be encountered; since the Hermite and Smith forms can be obtained for nonsquare matrices, no difficulties in principle seem likely.

Of more importance is the task of using methods of design based on diagonal dominance. Here, some results are in hand. For example, we have obtained theorems concerning the existence and construction of precompensators such that the compensated plant is diagonally dominant. (There is however no direct control over the McMillan degree of the compensators.)

As has been emphasized, the ideas of this paper need validation by examples. One of the many aspects which need to be checked is the feasibility of exchanging ease of realization of the diagonalizing or triangularizing compensators with the ease of realizing the compensators for the separate scalar systems. Such exchange would be achieved presumably by varying the region S in which the diagonalizing or triangularizing compensator poles or zeros could lie. This sort of exchange seems to be novel, and might even be able to be carried over to other design procedures.

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