APPLIED AND COMPUTATIONAL CONTROL, SIGNALS, AND CIRCUITS

Recent Developments

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(Editor)

Contents

Preface

This is an interdisciplinary book blending mathematics, computational mathematics, scientific computing and software engineering with control and systems theory, signal processing and circuit simulations. The book contains six technical chapters: three in control, communication and power systems, one in signal processing and two in circuit design and simulations. Besides these technical chapters, the software section contains a chapter on the description and analysis of a software module for model reduction contained in the newly developed Fortran-based software library, called SLICOT, for control systems design and analysis.

The chapters present the state-of-the-art reviews of some of the recent developments in three inter-related areas of control, signal processing and circuits. They are written by leading experts in these fields on invitation by the editor-in-chief. The invitations were extended to the authors based on recommendations made by our distinguished editorial board.

The chapters should be accessible to a wide interdisciplinary audience: from experts to beginning researchers and graduate students. It is expected that the book will be an important reference for research scientists, practicing engineers, as well as students and teachers in control, power systems, signals, and circuit theory. The book also seems to be suitable for advanced graduate topic-courses.

Overview Of The Chapters

Chapter 1 – Constant disturbance rejection and zero steady state tracking error for nonlinear systems design
Steven W. Su, Brian D. O. Anderson, Thomas S. Brinsmead

The problem of disturbance rejection arises in many industrial fields, such as motion-control, active noise control and vibration control. Classically, for linear systems, the problem is solved by including an integrator in the controller. This paper extends this idea to nonlinear systems. Singular perturbation methods are used to guarantee stability. The method is implemented in the control of a simulated helicopter model.

Chapter 2 – Control Problems in Telecommunications: The Heavy Traffic Approach

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Constant disturbance rejection and zero steady state tracking error for nonlinear systems design

Steven W. Su[1], Brian D. O. Anderson, Thomas S. Brinsmead

ABSTRACT A relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is presented. By adding an integrator to a stabilising controller, it is possible to achieve both constant disturbance rejection and zero tracking error. Sufficient conditions for the rejection of a constant input disturbance are given. We give both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. Furthermore, we also present some alternative locations for adding an integrator into the closed loop system and extend these methods to deal with Multiple-input Multiple-output nonlinear systems. Finally, we implement our method in the control of a simulated helicopter model. The simulation results show that this method achieves satisfactory performance.

1.1 Introduction

An important objective of control system design is to minimise the effects of external disturbances. The problem of disturbance rejection (especially constant disturbance rejection) arises in many industrial fields, such as motion-control, active noise control and vibration control. For linear systems, the classical method of rejecting a constant disturbance is to include an integrator in the controller. This paper extends this idea to nonlinear systems, using singular perturbation methods to guarantee stability.

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Although the method presented in this paper extends classical methods for linear constant disturbance rejection, it is also related to nonlinear $\mathcal{H}_\infty$ methods presented in [12]. That paper extended the concept of comprehensive stability for linear systems [8] [7] to deal with the nonlinear constant disturbance suppression problem. As an $\mathcal{H}_\infty$ mixed sensitivity problem , the constant disturbance suppression problem is a nonstandard due to the existence of un-stabilisable states.

The main bottleneck for nonlinear state feedback $\mathcal{H}_\infty$ control, which is similar to the problem encountered in nonlinear optimal control, is the need to solve a Hamilton-Jacobi (HJ) partial differential equation (PDE) [6]. Paper [12] presents a method of simplifying (via order reduction) the Hamilton-Jacobi partial differential equation for the nonlinear disturbance rejection problem by using the concept of *comprehensive stability*, a concept which is extended from the linear case [7]. Because the states which are related to the disturbance are not directly measurable, they cannot be directly used. This forces us to consider nonlinear $\mathcal{H}_\infty$ output feedback control.

Nonlinear $\mathcal{H}_\infty$ output feedback control is particularly difficult. The standard solution of the linear $\mathcal{H}_\infty$ output feedback control problem normally involves solving two Riccati equations [14]. One of these, which arises in the state feedback control problem, is replaced by a Hamilton-Jacobi partial differential equation in the nonlinear case. The other, however, is replaced by a still more complicated equation (involving an information state)[1]. Practical approaches to the solution of this latter equation are so far lacking. Alternatively, one can draw on ideas of nonlinear observer theory [3] [5], and replace the state x with a state estimate $\hat{x}$ in a state feedback controller, retrospectively checking the γ-dissipativity and stability of the closed-loop system. In this case, the controller remains finite-dimensional, which is not always the case when information state methods are used.

Paper [12] also demonstrates that for disturbance suppression, an output feedback controller must contain an integrator. In this paper we ask whether we can directly add an integrator to an already existing controller to achieve constant disturbance rejection, while still retaining the stability of the system. Often that would be both a simpler and more practical way to deal with the nonlinear constant disturbance suppression problem.

This paper not only gives the affirmative answer but also suggests several locations where an integrator with low gain away from DC ($\frac{\varepsilon}{s}$ for short) maybe included, in order to deal with the constant input disturbance rejection problem. Furthermore, this method can also be applied to cope with the constant reference tracking problem, even for nonlinear MIMO systems, such as the helicopter system of [4].

In the next section, we give a description of the problem. In Section 3, we present a proof that an exponentially stabilising nonlinear controller appropriately augmented with a small integrator (a linear transfer function $\frac{\varepsilon}{s}$) can yield constant disturbance suppression. In Section 4, we will give both local and global conditions for the existence of a gain of the integrator that is sufficiently small to guarantee stability. Section 5, by using singular perturbation methods, gives a upper bound on a value of the gain that guarantees closed loop stability. In Section 6, we suggest alternative locations for adding an integrator into the system. Section 7 extends our method to deal with the constant disturbance rejection problem and constant reference tracking problem for Multiple-input Multiple-output(MIMO) systems. Finally in Section 8, we present simulation results obtained by implementing constant disturbance rejection and zero steady state tracking error control for a helicopter model by using this method.

1.2 Problem Description

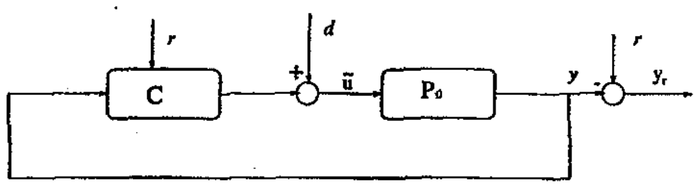

FIGURE 1.1. A nonlinear constant disturbance suppression problem

Firstly, let us consider a nonlinear input disturbance rejection problem as shown in Figure 1.1. This depicts a nonlinear single-input single output (SISO) system (We will extend our methods to MIMO systems later). It consists of the interconnection of a nonlinear plant P_0 and controller C, forced by a constant command signal r, as well as a constant input disturbance d. Here, y_r is the reference tracking error, and $\bar{u}$ is the input to the plant. What we are concerned with here is how to design a controller C which possesses the ability to both reject a constant input disturbance d, and to give zero steady state tracking error for a constant reference input r.

More precisely, we consider the question of how we might modify a pre-existing controller C_0 not achieving these properties, so that the properties are secured throughout the modification (See Figure 1.2).

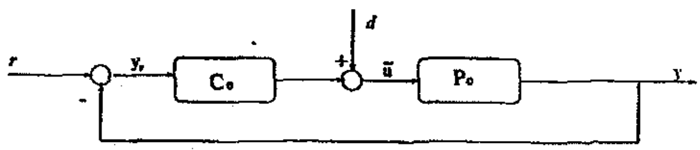

FIGURE 1.2. A nonlinear system with an existing stabilising controller C_0

In the case of a linear plant, the classical method employed to reject a constant disturbance is to include an integrator in the controller. Here, we extend this idea to deal with the nonlinear constant disturbance rejection problem.

Consider Figure 1.3. Suppose that we have already designed a controller C_0 which stabilises the plant P_0 (Later, we shall be precise concerning the type of stability). We then augment the closed loop with the addition of a small gain integrator. The original controller C_0 and small gain integrator $\frac{\epsilon}{s}$ in Case 1 of Figure 1.3 represents a solution to the problem of designing C in Figure 1.1. Then, the interconnection is equivalent to a single stable plant P as shown in Figure 1.3. By stating that the two cases in Figure 1.3 are equivalent, we mean that if the exogenous input signals d and r in the two cases are equal, then all labelled signals (including the output signals) will also be equal(given suitable matching of initial conditions, or after clear of initial condition effects). Hence, we can focus our attention on the simplified second case.

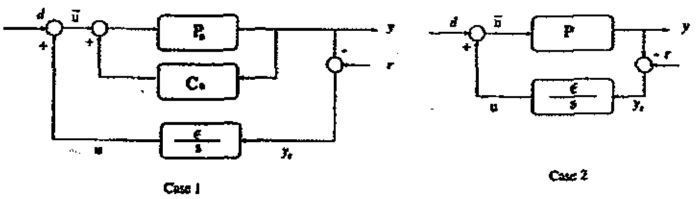

FIGURE 1.3. Two Equivalent Cases

In the second case of Figure 1.3, we suppose that the state equation of the plant P is modelled as follows.

$$P: \begin{cases} \dot{x} &= f(x, \bar{u}) \\ y &= g(x, \bar{u}). \end{cases} \tag{1.1}$$

If there is no particular declaration in this paper, we suppose that $f: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^l$ are unbiased in the sense that

$$\begin{cases} f(0,0) &= 0 \\ g(0,0) &= 0. \end{cases} \tag{1.2}$$

The state equation for the small integrator is expressed as a transfer function block $\frac{\epsilon}{s}$:

$$\frac{\epsilon}{s}: \begin{cases} \dot{\xi} &= \epsilon y_r \\ u &= \xi. \end{cases} \tag{1.3}$$

In the above, the reference tracking error y_r is equal to $y - r$. We suppose that the disturbance d and the reference input r are both constant.

The following parts of this paper will focus on two key questions. The first question is whether a controller that is augmented with an integrator will reject the constant disturbance. The second question is how to ensure the stability of the closed loop. Another but nevertheless important question is whether constant reference trajectory following occurs, with zero steady state error.

1.3 Sufficient Conditions for Constant Disturbance Rejection

In [12], it was shown that for input disturbance suppression an output feedback $\mathcal{H}_\infty$ controller must contain an integrator in the controller. In this section, we will still start our discussion from the point of view of an $\mathcal{H}_\infty$ treatment.

As in [12], we also extend the constant input disturbance rejection problem to a mixed sensitivity $\mathcal{H}_\infty$ problem (Figure 1.4). We introduce an integrator into one of the input weights(the disturbance weight), and choose cost variable $z = \tilde{u}$. The input $\hat{w}_1$ gives rise to the input disturbance d. The introduction of the input $\hat{w}_2$ can be interpreted as a way of capturing modelling uncertainty or as a reference input signal. Without an integrator weight function, the introduction of $\hat{w}_2$ is necessary for ensuring that the $\mathcal{H}_\infty$ problem is standard. Here, the input weighting function W_{d2} of $\hat{w}_2$ and the output weighting function W_z are both stable.

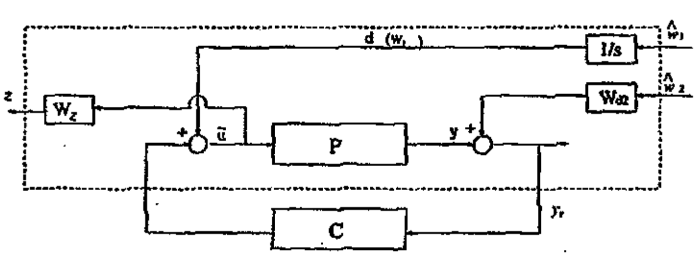

FIGURE 1.4. The mixed sensitivity $\mathcal{H}_\infty$ form

In order to set up the relationships between input-output stability [13] and Lyapunov stability for this constant disturbance rejection problem, we present a theorem from [13].

We will later identify the controller C in Figure 1.4 with the small gain integrator ($\frac{\epsilon}{s}$).

Definition 1.1. *A system is globally exponentially stable (GES) iff there* *exists a Lyapunov function $U(x) \leq 0$ such that*

$$\rho_1|x|^2 \leq U(x) \leq \rho_2|x|^2$$

and with zero input

$$\frac{d}{dt}U(x(t)) \leq -\rho_3|x|^2.$$

where $\rho_i > 0$, $i = 1,2,3$ are suitable scalar constants. If these conditions hold, it follows that there exists some constant $\rho \geq 0$ such that with $x(0) = x_0$, $|x(t)| \leq \rho|x_0|e^{-\rho_3 t/2}$ for all $t \geq 0$.

By local exponential stability (LES) we mean that this definition is valid at least for x in a neighbourhood of $x = 0$.

Definition 1.2. *Consider the nonlinear system of the form*

$$\begin{cases} \dot{x} &= f(x,u) \\ y &= g(x,u). \end{cases} \tag{1.4}$$

The system (1.4) is said to be "$\mathcal{L}_p$-stable with finite gain" if there exist constants b_p and $\gamma_p < \infty$ such that $u \in \mathcal{L}_p^m \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p\|u\|_p + b_p$. If $p = 2$, γ_p is said to be the $\mathcal{L}_2$ bound from u to y.

The system (1.4) is said to be "$\mathcal{L}_p$-stable without bias" if there exists a constant $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p\|u\|_p$.

The system (1.4) is "small signal $\mathcal{L}_p$-stable without bias" if there exist constants $r_p > 0$ and $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m$ with $\|u\|_p \leq r_p \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p\|u\|_p$.

As in the linear case, it is possible to establish a connection between these two types of stability [13].

Theorem 1.1. *Consider the system described by equation (1.4). Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^l$ are unbiased in the sense that*

$$\begin{cases} f(0,0) &= 0 \\ g(0,0) &= 0. \end{cases} \tag{1.5}$$

which ensures that $x = 0$ is an equilibrium of the unforced system

$$\dot{x} = f(x,0). \tag{1.6}$$

Suppose that $x = 0$ is an exponentially stable equilibrium of (1.6), and that f is C^1. Suppose also that f and g are locally Lipschitz continuous at $(0,0)$, that is, suppose there exist finite constants k_f, k_g, r such that

$$\|f(x,u) - f(z,v)\|_2 \leq k_f[\|x - z\|_2 + \|u - v\|_2], \forall (x,u)(z,v) \in B_r, \tag{1.7}$$

$$\|g(x,u) - g(z,v)\|_2 \leq k_g[\|x - z\|_2 + \|u - v\|_2], \forall (x,u)(z,v) \in B_r. \quad (1.8)$$

Here, B_r is the open ball of the radius r, that is, $B_r = \{x : \|x - x_0\| < r\}$. Then the system (1.4) is small signal $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. If $x = 0$ is a globally exponentially stable equilibrium, and (1.7) and (1.8) hold with B_r replaced by $\mathbb{R}^{(m+n)}$, then the system (1.4) is $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. Furthermore, there exists a Lyapunov function $U(x) \geq 0$ which satisfies the requirements of exponential stability of Definition 1.1, and the gain γ_p is related to the constants ρ_i defining the properties of $U(x)$ by

$$\|y\|_p \leq k_g[(\rho_3 k_f / 4\rho_1{}^2\rho_2{}^2) + 1]\|u\|_p.$$

Proof: See pages 286-289 of [13]. □

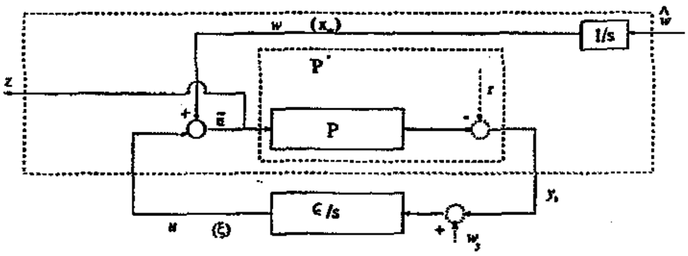

FIGURE 1.5. The simplified mixed sensitivity $\mathcal{H}_\infty$ form

Now, let us consider the mixed sensitivity $\mathcal{H}_\infty$ problem depicted by Figure 1.4. One design goal of $\mathcal{H}_\infty$ methods is to ensure that a finite $\mathcal{L}_2$ gain $\gamma_{\hat{w}z}$ exists from input $[\hat{w}_1\ \hat{w}_2]^T$ to output z, in other words to ensure that the system is "$\mathcal{L}_2$-stable with finite gain" (see Definition 1.2.). In this section, in order to emphasise the problem of constant input disturbance rejection as opposed to reference tracking and to simplify our discussion, we will not consider the input $\hat{w}_2$, that is, we set $\hat{w}_2 = 0$. We also assume that the weight function W_z is unity. Because the weighting functions W_{d2} and W_z are both stable, we can use Theorem 1.1 to see that these simplifications will not influence the existence of $\gamma_{\hat{w}z}$ and our further discussion.

We set the controller C in Figure 1.4 to be $\frac{\epsilon}{s}$. The system is then as depicted in Figure 1.5.

Theorem 1.2. *Consider the system depicted in Figure 1.5. The plant P and $\frac{\epsilon}{s}$ blocks are respectively described by equations (1.1),(1.2) and (1.3). Suppose that $(0,0)$ is an exponentially stable equilibrium of the* unforced closed loop

$(P, \frac{\epsilon}{s})$. Further, assume that f is C^1, and that f, g are locally Lipschitz continuous at $(0,0)$ with Lipschitz constants k_f and k_g to the Euclidean norm $\|.\|_2$ (See Definition 1.1 and Theorem 1.1.).

Then the system depicted in Figure 1.5 is small signal $\mathcal{L}_2$ stable without bias from $\hat{w}$ to z.

If $(0,0)$ is a globally exponentially stable equilibrium, and f, g are globally Lipschitz continuous at $(0,0)$, then the system is $\mathcal{L}_2$ stable without bias.

Proof:

Consider Fig 1.5, and suppose that there is an additional input w_3 to the integrator $\frac{\epsilon}{s}$. If we set $w_3 = \frac{\hat{w}}{\epsilon}$ and then replace $\hat{w}$ by zero, the input w_3 is equivalent to the input of the signal $\hat{w}$. That is, we can replace the disturbance input $\hat{w} \in \mathcal{L}_2$ of the system depicted in Fig 1.5 by the equivalent signal $w_3 \in \mathcal{L}_2$. Because $(0,0)$ is an exponentially stable equilibrium of the unforced closed loop $(P, \frac{\epsilon}{s})$, then we will see that according to Theorem 1.1 a finite gain $\gamma_{w_3 y}$ from w_3 to z exists.

More precisely, the augmented system with input w_3 and output z can be described as below.

$$\begin{cases} \dot{x} &= f(x,\xi) \\ \dot{\xi} &= \epsilon(g(x,\xi) + w_3) \\ z &= \xi. \end{cases} \tag{1.9}$$

Let $x_a = [x^T\ \xi]^T$, then the above equation can be rewritten in the form:

$$\begin{cases} \dot{x}_a &= f_a(x_a, w_3) \\ z &= g_a(x_a) \end{cases} \tag{1.10}$$

Here, $f_a(x_a, w_3) = \begin{bmatrix} f(x_a) \\ \epsilon(g(x_a) + w_3) \end{bmatrix}$, $g_a(x_a) = \xi$.
Then, $\forall (x_a, w_3), (x_a', w_3') \in \mathbb{R}^{(n+2)}$

$$\begin{aligned} & \|f_a(x_a, w_3) - f_a(x_a', w_3')\|_2 \\ = & \left\| \begin{matrix} f(x_a) - f(x_a') \\ \epsilon(g(x_a) - g(x_a')) + \epsilon(w_3 - w_3') \end{matrix} \right\|_2 \\ \le & k_f(\|x - x'\|_2 + \|\xi - \xi'\|_2) + k_g(\|x - x'\|_2 + \|\xi - \xi'\|_2) + \epsilon\|w_3 - w_3'\|_2 \\ \le & \sqrt{2}(k_f + k_g)\|x_a - x_a'\|_2 + \epsilon\|w_3 - w_3'\|_2 \\ \le & k_{f_a}(\|x_a - x_a'\|_2 + \|w_3 - w_3'\|_2). \end{aligned} \tag{1.11}$$

Here, $k_{f_a} = max\{\sqrt{2}(k_f + k_g), \epsilon\}$.

Similarly, it is obvious that $\|g_a(x_a) - g_a(x_a')\|_2 \le k_{g_a}\|x_a - x_a'\|_2$, where $k_{g_a} = 1$.

In view of the assumption that $(0,0)$ is an exponentially stable equilibrium of the unforced closed loop $(P, \frac{\epsilon}{s})$, there exists a Lyapunov function $U(x) \ge 0$, which satisfies the requirements of Definition 1.1. According to Theorem 1.1 the finite gain $\gamma_{\hat{w}_3 z}$ from w_3 to z is $\gamma_{w_3 z} = [(\rho_3 k_{f_a}/4{\rho_1}^2{\rho_2}^2) + 1]$, where the constants ρ_i are defined by the properties of $U(x)$.

Then, in view of the equivalence of the $\hat{w}$ and w_3 described at the beginning of the proof, we see that the bound from $\hat{w}$ to z is $\gamma_{\hat{w}z} \leq \frac{1}{\epsilon}[(\rho_3 k_{f_a}/4{\rho_1}^2{\rho_2}^2) + 1]$. □

Remarks:

- The significance of Theorem 1.2 is that it shows that if a controller is augmented with an integrator, and the closed loop is exponentially stable (we will present sufficient conditions for the stability of the closed loop in next section), then input-output stability from $\hat{w}$ to z is ensured. In fact, the $\mathcal{H}_\infty$ norm from $\hat{w}$ to z is less than the given bound $\gamma_{\hat{w}z}$. Note that there is an integrator weight function between $\hat{w}$ and w which ensures that even for a constant disturbance w, the output signal z is in $\mathcal{L}_2$ and hence asymptotically goes to zero. That is, the controller augmented with a low gain integrator $\frac{\epsilon}{s}$ will reject a constant input disturbance.
- For the mixed sensitivity $\mathcal{H}_\infty$ problem (which includes an additional input $\hat{w}_2$) depicted by Figure 1.4, if a controller contains $\frac{\epsilon}{s}$ and the closed loop is exponentially stable, then it is easy to see that input-output stability from $[\hat{w}_1\ \hat{w}_2]$ to z is also ensured, based on Theorem 1.1. That is, the controller with $\frac{\epsilon}{s}$ will robustly reject a constant input disturbance.

Note:

Any equilibrium x_e under investigation can be translated to the origin by redefining the state x as $x - x_e$ [10]. For simplicity, in most of the exposition following we will assume that such a translation has already been performed. Thus, for most parts of this paper, the equilibrium under investigation will be $x_e = 0$. When we need to emphasise a non-zero equilibrium, we will use $x = x_e$ as the equilibrium point instead of $x = 0$.

Consider the plant P' in Figure 1.5. If we have a nonzero constant reference input r, we can consider the original plant P and reference input r to be equivalent to a new plant P' with an equilibrium point (x_e, ξ_e), where $g(x_e, \xi_e) = r$. Sufficient conditions for stability in this situation are that the conditions of Theorem 1.2 are satisfied for the new equilibrium point. We will investigate the constant reference tracking problem in more detail later.

1.4 Guaranteeing Stability with Integrator Augmentation

We have established that a controller augmented with an integrator will reject a constant input disturbance provided that the combination is stabilisation. We are now concerned with the problem of how to design such a controller so as to ensure the stability of the closed loop $(P, \frac{\epsilon}{s})$. In this section, using singular perturbation theory, we will investigate both local and global conditions for the existence of a small scalar ϵ^* such that when $0 < \epsilon < \epsilon^*$ the closed loop $(P, \frac{\epsilon}{s})$ is stable.

Consider the set up of Figure 1.3 described by equations (1.1) and (1.3). If we set the constant input signal r and d to zero in order to analyse the Lyapunov stability of the unforced closed loop $(P, \frac{\epsilon}{s})$, then the state equation for the closed loop $(P, \frac{\epsilon}{s})$ can be expressed as:

$$(P, \frac{\epsilon}{s}) : \left\{ \begin{array}{lcl} \dot{x} & = & f(x, \xi) \\ \dot{\xi} & = & \epsilon g(x, \xi). \end{array} \right. \tag{1.12}$$

In order to use the singular perturbation method, we first transform equation (1.12) to its standard singular perturbation form [9].

Let $\tau = \epsilon(t - t_0)$, so that $\tau = 0$ at $t = t_0$. That leads to $\frac{d\tau}{dt} = \epsilon$. Then, we have

$$\left\{ \begin{array}{lcl} \epsilon \frac{d}{d\tau} x & = & f(x, \xi) \\ \frac{d}{d\tau} \xi & = & g(x, \xi). \end{array} \right. \tag{1.13}$$

It should be noticed that x is a vector; on the other hand, with a SISO problem, ξ is a scalar.

In order to be consistent with standard singular perturbation notation, we will for the moment use the notation $\dot{x}$ to denote the derivative on the *slow* time scale τ when we analyse singular perturbation models.

Theorem 1.3. *(Global conditions for the existence of ϵ^*)*

Consider the second case depicted in Figure 1.3 described by equation (1.13) which satisfies the requirement of equation (1.2), and suppose that the following assumptions are satisfied:

(i) The equation $0 = f(x, \xi)$ obtained by setting $\epsilon = 0$ in equation (1.13) implicitly defines a unique C^2 function $x = h(\xi)$.

(ii) For fixed $\xi \in R$, the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x, \xi)$ is Globally Asymptotically Stable (GAS) [10] and Locally Exponentially Stable (LES).

(iii) The equilibrium $\xi = 0$ of the reduced model (slow time scale) $\dot{\xi} = g(h(\xi), \xi)$ is GAS and LES (See Definition 1.1). A sufficient condition is that $g(h(\xi), \xi)\xi < 0$ (when $\xi \neq 0$) and $g(h(\xi), \xi)\xi \leq -\rho|\xi|^2$ for ξ in a neighbourhood of $\xi = 0$.

Then there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x, \xi) = (0, 0)$ is GAS. Furthermore if the conditions in (ii) and (iii) involve GES instead *of GAS, then the equilibrium* $(x, \xi) = (0, 0)$ *is GES.*

Proof:

This follows from Theorem 3.18 in page 90 of [10] and Corollary 2.2 in page 297 of [9].

Consider $V(\xi) = \frac{1}{2}\xi^2$ as a Lyapunov function candidate for the "slow time scale ". Then, $\dot{V}(\xi) = \xi\dot{\xi} = g(h(\xi), \xi)\xi$. This will satisfy the requirements for GAS and LES given that $g(h(\xi), \xi)\xi < 0$ (when $\xi \neq 0$) and $g(h(\xi), \xi)\xi \leq -\rho|\xi|^2$ (for some scalar $\rho > 0$) for ξ in a neighbourhood of $\xi = 0$. On the other hand, the "fast time scale" mode is GAS or ES by assumption.

□

Remarks:

- Condition (i) will usually be satisfied in practical situations.
- For linear systems the quantity $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ has an interpretation as the (incremental) DC gain .
- Our earlier assumption that the plant P is stable (that is, that P_0 is stabilised by C_0) with the extra requirement that P is LES, is sufficient for Condition (ii) to be satisfied.
- Although Condition (iii) nominally requires that $g(h(\xi), \xi)\xi < 0, \forall \xi \neq 0$, if instead it is the case that $g(h(\xi), \xi)\xi > 0, \forall \xi \neq 0$ then we can just change the sign of feedback to achieve closed loop stability. That is, if $g(h(\xi), \xi)\xi \geq \rho|\xi|^2$ for some $\rho > 0$, then there exists a negative value $\epsilon^* < 0$, such that for all $\epsilon^* \leq \epsilon \leq 0$, the equilibrium $(x, \xi) = (0, 0)$ is GAS.
- Condition (iii) may not be satisfied globally. However, if this condition is locally satisfied, then we can instead establish *local* closed loop stability by using Theorem 1.5 to follow.

Note:

If we consider the more general case that the equilibrium point ξ is not zero but fixed at $\xi = \xi_e$ by the influence of a constant reference input r, we require a slight adjustment to Condition (iii). In particular, we require that the equilibrium ξ_e of the reduced model (slow time scale) is GAS and LES (i.e. we should have that $[g(h(\xi)) - g(h(\xi_e))](\xi - \xi_e) \leq 0$, and $[g(h(\xi)) - g(h(\xi_e))](\xi - \xi_e) \leq \rho|\xi - \xi_e|^2$ is valid for ξ in a neighbourhood of $\xi = \xi_e$.). This will be satisfied for all ξ_e if $\frac{\partial g(h(\xi),\xi)}{\partial \xi} < -\rho < 0$, that is, if the "incremental DC gain" of the nonlinear plant is uniformly bounded away from zero.

We now introduce a theorem from [2] which gives sufficient conditions to guarantee the local stability of a standard singularly perturbed system.

Theorem 1.4. *(Conditions for the local stability of a general singular perturbed system)*

Consider a nonlinear differential equation

$$\begin{cases} \epsilon\dot{x} &= f(x,\xi), \quad f:\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}^n, \\ \dot{\xi} &= g(x,\xi), \quad g:\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}^m, \end{cases} \tag{1.14}$$

where $f(.,.)$ and $g(.,.)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. Define:

$$A_{11} = \frac{\partial g}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{12} = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)},$$

$$A_{21} = \frac{\partial f}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{22} = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)},$$

and suppose that A_{22} is nonsingular. Suppose further that the solution of the equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (1.13) implicitly defines a C^2 function $x = h(\xi)$. Then the following statements are true.

(i)If all eigenvalues of A_{22} and of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts, there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x_e = 0, \xi = 0)$ is an asymptotically stable equilibrium point.

(ii)If an eigenvalue of A_{22} or of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ has positive real part, there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x_e = 0, \xi = 0)$ is an unstable equilibrium point.

Proof: The proof is based on the indirect method of Lyapunov and the linear version of the singular perturbation result [2]. □

We now specialise the above theorem to the case depicted in Figure 1.3.

Theorem 1.5. *(Local conditions for the existence of ϵ^*)*

Consider the second case in Figure 1.3 described by equation (1.13), and suppose that $f(.,.)$ and $g(.,.)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. Define:

$$A_{11} = \frac{\partial g}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{12} = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)},$$

$$A_{21} = \frac{\partial f}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{22} = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)},$$

and suppose that A_{22} is nonsingular. Suppose further that the equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (1.13) has a unique C^2 solution $x = h(\xi)$, and that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonzero. Then if all eigenvalues of A_{22} and of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts, there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x_e = 0, \xi_e = 0)$ is an asymptotically stable equilibrium point.

Proof: This theorem is a special case of Theorem 1.4. □

Remarks:

- Note that if $\dot{x} = f(x,\xi)$ is stable when ξ is fixed, then all the eigenvalues of A_{22} (when ξ is fixed at $\xi = 0$) have negative real parts.
- Note that having $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ nonzero implies that $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is a non zero scalar (See Appendix 1.B for the proof of a more general case). The sign of the only eigenvalue can therefore be changed by changing the sign of $g(.,.)$. That is, if $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is positive, then closed loop stability can be achieved by changing the feedback from negative to positive. This argument will be generalised to the MIMO case in Section 7.

  In such a case, it is equivalent to see that if $\dot{x} = f(x,\xi)$ is stable for fixed ξ_e, then there exists $\epsilon^* < 0$, such that for all $\epsilon^* < \epsilon < 0$, the equilibrium (x_e, ξ_e) is locally stable for the closed loop $(P, \frac{\epsilon}{s})$.

1.5 An Integrator Gain Bound

In the last section, we gave sufficient conditions for the existence of a bound on the integrator gain that will guarantee closed loop stability. Here, we will give an explicit expression for such an ϵ^*, based on singular perturbation theory.

Theorem 1.6. *(An integrator gain bound)*
Consider the second case in Figure 1.3 described by equation (1.13) which satisfies the requirement of equation (1.2), and suppose that the following conditions are satisfied:
(i) There exists a function h such that $x = h(\xi)$ is the unique root of $0 = f(x,\xi)$ in $(x,\xi) \in B_x \times B_\xi$ (Here, B_x and B_ξ are some open balls on x and ξ space respectively).
(ii) There exists a Lyapunov function $W(x,\xi)$ such that for all $(x,\xi) \in B_x \times B_\xi$:
a. $W(x,\xi) > 0$ for all $x \neq h(\xi)$ and $W(h(\xi),\xi) = 0$.
b. There exists some $\alpha_2 > 0$, such that $\frac{\partial W}{\partial x} f(x,\xi) \leq -\alpha_2[\phi(x - h(\xi))]^2$.

c. There exists some γ and β_2 such that $\frac{\partial W}{\partial \xi} g(x,\xi) \leq \gamma[\phi(x - h(\xi))]^2 + \beta_2 \psi(\xi)\phi(x - h(\xi))$.

In the above, $\psi(.)$ and $\phi(.)$ are scalar functions of vector arguments which vanish only when their arguments are zero, e.g. $\psi(\xi) = 0$ iff $\xi = 0$.

(iii) There exists a Lyapunov function $V(\xi)$ such that:

d. $\frac{\partial V}{\partial \xi} g(h(\xi),\xi) \leq -\alpha_1 \psi^2(\xi)$, for some $\alpha_1 > 0$.

e. There exist some β_1 such that $\frac{\partial V}{\partial \xi}[g(x,\xi) - g(h(\xi),\xi)] \leq \beta_1 \psi(\xi)\phi(x - h(\xi))$.

Then, when $0 < \epsilon < \epsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2}$, there exists a Lyapunov function for the closed loop system $(P, \frac{\epsilon}{s})$ of the form:

$W_\gamma(x,\xi) = (1-d)V(\xi) + dW(x,\xi)$, where d is allowed to be any fixed value in the range $(0,1)$.

Furthermore, the origin is an asymptotically stable equilibrium of $(P, \frac{\epsilon}{s})$.

Proof: This theorem is a special case of Theorem 2.1 in page 297 of [9]. □

Here, the parameters β_1, β_2 and γ could, in general, be positive, negative or zero. In most problems, however, one arrives at inequalities c and e (in Theorem 1.6) using norm inequalities, leading automatically to nonnegative values for β_1, β_2 and γ [9].

In Section 4, sufficient conditions for the existence of an integrator gain bound that guarantees stability were given, and in Theorem 1.6, a value of such an ϵ^* is calculated. However, the relationship between the two theorems is not necessarily obvious. In Theorem 1.7 following, we determine the value of an ϵ^* directly in terms of the parameters of the unaugmented closed loop (P_0, C_0) and the conditions given in Theorem 1.3.

Theorem 1.7.

Consider the second case in Figure 1.3 described by equation (1.13) which satisfies the requirement of equation (1.2), and suppose that the following assumptions are satisfied:

(i) f and g are globally Lipschitz continuous with Lipschitz constants k_f and k_g for the Euclidean norm $\|.\|_2$.

(ii) The equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (1.13) implicitly defines a unique C^2 function $x = h(\xi)$.

(iii) For any fixed $\xi \in \mathbf{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x,\xi)$ (that is, the original unaugmented system P) is Globally Exponentially Stable (GES).

(iv) There exists a scalar $\alpha_1 > 0$ such that $g(h(\xi),\xi)\xi \leq -\alpha_1 \xi^2, \forall \xi$. This ensures that the equilibrium $\xi = 0$ of the reduced model (slow time scale) $\dot{\xi} = g(h(\xi),\xi)$ is GES.

Then, there exist some $\alpha_2 > 0$, β_1, β_2 and γ such that when $0 < \epsilon < \epsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2}$, the origin is an asymptotically stable equilibrium of the unforced closed loop $(P, \frac{\epsilon}{s})$.

Proof:

For the reduced model (slow time scale), we choose $V(\xi) = \frac{1}{2}\xi^2$, as a Lyapunov function candidate. Then $\frac{\partial V}{\partial \xi} g(h(\xi), \xi) = \xi g(h(\xi), \xi) \leq -\alpha_1 \xi^2$ (This satisfies condition d of Theorem 1.6).

According to the Lipschitz continuity of $g(x, \xi)$, there exists some β_1 such that $\frac{\partial V}{\partial \xi}[g(x, \xi) - g(h(\xi), \xi)] \leq k_g \xi \|x - h(\xi)\| \leq \beta_1 \xi \phi(x - h(\xi))$ (The condition e of Theorem 1.6 is met).

¿From the condition that for fixed $\xi \in \mathbb{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x, \xi)$ is GES, we conclude that there exists a Lyapunov function $W(x, \xi)$ such that

a. $W(x, \xi) > 0 \; \forall x \neq h(\xi)$ and $W(h(\xi), \xi) = 0$.

b. $\frac{\partial W}{\partial x} f(x, \xi) \leq -\alpha_2 \phi^2(x - h(\xi))$ (From the condition that the equilibrium $x_e = h(\xi)$ is GES).

c. $\frac{\partial W}{\partial \xi} g(x, \xi) \leq \gamma[\phi(x - h(\xi))]^2 + \beta_2 \xi \phi(x - h(\xi))$(From the condition that the equilibrium $x_e = h(\xi)$ is GES and the continuity and derivative of $g(.)$ and $h(.)$).

According to Theorem 1.6, we achieve that $\epsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2}$.

□

1.6 Alternative Locations for Including an Integrator

In the above, we have only discussed the case where the small gain integrator is connected in parallel with the original controller C_0 (See Figure 1.3). Actually, there are also other options for adding such an integrator to the system which will achieve a similar effect. In this section, we discuss alternative options which are depicted in Figures 1.6 and 1.7.

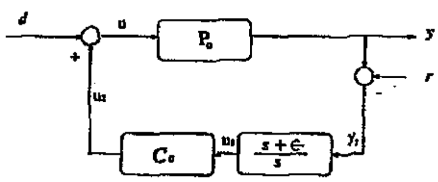

FIGURE 1.6. Alternative integrator location: Case 3

In Figure 1.6, we have started with a stable closed loop system as in Figure 1.2 and have serially connected a transfer function block $\frac{s+\epsilon}{s}$ between the output y_r and the original controller C_0.

We will use affine differential equations to simplify our discussion, and therefore assume(with some loss of generality) that we can express P_0, C_0 and $\frac{s+\epsilon}{s}$ as:

$$P_0 : \begin{cases} \dot{x}_0 &= f_1(x_0) + f_2(x_0)u \\ y &= g_1(x_0) + g_2(x_0)u, \end{cases} \tag{1.15}$$

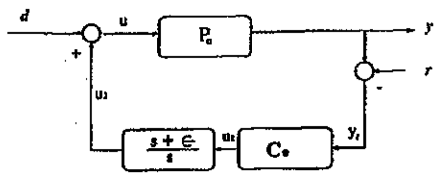

FIGURE 1.7. Alternative integrator location: Case 4

$$C_0 : \begin{cases} \dot{\eta} &= l_1(\eta) + l_2(\eta)u_1 \\ u_2 &= m_1(\eta) + m_2(\eta)u_1, \end{cases} \tag{1.16}$$

$$\frac{s+\epsilon}{s} : \begin{cases} \dot{\xi} &= \epsilon y_r \\ u_1 &= y_r + \xi. \end{cases} \tag{1.17}$$

Here, we assume that $f_1(0) = 0$, $l_1(0) = 0$, $g_1(0) = 0$ and $m_1(0) = 0$.

In order to avoid ill-posedness, it is sufficient to require that $g_2(x_0) = 0$. Then, the equations (1.15) can be simplified as:

$$P_0 : \begin{cases} \dot{x}_0 &= f_1(x_0) + f_2(x_0)[u_2 + d] \\ y &= g_1(x_0). \end{cases} \tag{1.18}$$

The state equation of the combined system is:

$$Case3 : \begin{cases} \dot{x}_0 &= f_1(x_0) + f_2(x_0)[m_1(\eta) + m_2(\eta)(\xi - r + g_1(x)) + d] \\ \dot{\eta} &= l_1(\eta) + l_2(\eta)(\xi - r + g_1(x)) \\ \dot{\xi} &= \epsilon[g_1(x) - r]. \end{cases} \tag{1.19}$$

If we let $x = [x_0^T \; \eta^T]^T$, then the above equation can be rewritten in the form:

$$Case\ 3 : \begin{cases} \dot{x} &= \bar{f}_1(x) + \bar{f}_2(x)d + \bar{f}_3(x)[\xi - r + g_1(x)] \\ \dot{\xi} &= \epsilon[g_1(x) - r]. \end{cases} \tag{1.20}$$

Here, $\bar{f}_1(x) = \begin{bmatrix} f_1(x_0) + f_2(x_0)m_1(\eta) \\ l_1(\eta) \end{bmatrix}$, $\bar{f}_2(x) = \begin{bmatrix} f_2(x_0) \\ 0 \end{bmatrix}$, $\bar{f}_3(x) = \begin{bmatrix} f_2(x_0)m_2(\eta) \\ l_2(\eta) \end{bmatrix}$.

Alternatively let us consider Case 4 as shown in Figure 1.7. See equations (1.15) and (1.16), where we assume that $m_2(\eta) = 0$ in order to avoid ill-posedness. Similarly to case 3, the state equation of the combined system can be written as:

$$Case\ 4 : \begin{cases} \dot{x} &= \tilde{f}_1(x) + \tilde{f}_2(x)r + \tilde{f}_3(x)[\xi + d + m_1(x)] \\ \dot{\xi} &= \epsilon m_1(x). \end{cases} \tag{1.21}$$

Here, $\tilde{f}_1(x) = \begin{bmatrix} f_1(x_0) \\ l_1(\eta) + l_2(\eta)g_1(x_0) \end{bmatrix}$, $\tilde{f}_2(x) = \begin{bmatrix} 0 \\ -l_2(\eta) \end{bmatrix}$,
$\tilde{f}_3(x) = \begin{bmatrix} f_2(x_0) \\ l_2(\eta)g_2(x_0) \end{bmatrix}$.

The methods for dealing with cases 3 and 4 are very similar to those for the parallel connection discussed in Sections 4 and 5. Hence, rather than a full analysis, we just give sufficient conditions for the existence of a scalar ϵ^* such that $0 < \epsilon < \epsilon^*$ guarantees stability of the closed loop. As in case 2, when we analyse the stability of the augmented system, we first neglect the constant input signal r and d [1]. By setting $r = 0$, $d = 0$, equations (1.20) and (1.21) can be analysed both together. We can choose equation (1.22) following as a common model for cases 3 and 4 to analyse the stability of the augmented system.

$$\begin{cases} \dot{x} &= f_1'(x) + f_3'(x)[\xi + g'(x)] \\ \dot{\xi} &= \epsilon g'(x) \end{cases} \tag{1.22}$$

In the above, f' is the $\tilde{f}$ of equation (1.20), or the $\tilde{f}$ of equation (1.21). Similarly, $g'(x)$ is the $g_1(x)$ of equation (1.20), or the $m_1(x)$ of equation (1.21).

In order to use singular perturbation theory, we change equation (1.22) to its standard singular perturbation form .

Let $\tau = \epsilon(t - t_0)$, $\tau = 0$ at $t = t_0$, $\frac{d\tau}{dt} = \epsilon$. It then follows that

$$Cases\ 3,4: \begin{cases} \epsilon\dot{x} &= f_1'(x) + f_3'(x)[\xi + g'(x)] \\ \dot{\xi} &= g'(x) \end{cases} \tag{1.23}$$

The dot in equation(1.23) means the derivative with respect to τ.

We do not absorb $f_3'(x)g'(x)$ into $f_1'(x)$ in order to emphasise the dependence of the state evolution equation $\dot{x}$ on the plant "output" $g'(x)$.

Theorem 1.8. *Consider equation (1.23), which represents either of the augmented systems in Figures 1.6 and 1.7 with $d = 0$ and $r = 0$. Let the following assumptions be satisfied:*

(i) The equation $0 = f_1'(x) + f_3'(x)[\xi + g'(x)]$ obtained by setting $\epsilon = 0$ has a unique C^2 solution $x = h(\xi)$.

(ii) For a fixed $\xi \in \mathbf{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem (1.23-1) is Globally Asymptotically Stable (GAS) and Locally Exponentially Stable (LES).

(iii) The equilibrium $\xi = 0$ of the reduced model $\dot{\xi} = [g'(h(\xi))]$ is GAS and LES.

It then follows that there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x, \xi) = (0, 0)$ is GAS.

---

[1] A nonzero constant reference input r will merely alter the equilibrium state of both P_0 and C_0 as in case 2. A constant disturbance d can also be rejected, if the augmented system is GES.

Proof: The proof is similar to Theorem 1.3. It is also helpful to see p.90 of [10]. □

Remarks:

- A sufficient condition for (ii) is that under any constant but arbitrary inputs r and d the closed loop (P_0, C_0) is GAS and LES to some equilibrium $x = x_e$. We now explain why this is the case.

  Note that, if we merely assume that the closed loop (P_0, C_0) is GAS and LES only for zero inputs r and d, then the unperturbed system equation $\dot{x} = f_1'(x) + f_3'(x)[g'(x)]$ is GAS and LES, but we can not ensure that for arbitrary fixed ξ, $\dot{x} = f_1'(x) + f_3'(x)[\xi + g'(x)]$ is also GAS and LES. On the other hand, a fixed arbitrary ξ is equivalent to a constant reference input r (in equation(1.20)) or a constant disturbance input d (in equation(1.21)). Hence, if we assume that the unperturbed system $\dot{x} = f_1'(x) + f_3'(x)[g'(x)]$ is GAS and LES under arbitrary constant inputs r and d then condition (ii) is satisfied.

- A sufficient condition for (iii) may be determined by considering a Lyapunov function candidate $V(\xi) = \frac{1}{2}\xi^2$. We just need that $V(\xi)$ satisfies the following requirements to ensure the reduced model $\dot{\xi} = g'(h(\xi))$ is GAS and LES.

  There exist positive constants $\rho_i, i = 1, 2, 3$ such that

  (a) $\rho_1|\xi|^2 \leq V(\xi) \leq \rho_2|\xi|^2$,

  (b) $\xi\dot{\xi} = g'(h(\xi))\xi \leq -\rho_3|\xi|^2$.

- We can also use methods similar to those in Section 5 to give a particular value for such an ϵ^*.

Note:
It also should be emphasised that if we consider the case that the equilibrium point ξ is not zero but fixed at $\xi = \xi_e$ by the influence of a nonzero reference input r, we can add a condition that is more strict than conditions (a) and (b) to ensure the reduced model $\dot{\xi} = g'(h(\xi))$ is GAS and LES for all fixed equilibrium point $\xi = \xi_e$. In particular, for the equilibrium point $\xi = \xi_e$, we assume that $[g'(h(\xi)) - g'(h(\xi_e))](\xi - \xi_e) \leq -\rho_3|\xi - \xi_e|^2$.

1.7 MIMO Systems

So far in the development, we have concentrated our attention on SISO systems. In this section, we will extend our results to MIMO systems as well.

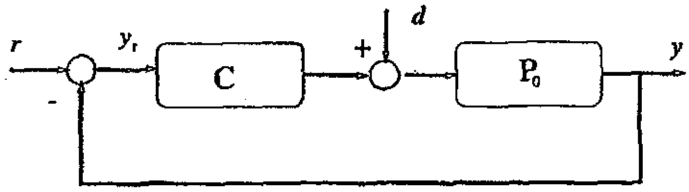

FIGURE 1.8. The nonlinear MIMO constant disturbance suppression problem

For linear time invariant (LTI) SISO systems, it has been shown that[12] an integrator should be included in the controller to ensure constant input disturbance rejection, regardless of whether the plant itself also has an integrator. For nonlinear MIMO systems, however, it is sometimes complicated to check whether sufficient integrators are included in the controller or plant. We present Theorems 1.9 and 1.10 following to give sufficient conditions for ensuring both constant input disturbance rejection and zero steady state tracking error for nonlinear MIMO systems.

Theorem 1.9.

Consider a closed loop system depicted by Figure 1.8 (which may be MIMO system). If one can find or construct a sub-state [2] $x_{itg_i} \in \mathbb{R}$ *for each reference tracking error* $y_{r_i} \in \mathbb{R}$ *(see figure 1.8) such that*

i) $x_{itg_i} = \int \phi_i(y_{ri})dt$, $i = 1, 2, ..., p$, *where,* ϕ *is a scalar function such that* $\phi_i(y_{r_i}) = 0$ *iff* $y_{r_i} = 0$.

ii)The whole system is stabilised.

Then this closed loop system will reject the constant input disturbance d and ensure zero steady state tracking error for constant reference input.

Proof: It is easy to see that if each sub-state x_{itg_i} is stabilised (that is when $t \mapsto +\infty$, x_{itg_i} approaches a constant.), then y_r will go to zero no matter whether the constant input disturbance exist or not. □

Remarks

- It should be noted that every sub-state x_{itg_i} should be just the integration of a function of the reference tracking error y_{r_i} which vanishes only when its arguments are zero.
- If there already exists such a sub-state in the open loop plant, then it is not necessary to include a small gain integrator $\frac{\varepsilon}{s}$ to construct such a sub-state. In the control of a MIMO helicopter model which

[2]Given an original state vector of a system satisfying a nonlinear differential equation, a substate of the system is defined as a sub-vector of any Lyapunov transformation of the original state vector.

we present in the next section, the velocity v_y (when ϕ, θ and y_{r_ψ} is small, $\dot{v}_y \approx -\frac{f_{bx}}{m} sin(y_{r_\psi}) \approx -\frac{f_{bx}}{m} y_{r_\psi}$) is just the sub-state of the yaw angle(See Section 8). Hence, we have not added $\frac{\epsilon}{s}$ in the yaw angle channel but still acquired constant input disturbance rejection and zero steady state tracking error.

In Theorem 1.9, we have assumed that the MIMO system with the integrators is stable. We have not provided any conditions to ensure stability. In the following, we will give some sufficient conditions to guarantee local stability for a MIMO system augmented with low gain integrators.

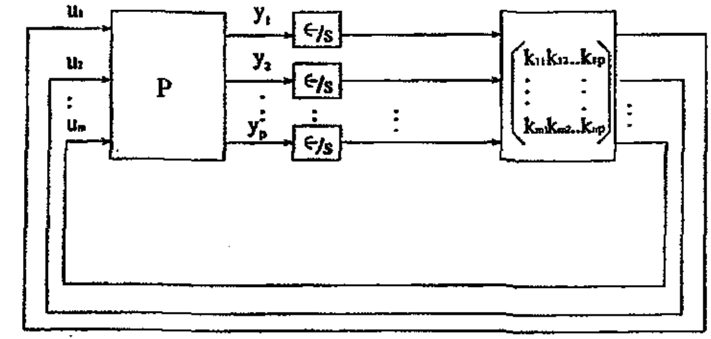

FIGURE 1.9. MIMO system augmented with small gain integrators

As for the SISO analysis, we neglect the constant input signals r and d when we analyse the stability.

Consider Figure 1.9. Let P be a MIMO system with m inputs and p outputs (here, $m \geq p$) described by the following differential equation.

$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases} \quad (1.24)$$

We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^p$ are unbiased in the sense that

$$\begin{cases} f(0,0) = 0 \\ g(0,0) = 0. \end{cases} \quad (1.25)$$

The state equation of the augmented system can be described as below.

$$\begin{cases} \dot{x} = f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} = \epsilon g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \quad (1.26)$$

Again, we change equation (1.26) to its standard singular perturbation form .

$$\begin{cases} \epsilon\dot{x} = f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} = g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \tag{1.27}$$

In equation (1.27), the dot means the derivative with respect to τ.

Theorem 1.10.

Consider the system described by equations (1.24) and (1.26) and illustrated in Figure 1.9. Assume that $x = 0$ is an asymptotically stable equilibrium for the plant P, and that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. We assume that

(i) The equation $f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) = 0$ obtained by setting $\epsilon = 0$ in equation (1.27) has a unique C^2 solution $x = h(\xi)$,

(ii) The matrix $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular.

Then, there exists ϵ^* and a constant matrix $K = (k_{ij})_{m \times p}$ (see Figure 1.9) such that $(x = 0, \xi = 0)$ is an asymptotically stable equilibrium whenever $0 < \epsilon < \epsilon^*$.

Proof: The proof is in Appendix 1.A. □

Synthesis of a controller for global stability is more complicated than that for local stability. However, if a system is globally stable, it is also locally stable. One possible method for designing the constants (k_{ij}) is just to consider local stability, and then perform global stability analysis.

Another practical way to consider the global stability problem is to recursively apply Theorem 1.3 for SISO systems to MIMO systems. Actually, this method can also be applied to deal with the local stability problem (see Theorem 1.10) and achieves a diagonal feedback matrix K when $m = p$.

Specifically, we first view the MIMO system as a SISO system by considering only the input u_1 and output y_1, that is, we assume that the other inputs are zero and neglect the other outputs. Then, we make the connection of single small gain integrator $\frac{\epsilon}{s}$ and the "SISO" system globally stable if the "SISO" system satisfies the sufficient conditions of Theorem 1.3. Recursively, we connect a second small gain integrator to the augmented "SISO" system with input u_2 and output y_2. Then, the connection of the second single small gain integrator and the augmented "SISO" system is globally stable if the augmented "SISO" satisfies the sufficient conditions of Theorem 1.3, and so on. If the sufficient conditions of Theorem 1.3 are satisfied by each augmented "SISO" system, in this way, we can include all necessary integrators to the MIMO system while ensuring global stability.

1.8 Controller Design for a Nonlinear Helicopter Model

In this section, we implement our constant input disturbance rejection method on the simulated control of a helicopter model provided by [4]. In [4], an output tracking controller is designed based on approximate linearisation. However, this method does not completely suppress constant input disturbances. In order to deal with this shortcoming, we augmented an approximate linearised output tracking controller, by including an extra integrator block(See Figure 1.10). In Figure 1.10, the numbers near the lines are the dimension of the vectors depicted by the lines.

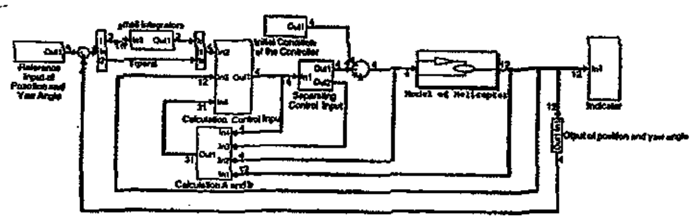

FIGURE 1.10. Approximate linearised output tracking control with integrator augmentation

The helicopter model appears as below in both [4] and [11].

$$\begin{bmatrix} \dot{P} \\ \dot{v}^p \\ \dot{R} \\ \dot{\omega}^b \\ \dot{T}_M \\ \dot{T}_T \\ \dot{a}_{1s} \\ \dot{b}_{1s} \end{bmatrix} = \begin{bmatrix} v^p \\ \frac{1}{m} R f^b \\ R\hat{\omega}^b \\ \mathcal{I}^{-1}(\tau^b - \omega^b \times \mathcal{I}\omega^b) \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \tag{1.28}$$

$$y = [p_x \; p_y \; p_z \; \phi \; \theta \; \psi]^T, \tag{1.29}$$

where $P \in \mathbb{R}^3$ and $v^p \in \mathbb{R}^3$ are the position and velocity vectors of centre of the mass in spatial coordinates, $R \in \mathcal{SO}(3)$ is the rotation matrix of the body axes relative to the partial axes, $w^b = [w_x^b, w_y^b, w_z^b]^T \in \mathbb{R}^3$ is the body angular velocity vector, $\hat{w}^b \in \mathbb{R}^{3\times3}$ is defined as in equation (1.30), $m > 0$ is the body mass, $\mathcal{I} \in \mathbb{R}^{3\times3}$ is the inertial matrix and $f^b, \tau^b \in \mathbb{R}^3$ are the body force and torque.

$$\hat{w}^b = \begin{bmatrix} 0 & -w_z^b & w_y^b \\ w_z^b & 0 & -w_x^b \\ -w_y^b & w_x^b & 0 \end{bmatrix} \tag{1.30}$$

The body forces and torques generated by the main rotor are controlled by T_M,a_{1s} and b_{1s}, in which a_{1s} and b_{1s} are respectively the longitudinal and lateral tilt of the tip path plane of the main rotor with respect to the shaft. The tail rotor is considered as the source of pure lateral force and anti-torque, which are controlled by T_T.

As in [4], we also assume that all the states are measurable. In order to present the helicopter system in an input-affine form, we define $w = [w_1\ w_2\ w_3\ w_4]^T$, which are the derivatives of T_M, T_T, a_{1s} and b_{1s} as auxiliary inputs to the system. Here the state $x \in R^{16}$, the inputs $w \in R^4$, the output $y \in R^6$.

It can be seen that the helicopter model (1.28) is marginally unstable, so we first need to design a stabilising controller. As in [4], we can design an approximate linearised output tracking controller. Based on this controller we then design a modified controller by augmentation with integrators, and achieve satisfactory disturbance rejection results.

The system equations of (1.28) and (1.29) have four control inputs so the maximum number of outputs for possibly applying an input-output linearisation procedure is four. We choose the outputs p_x, p_y, p_z, ψ as in [4].

Approximate linearisation is implemented by neglecting the coupling terms, a procedure which is presented very clearly in [4].

We define reference tracking error signals as follows:

$$y_{r_{p_i}} = r_{p_i} - y_{p_i},$$

$$y_{r_\psi} = r_\psi - y_\psi.$$

Here, $i = x, y, z$, while r_{p_i} and r_ψ are the reference inputs for position and yaw angle respectively.

We augment the approximate input-output linearisation controller with the small gain integrators of the reference tracking error of position $y_{r_{p_x}}, y_{r_{p_y}}$ and $y_{r_{p_z}}$ (see Figure 1.10). We define the output of the small gain integrators as

$$u_{p_i} = \frac{k_{p_i}}{s} y_{r_{p_i}}$$

According to Theorem 1.10, it is possible to choose values for k_{p_i} to retain the stability of the system while acquiring constant input disturbance rejection and zero steady state tracking error.

As we stated in last section, we have not integrated the yaw angle reference tracking error y_{r_ψ}, because when ϕ, θ and ψ is small, $\dot{v}_y \approx$

$-\frac{f_{bz}}{m} sin(y_{r_\psi}) \approx -\frac{f_{bz}}{m} y_{r_\psi}$. That is v_y is just the sub-state x_{itg_ψ} corresponding to the yaw angle reference tracking error y_{r_ψ} that is required in Theorem 1.9. So, we need not augment the controller with an integrator for y_{r_ψ}.

Figures 1.11 and 1.12 illustrate some simulation results, where the mass of the helicopter has changed 10 percent from the nominal case. Although a 10 percent change in mass is, strictly speaking, a change in the plant rather than a constant input disturbance, it has a similar effect of altering the (constant) control input required to achieve equilibrium. We consider such a "disturbance" in order to make comparison with the results reported in [4]. The augmented controller is still able to track the reference input without steady state errors. In contrast, the approximate input-output linearisation controller without the integrator augmentation does not reject such disturbances.

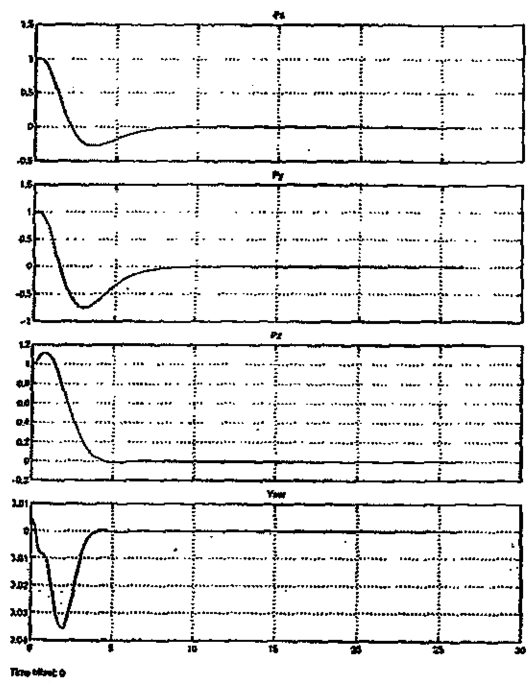

FIGURE 1.11. **The Output of Augmented System Under 10 percent Change of Mass**

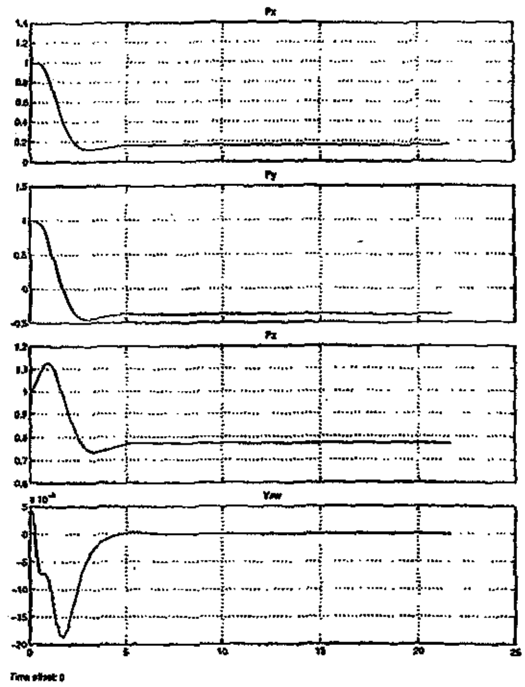

FIGURE 1.12. The Output of System without the Augmentation Under 10 percent Change of Mass

1.9 Conclusion

In this paper we have addressed the problem of achieving constant input disturbance rejection and constant reference tracking, for nonlinear systems. A relatively intuitive solution to this problem has been proposed: we simply augment an existing controller (which stabilises the nonlinear system) with (an) appropriately located integrator(s), with appropriately small gain. We can use singular perturbation theory to guarantee that, even with the addition of such an integrator, closed loop stability will be retained. It is also straightforward to deduce that the inclusion of an appropriately located integrator in the closed loop will ensure that constant input disturbances are in fact, rejected and that constant references will be tracked.

A performance tradeoff with respect to the integrator gain certainly holds for linear systems. We expect that such a tradeoff would also hold in general. This tradeoff is between the time constant associated with the suppression of the constant signals and the performance with respect to other disturbance signals for which the original controller was designed. The speed of the constant suppression (the slow time-scale system), in general, increases with the magnitude of the integrator gain. However, as the magnitude of this gain increases, the closed-loop performance is no longer guaranteed by singular perturbation theory to accurately approximate the ideal two time-scale system, and the closed loop may approach instability, yielding poor performance for some classes of disturbances.

Our simulation results on a nonlinear helicopter model indicate that satisfactory performance can be achieved in some circumstances, and that the proposed method is a simple but effective way to achieve the suppression of exogenous signals.

REFERENCES

[1] J. W. Helton and M. R. James. *Extending $\mathcal{H}_\infty$ Control to Nonlinear Systems: Control of Nonlinear Systems to Achieve Performance Objectives.* Society for Industral and Applied Mathematics, Philadelphia U.A.S., 1999.

[2] R. A. Kennedy. *Nonlinear Systems Analysis: Lyapunov and Input-Output Methods.* ANU, Canberra, 2000.

[3] H.W. Knobloch, A. Isidori, and D. Flockerzi. *Topics in Control Theory.* Birkhäuser, Berlin Germany, 1993.

[4] T.J. Koo and S.Sastry. Output tracking control design of a helicopter model based on approximate linearization. *Proceedings of the 37th Conference on Decision and Control,* 1998.

[5] H. J. Kushner. *Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems.* Birkhäuser, Boston U.S.A., 1990.

[6] W. C. A. Maas. *Nonlinear $\mathcal{H}_\infty$ control: the singular case.* Centrum voor Wiskunde en Informatica, Netherlands, 1996.

[7] T. Mita, M. Hirata, K. Murata, and H. Zhang. $\mathcal{H}_\infty$ control versus disturbance-observer-based control. *IEEE Transactions on Industrial Electronics,* pages 488–495, JUNE 1998.

[8] T. Mita, X. Xin, and B. D. O. Anderson. Extended $\mathcal{H}_\infty$ control—solving $\mathcal{H}_\infty$ servo and estimation problems. *Proceedings of the 36th Conference on Decision and Control,* pages 4653–4658, 1997.

[9] P.Kokotovic, H.K.Khalil, and J.O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press Inc, U.S.A., 1986.

[10] R.Sepulchre, M.Jankovic, and P.Kokotovic. *Constructive Nonlinear Control*. Springer Verlag, New York, 1996.

[11] S.Sastry. *Nonlinear Systems*. Springer Verlag, New York, 1999.

[12] S. W. Su, B. D. O. Anderson, and T. S. Brinsmead. Robust disturbance suppression for nonlinear systems based on $\mathcal{H}_\infty$ control. *Proceedings of the 39th Conference on Decision and Control*, 2000.

[13] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, New Jersey, 1993.

[14] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, New Jersey, 1996.

1.A Proof of Theorem 1.10

Proof: For augmented system (1.26), we can apply Theorem 1.4 by making the following matrix identifications.

$A_{11} = G_u K$,
$A_{12} = G_x$,
$A_{21} = F_u K$,
$A_{22} = F_x$.
Here,

$$G_u = \frac{\partial g}{\partial u}|_{(x,\xi)=(0,0)} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_p}{\partial u_1} & \frac{\partial g_p}{\partial u_2} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & \cdots & k_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ k_{m1} & k_{m2} & \cdots & k_{mp} \end{bmatrix},$$

$$G_x = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_p}{\partial x_2} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$F_u = \frac{\partial f}{\partial u}|_{(x,\xi)=(0,0)} \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$F_x = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x,\xi)=(0,0)}.$$

As we have assumed that $x = 0$ is an asymptotically stable equilibrium for plant P, the eigenvalues of A_{22} have negative real parts.

¿From the assumption that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular, we can conclude that $(G_u - A_{12}A_{22}^{-1}F_u)K$ is also nonsingular (See Appendix 1.B). This implies that $(G_u - A_{12}A_{22}^{-1}F_u)$ is full row rank.

Because $A_{11} - A_{12}A_{22}^{-1}A_{21} = (G_u - A_{12}A_{22}^{-1}F_u)K$, we can choose a K to ensure that all eigenvalues of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts.

According to Theorem 1.4, there exists ϵ^* and a matrix $K_{m\times p}$ such that $(x = 0, \xi = 0)$ is an asymptotically stable equilibrium whenever $0 < \epsilon < \epsilon^*$.

Furthermore, when $m = p$, the stabilising control matrix K can be chosen to be triangular or even diagonal (See Appendix 1.C).

□

1.B Proof of Non-singularity

Lemma 1.1.

Consider the system described by equations (1.24) and (1.27) and illustrated in Figure 1.9. Assume that $x = 0$ is an asymptotically stable equilibrium for the plant P, and that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. We assume that

(i) The equation $f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) = 0$ obtained by setting $\epsilon = 0$ in equation (1.27) has a unique C^2 solution $x = h(\xi)$,

(ii) The matrix $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular.

Then, $(G_u - A_{12}A_{22}^{-1}F_u)K$ (G_u, A_{12}, A_{22} and F_u are defined as in Theorem 1.10) is nonsingular.

Proof: Consider that $f(h(\xi), \xi) = 0$, we have
$\frac{\partial f}{\partial x}\frac{\partial h}{\partial \xi} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial \xi} = 0$.
That is
$F_x\frac{\partial h}{\partial \xi} + F_u K = 0$, so
$\frac{\partial h}{\partial \xi} = -F_x^{-1}F_u K$.
Now, we have
$\frac{\partial g(h(\xi),\xi)}{\partial \xi} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial \xi} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial \xi} = (G_u - A_{12}A_{22}^{-1}F_u)K$.

¿From the assumption that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}$ is nonsingular, we conclude that $(G_u - A_{12}A_{22}^{-1}F_u)K$ is also nonsingular.

□

1.C Proof of the Existence of a Stabilising Diagonal Matrix K

Lemma 1.2. *Define A_i, the $i \times i$ (upper left) sub-matrix of A as*

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i} \\ a_{21} & a_{22} & \dots & a_{2i} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ii} \end{bmatrix}. \tag{1.31}$$

For any nonsingular matrix $A_0 \in \mathbb{R}^{n\times n}$, it is possible to reorder the columns of A_0 to ensure the reordered matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ satisfies the property that for each $i = 1, 2, \dots n$, $\det(A_i) \neq 0$.

Proof: It is easy to see this if we consider each column as a vector. □

Lemma 1.3. *If all the roots of the equation $s^n + a_1 s^{n-1} + \dots + a_0 = 0$ have negative real parts, then there exists an ϵ^* such that when $0 < \epsilon < \epsilon^*$ all the roots of the equation $s^{n+1} + a_1 s^n + \dots + a_0 s + \epsilon = 0$ have negative real parts.*

Proof: This is a direct result of linear singular perturbation theory. □

Theorem 1.11. *For a nonsingular square matrix $A \in \mathbb{R}^{n\times n}$ with A_i defined as in (1.31) and $\det(A_i) \neq 0$ for each $i = 1, 2, \dots n$ there exists a diagonal matrix $K \in \mathbb{R}^{n\times n}$ such that all the eigenvalues of the matrix AK have negative real parts*

Proof: Write K as

$$K = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_n \end{bmatrix}. \tag{1.32}$$

Then,

$$AK = \begin{bmatrix} k_1 a_{11} & k_2 a_{12} & \dots & k_n a_{1n} \\ k_1 a_{21} & k_2 a_{22} & \dots & k_n a_{2n} \\ \dots & \dots & \dots & \dots \\ k_1 a_{n1} & k_2 a_{n2} & \dots & k_n a_{nn} \end{bmatrix},$$

$$
\begin{aligned}
& det([sI - AK]) \\
= \;& s^n + (-1)^1(k_1a_{11} + k_2a_{22} + ... + k_na_{nn})s^{n-1} \\
+ \;& (-1)^2(k_1k_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + k_1k_3 \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + ... \\
+ \;& k_{n-1}k_n \begin{bmatrix} a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n\ n-1} & a_{n\ n} \end{bmatrix})s^{n-2} \\
+ \;& (-1)^3\Big(k_1k_2k_3 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + ... \\
& +k_{n-2}k_{n-1}k_n \begin{bmatrix} a_{n-2\ n-2} & a_{n-2\ n-1} & a_{n-2\ n} \\ a_{n-1\ n-2} & a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n\ n-2} & a_{n\ n-1} & a_{n\ n} \end{bmatrix}\Big)s^{n-3} \\
+ \;& ... \\
+ \;& (-1)^n k_1k_2...k_n \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}
\end{aligned}
\tag{1.34}
$$

Now, we set $|\frac{k_2}{k_1}| = \varepsilon_1$, $|\frac{k_3}{k_2}| = \varepsilon_2$, ... $|\frac{k_n}{k_{n-1}}| = \varepsilon_{n-1}$, and $\bar{\varepsilon} = max\{\varepsilon_1, \varepsilon_2, ...\varepsilon_{n-1}\}$.
Let $\bar{\varepsilon} \ll 1$, then

$$
\begin{aligned}
& det([sI - AK]) \\
= \;& s^n + (-1)^1(k_1det(A_1) + O(\varepsilon_1))s^{n-1} \\
+ \;& (-1)^2(k_1k_2det(A_2) + O(\varepsilon_1\varepsilon_2))s^{n-2} \\
+ \;& (-1)^3(k_1k_2k_3det(A_3) + O(\varepsilon_1\varepsilon_2\varepsilon_3))s^{n-3} \\
+ \;& ... \\
+ \;& (-1)^n k_1k_2k_3...k_n det(A_n)
\end{aligned}
\tag{1.35}
$$

Consider that $det(A_i) \neq 0$, for all $i = 1, 2, ...n$ and apply lemma 1.3 repeatedly. First, we choose a value for k_1 and a small enough ε_1^* such that when $0 < \varepsilon_1 < \varepsilon_1^*$ all the roots of $s + (-1)^1(k_1det(A_1) + O(\varepsilon_1)) = 0$ have negative real parts. Secondly, we choose a value for k_2 (after the choice of k_2, ε_1 can be fixed) and a small enough ε_2^* such that when $0 < \varepsilon_2 < \varepsilon_2^*$ all the roots of $s^2+(-1)^1(k_1det(A_1)+O(\varepsilon_1))s+(-1)^2(k_1k_2det(A_2)+O(\varepsilon_1\varepsilon_2)) = 0$ have negative real parts. We continue this procedure. Finally, we choose a small enough k_n such that all the roots of the nth-order equation have negative real parts. □