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# *Nonlinear Control in the Year 2000*

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# Preface

These two volumes contain papers based on talks delivered at the 2nd Workshop of the Nonlinear Control Network (`http://www.supelec.fr/lss/NCN`), held in Paris, June 5-9, 2000. The Authors of the presented papers, as well as the Editors of these two volumes, hope that "Nonlinear Control in the Year 2000" is not only one more book containing proceedings of a workshop. Two main reasons justifying our hope are that, firstly, the end of the century is a natural moment to think about past developments in nonlinear control and about its perspectives in the twenty-first century; and, secondly, we believe that nonlinear control has reached an age of maturity which enables the community to sketch a state-of-the-art of the field. We hope that papers contained in the two volumes serve to fulfill these goals: many of them have their roots and their origins in nonlinear control theories which have been developed in past three decades and which, by now, form a basis of canonical results in the field. Such papers form a bridge between the actual theory and its future developments. Many other papers contained in the book present completely new ideas and suggest new directions and, in this sense, they are directed towards the future of the nonlinear control.

We would like to emphasize one peculiarity of our field: nonlinear control is an example of a theory situated at a crossroad between mathematics and engineering science. Due to this position, nonlinear control has its roots in both fields and, as we deeply believe, can bring new ideas and new results for both domains. The book reflects very well this "double character" of nonlinear control theory: the reader will find in it results which cover a wide variety of problems: starting from pure mathematics, through its applications to nonlinear feedback design, all way to recent industrial advances.

Eight papers contained in the book are based on invited talks delivered at the Workshop by:

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| *Alessandro Astolfi,* | *John Baras,* | *Christopher Byrnes,* |
| *Bronisław Jakubczyk,* | *Anders Rantzer,* | *Kurt Schlacher,* |
| *Eduardo Sontag,* | *Hector Sussmann.* | |

Altogether the book contains 80 papers and therefore it is impossible to mention all discussed topics: to give a flavour of the presented material let us mention a few of them. For many theoretical papers a common factor is optimal control, for example subriemannian geometry (in particular fundamental results on (non)subanalyticity of small balls and of the distance function), the use of generalized differentials in generalized Maximum Principle, singular and constrained optimal control problems. Another subdomain of nonlinear control attracting many contributions to the book is stability and asymptotic behavior. In this area, stemming from traditional Lyapunov techniques, the methods based on the concept of input-to-state stability have established a leading role in analysis and design, and new ideas have emerged, such as those aiming at the analysis of non-equilibrium steady-state behaviors, or at the evaluation of asymptotic convergence for almost all initial conditions via dual Lyapunov analysis. Applications of these ideas in nonlinear control are widespread: stabilization of nonlinear systems, trajectory tracking, adaptive control. Other subdomains of nonlinear control attracting a lot of attention, and represented in the book, are that of observability and observers for nonlinear systems, sliding mode control, theory of nonlinear feedback (invariants, classification, normal forms, flatness and dynamic feedback), recursive design (backstepping and feedforwarding). The papers present in the two volumes cover various aspects of all just mentioned topics. Moreover, the book contains also papers discussing main results of a plenary Session devoted to industrial applications.

We wish to thank all invited speakers and all contributors to the 2nd Nonlinear Control Network Workshop for making this conference an outstanding intellectual celebration of the area of Nonlinear Control at the turn of the century. The Editors are grateful to all the chairpersons:

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| *Fritz Colonius,* | *Emmanuel Delaleau,* | *Michel Fliess,* |
| *Halina Frankowska,* | *Jean-Paul Gauthier,* | *Henri Huijberts,* |
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| *Frédéric Mazenc,* | *Gérard Montseny,* | *Claude Moog,* |
| *Philippe Mullhaupt,* | *Henk Nijmeijer,* | *Romeo Ortega,* |
| *Elena Panteley,* | *Laurent Praly,* | *Anders Rantzer,* |
| *Pierre Rouchon,* | *Joachim Rudolph,* | *Andrey Sarychev,* |
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They have excellently played their role during the presentations and also have helped us in reviewing the contributed papers accepted for publication in this book. We would like to thank the TMR Program for the financial support which, in particular, helped numerous young researchers in attending the Workshop. We express our thanks to *Christiane Bernard* (Mathematics and Information Sciences Project Officer at the European Community) and *Radhakisan Baheti* (Program Director for Control at NSF) for their participation at the Session on Founding to Academic Research on Control. We also thank the CNRS staff as well as PhD students and Postdocs at L2S in Gif-sur-Yvette for their help in the organization of the Workshop.

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# Algorithms for identification of continuous time nonlinear systems: a passivity approach. Part I: Identification in open-loop operation Part II: Identification in closed-loop operation

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**Keywords** : Recursive Identification, Nonlinear Systems, Adaptive Systems, Output Error

**Abstract.** Algorithms for the identification of continuous time nonlinear plants operating in open-loop and in closed-loop are presented. An adjustable output error type predictor is used in open-loop operation. An adjustable output error type predictor parametrized in terms of the existing controller and the estimated plant model is used in closed-loop operation. The algorithms are derived from stability considerations in the absence of noise and assuming that the plant model is in the model set. Some convergence results based on passivity concepts are presented. Subsequently the algorithms are analyzed in the presence of noise and when the plant model is not in the model set. Examples illustrate the use of the various algorithms.

## General introduction

Output error identification algorithms for linear systems have been known for a long time to offer excellent performances in the presence of output noise. In particular they do not require a dynamic model for the noise, the only requirements relate to independence with respect to the excitation signal and its boundedness [7,12].

It seems therefore interesting to extend this type of algorithm to the identification of continuous time nonlinear systems operating in open-loop. In the present paper we focus on the recursive identification of *nonlinear* plants whose outputs cannot be expressed linearly in terms of the unknown parameters (i.e. $y \neq \theta_0^T \psi$ where $y$ is the output, $\theta_0$ is the vector of parameters and $\psi$ is a vector of nonlinear functions of various variables).

Therefore the first part of the paper will be dedicated to this subject.

The development of algorithms for plant model identification in closed-loop has been an important line of research in the last few years.

This line of research has been motivated by several factors: 1) the fact that in a number of situations identification in open-loop is difficult or simply not feasible (unstable plants, drift), 2) the presence of a controller in the loop (which has to be re-tuned), 3) the possibility of capturing the dynamic characteristics of the plant model which are critical for control design.

In the context of linear models, recursive and batch algorithms for plant model identification in closed-loop have been proposed, analyzed and evaluated experimentally [18,10,8,17,13,9]. Such algorithms have already moved towards standard use in industry.

One of the successful ways to develop algorithms for identification in closed-loop is to consider "closed-loop output error" schemes. [8,17].

The problem of closed-loop identification of nonlinear time-varying systems in the presence of a linear or a nonlinear controller has been discussed in [3,10] using the Hansen scheme and in [18] using a gradient approach. The convergence of the algorithms is not discussed.

In the present paper we focus on the recursive identification of *nonlinear* plants operating in closed-loop with a *nonlinear* controller using a closed-loop output error identification scheme. Preliminary results can be found in [19].

Passivity properties of various linear time-varying input-output operators play an important role in assessing the convergence properties for the various algorithms.

The paper is organized as follows.

Part I is dedicated to open-loop identification. Part II is dedicated to closed-loop identification. An example of identification of an open-loop unstable nonlinear plant model in closed-loop operation is given in Section 2.4. The concept of strong strict passivity and related properties are used extensively in this paper and outlined in Appendix A.

# Part I. Identification in open-loop operation

## 1.1 Open-loop output error identification. The basic equations and problem setting

The objective is to estimate the parameters of a single input single output (SISO) nonlinear time invariant system described by

$$\mathcal{S}: \quad y = P_0(u, v) \tag{1}$$

where $P_0$ is an unknown causal nonlinear operator, $u$ is the control input signal, $y$ is the achieved output signal and $v$ is the disturbance signal allowed to enter the system nonlinearly. It is not assumed that the output $y$ can be expressed linearly in terms of some parameter vector $\theta_0$. For ease of notation the time argument will be omitted when there are no ambiguities.

It is required that the system $P_0$ is Bounded Input Bounded Output (BIBO) stable. In the sequel we often make use of linearizations of some nonlinear operators around their operating trajectories. We therefore require that the plant and the model (to be defined subsequently) are smooth functions of the input signal, the output signal and the disturbance signal. This means that if the operator is linearized around any (stable) trajectory, the resulting linear (time-varying) system is BIBO stable. See [5] for more details.

We consider the following adjustable model for the system defined by (1)

$$y(\theta) = P(\theta, u) \tag{2}$$

where $P(\theta, u)$ defines the adjustable plant model, $y(\theta)$ is the output of the predictor and $u$ is the plant model input.

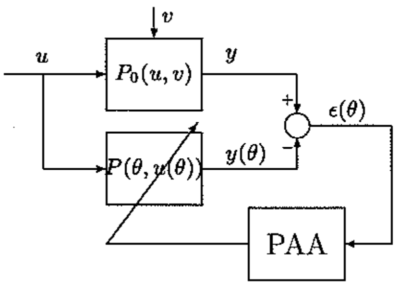


**Fig. 1.** Open-Loop Output Error (OLOE) identification scheme

The output error is defined as

$$\varepsilon = y - y(\theta). \tag{3}$$

The following assumptions will be made until further notice:

(i) $\exists\, \theta_0$ such that $P(\theta_0, u) = P_0(u, 0)$ for all $u \in \mathcal{L}_{2e}$ and $v \equiv 0$ (subsequently in the case $v \equiv 0$ the argument $v$ will be deleted)
(ii) **Notation:**
The partial derivative of $P(\theta, u)$ with respect to $\theta_j$ is denoted by $P'_{\theta_j}(\theta, u)$ for $j = 1, \cdots, d$ where $d$ is the dimension of the parameter vector $\theta$.

The operator $P'_{\theta_j}(\theta, u)$ and its time derivatives exist and are norm-bounded $\forall j$ along the trajectories of the predictor which requires $\dot{u}$ to be bounded. This assumption is not particularly restrictive as $P$ and $P(\theta)$ are assumed to be smooth operators.
(iii) The input $u$ and the stochastic disturbance $v$ are independent.

Assumption (i) means that at least for $\theta = \theta_0$ and in the absence of noise, the plant is in the model set. (The case when this is not true will be discussed separately in Section 1.4).

The generic parameter adaptation algorithm (PAA) which will be used throughout the paper is the continuous time version of the general PAA used in [9]:

$$\dot{\theta}(t) = F(t)\phi(t)\varepsilon(t) \tag{4}$$
$$\dot{F}^{-1}(t) = -\left[1 - \lambda_1(t)\right] F^{-1}(t) + \lambda_2(t)\phi(t)\phi^T(t) \tag{5}$$
$$0 < \lambda_1(t) \leq 1, \quad 0 \leq \lambda_2(t) < 2, \; F(0) > 0, \; F^{-1}(t) > \alpha F^{-1}(0), \; 0 < \alpha < \infty$$

where $\theta(t)$ is the estimated parameter vector, $\varepsilon(t)$ is the open-loop output error (defined above), $\phi(t)$ is the observation vector, $F(t)$ is the adaptation gain matrix, $\lambda_1(t)$ is a time-varying forgetting factor and $\lambda_2(t)$ allows one to weight the rate of decrease of the adaptation gain. The two functions $\lambda_1(t)$ and $\lambda_2(t)$ allow one to have different laws of evolution of the adaptation gain. Some of the typical cases are:

1. $\lambda_1(t) \equiv 1$; $\lambda_2(t) \equiv 0$; $\dot{F}(t) = 0$; $F(t) = F(0)$ (the gradient algorithm);
2. $\lambda_1(t) \equiv 1$; $\lambda_2(t) \equiv 1$ (recursive least squares type algorithm);
3. $\lambda_1(t) = \text{const} < 1$; $\lambda_2(t) \equiv 1$ (least squares with forgetting factor);
4. $\lambda_1(t) < 1$; $\lim_{t\to\infty} \lambda_1(t) = 1$; $\lambda_2(t) \equiv 1$ (variable forgetting factor).

We will consider subsequently that the assumptions (i) through (iii) are valid and furthermore, for some analysis, that:

(iv) $v \equiv 0$

(v) The higher order terms in the Taylor series involving expansions in powers of $(y - y(\theta))$ and $(\theta_0 - \theta)$ along the trajectories of the system can be neglected

This will allow us to implement the appropriate parameter estimation algorithm to begin with (i.e. it allows us to find the observation vector $\phi(t)$) and to analyze its asymptotic properties. In the first stage we will use several expansions in Taylor series for the expression of the plant output and predictor output and we will neglect the terms of power higher or equal to 2. A subsequent analysis will discuss the case when these terms are not neglected. It will also treat the presence of disturbances and unmodeled dynamics, requiring Assumption (iii).

### 1.2 Nonlinear open-loop output error algorithms

One has the following result (the NLOLOE algorithm):

**Theorem 1.** *Under the assumptions (i) through (ii), (iv) and (v) one has for*

$$\phi(t) = [P'(\theta, u)]^T = [P'_{\theta_1}(\theta, u) \;\; \cdots \;\; P'_{\theta_d}(\theta, u)]^T \tag{6}$$

*that*

$$\lim_{t\to\infty} \varepsilon(t) = 0 \tag{7}$$

*and*

$$\lim_{t\to\infty} \phi^T(t)\,(\theta(t) - \theta_0) = 0. \tag{8}$$

**Remark I.1:**

1. For the particular case when one can write

$$y(\theta) = P(\theta, u) = \phi^T(t)\theta$$

where $\phi(t)$ is a vector of linear or nonlinear functions of $y(\theta)$ and $u$ one has

$$[P'(\theta, u)]^T = \phi(t).$$

2. The condition (8) assures that the estimated parameter vector $\theta$, converges to a set defined as

$$\mathcal{D}_c = \left\{\theta : \phi^T(t)\,(\theta - \theta_0) = 0\right\}\cdot \tag{9}$$

If

$$\phi^T(t)\,(\theta - \theta_0) = 0 \tag{10}$$

has a unique solution $\theta = \theta_0$, the parameter vector will converge toward this value. In fact this condition is a "persistence of excitation" condition for the nonlinear case.

**Proof of Theorem 1:** The proof will be done in several steps.

**Step I:** Establishing the expression $\varepsilon = \mathbf{f}(\theta_0 - \theta(t))$

From (3) one has:

$$\varepsilon = P(\theta_0, u) - P(\theta, u) \tag{11}$$

Using a series expansion around $\theta$, one has

$$\begin{aligned}\varepsilon = P(\theta_0, u) - P(\theta, u) &= P(\theta, u) + P'(\theta, u)\,(\theta_0 - \theta) - P(\theta, u)\\ &= P'(\theta, u)\,(\theta_0 - \theta),\end{aligned} \tag{12}$$

neglecting higher order terms in $(\theta_0 - \theta)$. Therefore (11) becomes

$$\varepsilon = P'(\theta, u)\,(\theta_0 - \theta) \tag{13}$$

**Step II:** (Stability proof) With $\phi(t)$ given by (6), (12) together with the P.A.A. given by (4) and (5) define an equivalent feedback system characterized by the following equations:

$$\varepsilon = y_1 = \left(-P'(\theta, u)\,\widetilde{\theta}(t)\right) = u_1 = -y_2 \tag{14}$$

$$\dot{\widetilde{\theta}}(t) = F(t)[P'(\theta, u)]^T\,\varepsilon = F(t)[P'(\theta, u)]^T\,u_2 \tag{15}$$

$$y_2 = P'(\theta, u)\,\widetilde{\theta}(t) \tag{16}$$

where

$$\widetilde{\theta}(t) = \theta(t) - \theta_0 \tag{17}$$

and $u_j$, $y_j$, $j = 1, 2$ define the inputs and outputs of the equivalent feedforward and feedback blocks, respectively. The feedforward block is characterized by a unit gain. Refer to Figure 2.

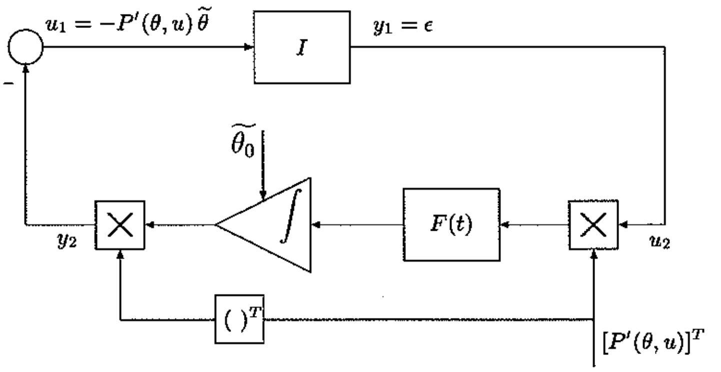


**Fig. 2.** Equivalent feedback representation of the identification scheme

In the general case with $F(t)$ time-varying, the feedback path is not provably passive and we have to use an extension of the passivity theorem, given in Appendix (Theorem 6) as well as the definitions of the systems belonging to the class $L(\Lambda)$ (excess of passivity) and $N(\Gamma)$ (lack of passivity) (see Appendix, Definitions 2 and 3).

Consider the equations (15) and (16) together with (17). Equations (15) and (16) correspond to a state space representation considered in Lemma 3 (appendix) with

$$\begin{aligned} &A = 0,\, B = F(t)\phi(t),\, C = \phi^T(t), \quad D = 0 \\ &x = \widetilde{\theta},\ u = \varepsilon = u_2, \quad y = y_2 = P'(\theta, u)\,\widetilde{\theta}(t). \end{aligned} \tag{18}$$

Taking $P(t) = F^{-1}(t)$ in Lemma 2 one gets using (5), (A.11)

$$[1 - \lambda_1(t)]\, F^{-1}(t) - \lambda_2(t)\phi(t)\phi^T(t) = Q(t) \tag{19}$$

also,

$$S(t) = 0 \quad \text{and} \quad R(t) = 0. \tag{20}$$

Notice that the positive semidefiniteness of (167) is not being claimed. Now using Lemma 3 one has

$$\int_{t_0}^{t} y_2^T u_2 \, d\tau = \int_{t_0}^{t} P'(\theta, u)\, \widetilde{\theta}\, \varepsilon \, d\tau$$

$$\geq \frac{1}{2}\widetilde{\theta}^T(t)F^{-1}(t)\,\widetilde{\theta}(t) - \frac{1}{2}\widetilde{\theta}^T(t_0)F^{-1}(t_0)\,\widetilde{\theta}(t_0)$$
$$- \frac{1}{2}\int_{t_0}^{t} \lambda_2(\tau)\,\|\widetilde{\theta}^T(\tau)\phi(\tau)\|^2\,d\tau$$
$$= \frac{1}{2}\widetilde{\theta}^T(t)F^{-1}(t)\,\widetilde{\theta}(t) - \frac{1}{2}\widetilde{\theta}^T(t_0)F^{-1}(t_0)\,\widetilde{\theta}(t_0)$$
$$- \frac{1}{2}\int_{t_0}^{t} \lambda_2(\tau)\,\|y_2(\tau)\|^2\,d\tau. \tag{21}$$

Therefore, it follows from Definition 3 that the equivalent feedback block belongs to the class $N(\Gamma)$ with $\Gamma = \lambda_2(t)$ (i.e. it falls short of being provably passive).

The feedforward block belongs to the class $L(\lambda(t))$ with $\lambda(t) > \lambda_2(t)$ and applying Theorem 6, (7) and (8) result.

### 1.3 Analysis of the Algorithms in the Presence of Noise

In the following analysis we will make the following assumptions:

- The noise signal $v$ may enter nonlinearly. However, we show later that one can only prove convergence w.p.1 if the noise is additive.
- The signal to noise ratio (SNR) is high.
- The noise is zero mean, finite power and independent of the external excitation $r$.
- The higher order terms in certain Taylor series expansions around the nominal trajectory are neglected (i.e. one assumes that they are small compared to the noise level).
- $\partial P_{0_v}(u,0) = \partial P_v(\theta_0, u, 0)$ is assumed to be a BIBO (asymptotically) stable operator. Here $\partial P_{0_v}(u,0)$ denotes the linearization of $P_0$ in response to a perturbation in $v$ around the trajectory $u$ and $v = 0$.

Denote by

$$y = P_0(u,0) \tag{22}$$

the values of the plant output obtained in the absence of noise (i.e. $v \equiv 0$).

Denote by

$$\overline{y} = P_0(u,v) \tag{23}$$

the values of the plant output obtained in the presence of noise.

Define

$$\overline{y} = y + w \tag{24}$$

where $w$ is the perturbation coming from the noise $v$.

Then

$$\overline{y} = P(u,0) + \partial P_{0_v}(u,0)\,v, \tag{25}$$

i.e. one has

$$w = \partial P_{0_v}(u, 0)\, v. \tag{26}$$

Therefore the effect of the noise can be considered to be additive in a small noise situation. We will also assume that $w(t)$ is zero mean (if it is not the case, the mean value can be removed). Note that for the case of an additive noise, one has $w = v$. If the noise is not additive, one can not guarantee $u$ and $w$ to be independent and therefore w.p.1 convergence of the parameters can not be assured in this case.

The analysis will be done in the context of a decreasing adaptation gain algorithm (i.e. $\lambda_1(t) \equiv 1$ and $\lambda_2(t) > 0 \quad \forall t$). Since the continuous time P.A.A. algorithm given in (4) and (5) is in this case a limiting case of a discrete time least squares type algorithm, one can use an averaging techniques for large $t$ and in particular the O.D.E. approach developed by Ljung [9,12].

We will prove the following theorem which establish the w.p.1. convergence conditions in the presence of noise for the algorithm presented in Section 1.2.

**Theorem 2.** *Consider the P.A.A*

$$\dot{\theta}(t) = F(t)\, \phi(t)\, \varepsilon(t) \tag{27}$$

$$\dot{F}^{-1}(t) = \lambda_2\, \phi(t)\, \phi^T(t); \quad \lambda_2 > 0;\ F(0) > 0 \tag{28}$$

*where $F(t)$ is the adaptation gain matrix, $\phi(t)$ is the observation vector and $\varepsilon(t)$ is the output error.*

- *Assume that the stationary processes $\phi(t,\theta)$ and $\varepsilon(t,\theta)$ can be defined for $\theta(t) \equiv \theta$ (i.e. $\theta$ is assumed to be constant).*
- *Assume that $\theta(t)$ generated by the algorithm (27), (28) belongs infinitely often to the domain $\mathcal{D}_S$ for which the stationary processes $\phi(t,\theta)$ and $\varepsilon(t,\theta)$ can be defined.*
- *Define the convergence domain $\mathcal{D}_C$ as*

$$\mathcal{D}_C = \{\theta :\ \phi^T(t,\theta)\,[\theta - \theta_0] = 0\}\cdot \tag{29}$$

- *Assume that $\varepsilon(t,\theta)$ can be expressed as*

$$\varepsilon(t,\theta) = H(\theta_0,\theta)\phi^T(t,\theta)[\theta_0 - \theta(t)] + w(t,\theta_0,\theta) \tag{30}$$

  *where $H(\theta_0,\theta)$ is a linear time varying causal operator[1] having the structure $H(\theta_0,\theta) = h(\theta_0,\theta)\, I$, where $h(\theta_0,\theta)$ is a scalar operator.*
- *Assume that $\phi(t,\theta)$ and $w(t,\theta_0,\theta)$ are independent.*

[1] The structural assumption on $H$ can be omitted at the cost of more complicated calculations.

*Then*

$$\text{Prob}\left\{\lim_{t\to\infty}\theta(t)\in\mathcal{D}_C\right\}=1 \tag{31}$$

*if*

$$H'(\theta_0,\theta)=H(\theta_0,\theta)-\frac{\lambda(t)}{2}I;\quad \lambda(t)>\lambda_2\quad \forall t>t_0 \tag{32}$$

*is a strong strictly passive operator*[2] *for all* $\theta$.

**Corollary 1.** *With the same hypotheses as for Theorem 2, if*

$$\phi^T(t,\theta)(\theta-\theta_0)=0 \tag{33}$$

*has a unique solution* $\theta=\theta_0$*, then the condition that* $H'(\theta_0,\theta)$ *be a strong strictly passive operator implies that*

$$\text{Prob}\left\{\lim_{t\to\infty}\theta(t)=\theta\right\}=1. \tag{34}$$

For the algorithms presented in Section 1.2, the equations of the output error for a fixed value $\theta$ takes the form:

$$\varepsilon(t,\theta)=\phi^T(t)(\theta_0-\theta)+w(t,\theta_0) \tag{35}$$

i.e. $H(\theta_0,\theta)=1$ and therefore condition (32) is automatically satisfied.

As can be observed, the stationary process $\phi(t,\theta)$ for all the algorithms will depend only on the external excitation $u$ (it will not be affected by the noise $v$). On the other hand, if $v(t)$ is an additive noise, $w(t)=v(t)$) and it is independent of $u$. As a consequence $w(t)$ and $u(t)$ are independent. Therefore the w.p.1. convergence of the parameters in the stochastic case is assured.

**Proof of Theorem 2**

Define

$$R(t)=\frac{1}{t}F^{-1}(t). \tag{36}$$

Using (5) one gets

$$\dot{R}(t)=\frac{1}{t}\left[\lambda_2\,\phi(t)\,\phi^T(t)-R(t)\right]. \tag{37}$$

The ordinary differential equation associated with the algorithm (27), (28) takes the form [9,12]

$$\dot{\theta}(\tau)=R^{-1}(\tau)\,f(\theta(\tau)) \tag{38}$$

$$\dot{R}(\tau)=\lambda_2\,G\left(\theta(\tau)\right)-R(\tau) \tag{39}$$

[2] The definition of a strictly strongly passive operator is given in appendix.

where

$$f(\theta) = E\{\phi(t,\theta)\varepsilon(t,\theta)\} \tag{40}$$

and

$$G(\theta) = E\{\phi(t,\theta)\phi^T(t,\theta)\}\cdot \tag{41}$$

Using (30) one gets for $\theta = \text{const}$,

$$f(\theta) = E\{\phi(t,\theta)\,H(\theta_0,\theta)\,\phi^T(t,\theta)\}(\theta_0 - \theta) + E\{\phi(t,\theta)\,w(t,\theta_0,\theta)\}\cdot \tag{42}$$

But $\phi(t,\theta)$ for $\theta = \text{const}$ and $w(t)$ are assumed to be independent. Therefore

$$E\{\phi(t,\theta)\,w(t,\theta_0,\theta)\} = 0 \tag{43}$$

The ODE defined by (40) and (41) becomes:

$$\begin{aligned}\dot{\widetilde{\theta}}(\tau) &= -R^{-1}(\tau)\,E\{\phi(t,\theta)\,H(\theta_0,\theta)\,\phi^T(t,\theta)\}\,\widetilde{\theta}(\tau)\\ &= -R^{-1}(\tau)\,\widetilde{G}(\theta)\,\widetilde{\theta}(\tau) \qquad (44)\\ \dot{R}(\tau) &= \lambda_2\,E\{\phi(t,\theta)\,\phi^T(t,\theta)\} - R(\tau) \qquad (45)\end{aligned}$$

where

$$\widetilde{\theta} = \theta - \theta_0. \tag{46}$$

The stationary (equilibrium) points of the ODE which correspond to the possible convergence points of the algorithm are given by

$$\mathcal{D}_C = \left\{\theta :\ \widetilde{G}(\theta)(\theta - \theta_0^*) = 0\right\} \tag{47}$$

or equivalently by

$$\mathcal{D}_C = \left\{\theta :\ \phi^T(t,\theta)(\theta - \theta_0) = 0\right\}\cdot \tag{48}$$

If there exist $\theta$ and $\phi(t,\theta)$ such that the condition (48) has a unique solution then one has a single equilibrium point $\theta = \theta_0$, which is the only possible convergence point of the algorithm.

The next step is to establish the stability properties of the equilibrium points of the ODE which will give the w.p.1. convergence property for the algorithm.

Define the candidate Lyapunov function

$$V(\widetilde{\theta}, R) = \widetilde{\theta}^T(\tau)\,R(\tau)\,\widetilde{\theta}(\tau). \tag{49}$$

Since $G(\theta) \neq 0$ (because of the implicit assumption that $r(t)$ is not identically null) one has $R(\tau) > 0$ and therefore, $V(\widetilde{\theta}, R)$ is a positive definite radially unbounded function outside $\mathcal{D}_C$ for all $t > t_0$.

Along the trajectories of (4.24) one gets

$$\frac{d}{d\tau} V(\widetilde{\theta}, R) = -\widetilde{\theta}^T\left[\widetilde{G}(\theta) + \widetilde{G}^T(\theta) - \lambda_2\,G(\theta)\right]\widetilde{\theta} - \widetilde{\theta}^T R(\tau)\,\widetilde{\theta} \tag{50}$$

and for concluding on the stability of the equilibrium points it is sufficient to show that

$$\widetilde{\theta}^T \left[\widetilde{G}(\theta) + \widetilde{G}^T(\theta) - \lambda_2 \, G(\theta)\right] \widetilde{\theta} > 0 \quad \forall \widetilde{\theta} \neq 0 \tag{51}$$

or equivalently that

$$\overline{G}(\theta) = E\left\{\phi(t,\theta)\left[H + H^T - \lambda_2 \, I\right] \phi^T(t,\theta)\right\} \tag{52}$$

$$= 2\,E\left\{\phi(t,\theta)\left[H - \frac{\lambda_2}{2} I\right] \phi^T(t,\theta)\right\} \tag{53}$$

is a positive definite matrix function (one takes into account that $H$ is a diagonal matrix $H = hI$ where $h$ is a scalar operator).

To prove this it is enough to show that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t l^T \phi(\tau,\theta)\left[H - \frac{\lambda_2}{2} I\right] \phi^T(\tau,\theta)\, l \, d\tau > 0 \tag{54}$$

for any constant vector $l \in \mathbf{R}^d$ where $d$ is the dimension of the parameter vector $\theta$.

But the integral (54) can be viewed as the input output product for a scalar operator $\left[h - \frac{\lambda_2}{2}\right]$ with input $u = \phi^T(t,\theta)\, l$ and output $y = \left[h - \frac{\lambda_2}{2}\right]$ $\phi^T(t,\theta)\, l$.

Using condition (32) of Theorem 2, (54) can be rewritten as

$$\lim_{t\to\infty} \frac{1}{t}\left[\int_0^t l^T \phi(\tau,\theta)\left[h - \frac{\lambda(t)}{2}\right] \phi^T(\tau,\theta)\, l \, d\tau + \right.$$
$$\left. \int_0^t l^T \phi(\tau,\theta)\left[\frac{\lambda(t) - \lambda_2}{2}\right] \phi^T(\tau,\theta)\, l d\tau\right] > 0 \tag{55}$$

since $\lambda(t) - \lambda_2 > 0 \; \forall t$, $h - \frac{\lambda(t)}{2}$ is a strictly passive operator (since $H = hI$), and one takes into account the fact that the effect of the initial conditions vanishes as $t \to \infty$ because of division by $t$. ∎

## 1.4 Robustness Analysis

It is important to analyze the robustness of the identification schemes when the plant is not in the model set, when the output is affected by a disturbance that is allowed to enter the system nonlinearly and when the higher terms in the Taylor series expansion around the nominal trajectory cannot be neglected.

The objective of the analysis is to show that norm boundedness and mean square boundedness of all signals is assured for a certain type of characterization of the mismatch between the model and the plant and of the terms of higher order in the Taylor series expansion.

The plant will be described by

$$y = P_0(u, v) + \Delta P(u, v) \tag{56}$$

where $P_0(u, v)$ is the "reduced" order plant, $v(t)$ is a zero mean bounded disturbance, and $\Delta P(u, v)$ is a BIBO operator that is due to the unmodeled part of the system. Note that the BIBO assumption might be unnecessarily restrictive.

The estimated model is assumed to be represented by:

$$y(\theta) = P(\theta, u) \tag{57}$$

with the property that $P_0(u, 0) = P(\theta_0, u)$.

To start with, we show that the effect of the noise and the unmodeled dynamics upon the system can be considered to be additive. Denote by

$$y = P(\theta_0, u) = P_0(u, 0) \tag{58}$$

the values of the output obtained for the reduced order plant in the absence of noise.

Denote by

$$\bar{y} = P_0(u, v) + \Delta P(u, v) \tag{59}$$

the values of the plant output in the presence of noise and with the unmodeled dynamics.

Define

$$\bar{y} = y + y_p \tag{60}$$

where $y_p$ is the perturbation coming from the noise $v$ and the unmodeled plant dynamics.

Then

$$\bar{y} = P(\theta_0, u) + \partial P_{0_v}(u, 0)\, v + \Delta P(u, 0) + \partial \Delta P_v(u, 0)\, v. \tag{61}$$

Here, $\partial P_{0_v}(u, 0)$ denotes the linearization of $P_0$ in response to a perturbation in $v$ around the trajectory $u$ and $v = 0$. Note that terms of order higher than one in the Taylor series expansion have been neglected; these are taken care of subsequently. Also, $\partial \Delta P_v(u, 0)$ denotes the linearization of $\Delta P$ in response to a perturbation in $v$ around the trajectory $u$ and $v = 0$. Therefore

$$y_p = [\partial P_{0_v}(u, 0) + \partial \Delta P_v(u, 0)]\, v + \Delta P(u, 0) \tag{62}$$

It is assumed that $\partial P_{0_v}(u, 0)$, $\Delta P(u, 0)$ and $\partial \Delta P_v(u, 0)$ are BIBO operators leading therefore to a bounded $y_p$. Therefore the equation of the output error will take the form

$$\varepsilon(t) = \phi(t, \theta)^T\, [\theta_0 - \theta(t)] + w(t) \tag{63}$$

with

$$w(t) = y_p(t) + \mathcal{O}(\theta_0 - \theta)$$

where $y_p$ reflects the perturbation due to the unmodeled part of the plant and the possible bounded output disturbances, and $\mathcal{O}(\theta_0 - \theta)$ reflects the effect of the high order terms in all Taylor series expansions.

One has the following result:

**Theorem 3.** • *Assume that the external excitation $u(t)$, the noise $v(t)$ are norm bounded*

$$\lim_{t\to\infty} \int_{\tau=0}^{t} u^2(\tau)\, d\tau \leq \alpha^2; \quad \alpha^2 < \infty, \tag{64}$$

$$\lim_{t\to\infty} \int_{\tau=0}^{t} v^2(\tau)\, d\tau \leq \beta^2; \quad \beta^2 < \infty. \tag{65}$$

- *Assume that $\mathcal{O}(\theta_0 - \theta)$ is norm bounded.*
- *Assume that the true system is stable.*
- *Assume that the PAA of (4) and (5) with $\lambda_1(t) \equiv 1$ is used.*

*Then, the output error $\varepsilon(t)$ and the predicted output $y(\theta, t)$ are norm bounded.*

**Corollary 2.** *If the external excitation $u(t)$ and the noise $v(t)$ are mean square bounded*

$$\lim_{t\to\infty} \int_{\tau=0}^{t} u^2(\tau)\, d\tau \leq \alpha^2\, t + k_u; \quad \alpha^2 < \infty; \quad 0 < k_u < \infty, \tag{66}$$

$$\lim_{t\to\infty} \int_{\tau=0}^{t} v^2(\tau)\, d\tau \leq \beta^2\, t + k_v; \quad \beta^2 < \infty; \quad 0 < k_v < \infty \tag{67}$$

*and $\mathcal{O}(\theta_0 - \theta)$ is mean square bounded, then $\varepsilon(t)$ and $y(\theta, t)$ are mean square bounded.*

**Proof:** Under the hypotheses of Theorem 3, $y_p$ will be norm bounded and therefore $w(t)$ will be bounded. The straightforward application of Theorem 5 in Part II (taking in account (63)) leads to the desired result.

### 1.5 Example

We refer to Section 2.4 for an example comparing NonLinear Open-Loop Output Error (NLOLOE) and NonLinear Closed-Loop Output Error (NLCLOE) identification.

# Part II. Identification in closed-loop operation

## 2.1 Closed-loop output error identification. The basic equations and problem setting

The objective is to estimate the parameters of a single input single output (SISO) nonlinear time invariant system described as in (1) by

$$\mathcal{S}: \quad y = P_0(u, v) \tag{68}$$

We will assume that the plant $P_0$ is a BIBO operator. We refer to [4] for a theory based on kernel representations which allows the recursive closed-loop identification of unstable nonlinear plants.

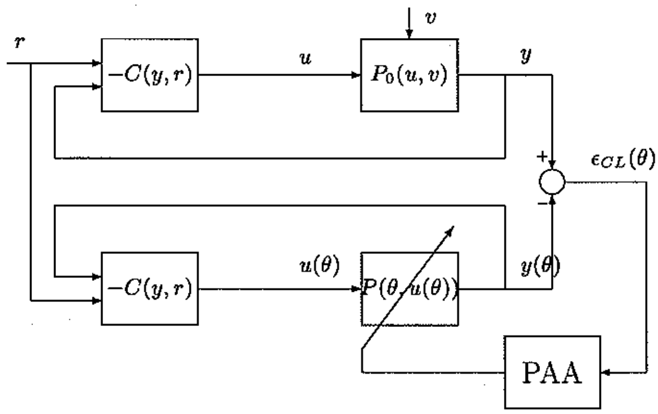


**Fig. 3.** Closed-loop output error identification scheme

The plant is operated in closed-loop with a known nonlinear controller, i.e.

$$\mathcal{C}: \quad u = -C(y, r) \tag{69}$$

where $r$ is an external reference which is assumed to be quasi-stationary and uncorrelated with $v$. The controller $C$ is a causal BIBO nonlinear operator of both $r$ and $y$.

The closed-loop operator from the measured reference signal $r$ to the measured output signal $y$, as defined in Figure 3 is denoted by

$$y = T_0(r, v). \tag{70}$$

It is required that the closed-loop system is Bounded Input Bounded Output (BIBO) stable. In the sequel we often make use of linearizations of some nonlinear operators around their operating trajectories. We therefore require that the plant, the model (to be defined subsequently), the controller and all closed-loop operators are smooth functions of the reference signal, the input signal, the output signal and the disturbance signal. This means that if the closed-loop operator is linearized around any (stable) trajectory, the resulting linear (time-varying) system is BIBO stable. See [5] for more details.

We consider the following adjustable model for the closed-loop system defined by (68) and (69) (See also Figure 3)

$$y(\theta) = P(\theta, u(\theta)) \tag{71}$$

$$u(\theta) = -C(y(\theta), r) \tag{72}$$

where $P(\theta, u)$ defines the adjustable plant model, $y(\theta)$ is the output of the closed-loop predictor and $u(\theta)$ is the plant model input.

The closed-loop output error is defined as

$$\varepsilon_{CL} = y - y(\theta). \tag{73}$$

The following assumptions will be made until further notice:

(i) $\exists\, \theta_0$ such that $P(\theta_0, u) = P_0(u, 0)$ for all $u \in \mathcal{L}_{2e}$ and $v \equiv 0$ (subsequently in the case $v \equiv 0$ the argument $v$ will be deleted)

(ii) **Notation:**
The operator $\partial P_u(\theta, u)$ is the linearization of $P(\theta, u)$ in response to a perturbation in $u$ along the input trajectory $u$. The operator $\partial C_y(r, y)$ is the linearization of $C(y, r)$ in response to a perturbation in $y$ along the trajectories produced $r$ and $y$.

It is assumed that $\partial P_u(\theta, u)$ and $\partial C_y(r, y)$ exist for all allowable $u$, $y$ and $r$. They are linear time-varying operators along the trajectories of the closed-loop system.

(iii) **Notation:**
The partial derivative of $P(\theta, u)$ with respect to $\theta_j$ is denoted by $P'_{\theta_j}(\theta, u)$ for $j = 1, \cdots, d$ where $d$ is the dimension of the parameter vector $\theta$.

The operator $P'_{\theta_j}(\theta, u)$ and its time derivatives exist and are norm-bounded $\forall j$ along the trajectories of the closed-loop predictor which requires $\dot{r}$ to be bounded. This assumption is not particularly restrictive as $P$ and $P(\theta)$ are assumed to be smooth operators.

(iv) Let us define the operator

$$P_{CL}(\theta) = [I + \partial P_u(\theta, u(\theta)) \partial C_y(r, y(\theta))]\,. \tag{74}$$

It is assumed that $P_{CL} = P_{CL}(\theta_0)$ and its inverse $P_{CL}^{-1}$ exist along every trajectory of the closed-loop system encountered during the identification process. Both operators are linear time-varying operators and $P_{CL}^{-1}$ is BIBO by assumption (smoothness of the closed-loop).

(v) The reference $r$ and the stochastic disturbance $v$ are independent.

Assumption (i) means that at least for $\theta = \theta_0$ and in the absence of noise, the plant is in the model set. (The case when this is not true will be discussed separately in Section 2.3).

The generic parameter adaptation algorithm (PAA) which will be used for identification in closed-loop is of the same form as the one given in (4) and (5) except that the open-loop output error $\varepsilon(t)$ will be replaced by the closed-loop output error $\varepsilon_{CL}(t)$:

$$\dot{\theta}(t) = F(t)\phi(t)\varepsilon_{CL}(t) \tag{75}$$

where $\theta(t)$ is the estimated parameter vector, $\varepsilon_{CL}(t)$ is the closed-loop output error, $\phi(t)$ is the observation vector, $F(t)$ is the adaptation gain matrix.

We will consider subsequently that the assumptions (i) through (iv) are valid and furthermore, for some analysis, that:

(vi) $v \equiv 0$

(vii) The higher order terms in the Taylor series involving expansions in powers of $(u-u(\theta))$, $(y-y(\theta))$ and $(\theta_0-\theta)$ along the trajectories of the system can be neglected

## 2.2 Nonlinear closed-loop output error algorithms

In this section, we present the derivations of the algorithm and we provide a stability analysis in a deterministic environment assuming that the system can be modeled exactly and that one can neglect terms of power higher than one in certain Taylor series expansions. The results in this section heavily rely on concepts of strong strict passivity outlined in the appendix.

One has the following result (the NLCLOE algorithm):

**Theorem 4.** *Under the assumptions (i) through (iv), (vi) and (vii) one has for*

$$\phi(t) = [P'(\theta, u(\theta))]^T = [P'_{\theta_1}(\theta, u(\theta)) \quad \cdots \quad P'_{\theta_d}(\theta, u(\theta))]^T \tag{76}$$

*that*

$$\lim_{t\to\infty} \varepsilon_{CL}(t) = 0 \tag{77}$$

*if the linear time-varying operator*

$$H = P_{CL}^{-1} - \frac{\lambda(t)}{2} I; \quad \lambda(t) > \lambda_2(t), \quad \forall t \tag{78}$$

*is strongly strictly passive*[3].

[3] It is assumed here that $H$ has the form (155)-(156). See Definition 1 in the appendix for a definition of strong strict passivity.

*If furthermore $P_{CL}^{-1}$ has a finite-dimensional description as in (162)-(163) one has also*

$$\lim_{t\to\infty} \phi^T(t)\,(\theta(t)-\theta_0)=0. \qquad (79)$$

**Remark II.1:**

1. The condition (78) assures that the closed-loop output error goes asymptotically to zero, and that the estimated parameter vector $\theta$, converges to a set defined as

   $$\mathcal{D}_c=\left\{\theta:\phi^T(t)\,(\theta-\theta_0)=0\right\}\cdot \qquad (80)$$

   If

   $$\phi^T(t)\,(\theta-\theta_0)=0 \qquad (81)$$

   has a unique solution $\theta=\theta_0$, the parameter vector will converge toward this value. In fact this condition is a "persistence of excitation" condition for the nonlinear case.
2. The passivity condition of Theorem 4 can be relaxed by making other choices for $\phi(t)$, as will be indicated later. Note that passivity conditions occur also in the linear case.

**Proof of Theorem 4:** The proof will be done in several steps.

**Step I:** Establishing the expression $\varepsilon_{CL}=\mathbf{f}(\theta_0-\theta(t))$
One has the following lemma:

**Lemma 1.** *Under the assumptions (i) through (iv), (vi) and (vii) the closed-loop output error is given by*

$$\varepsilon_{CL}=P_{CL}^{-1}P'(\theta,u(\theta))\,[\theta_0-\theta(t)]\,. \qquad (82)$$

**Proof:** From (68) with $v\equiv 0$ one gets

$$y=P(\theta_0,u)=P(\theta_0,u(\theta))+[P(\theta_0,u)-P(\theta_0,u(\theta))] \qquad (83)$$

and using a series expansion around $u$ while neglecting higher order terms in $(u-u(\theta))$ one gets

$$P(\theta_0,u)-P(\theta_0,u(\theta))=-\partial P_u(\theta_0,u).[C(y,r)-C(y(\theta),r)]. \qquad (84)$$

On the other hand $[C(y,r)-C(y(\theta),r)]$ can be expressed as

$$[C(y,r)-C(y(\theta),r)]=\partial C_y(r,y)\,(y-y(\theta)) \qquad (85)$$

(neglecting higher order terms in $(y-y(\theta))$) and therefore

$$P(\theta_0,u)-P(\theta_0,u(\theta))=-\partial P_u(\theta_0,u)\partial C_y(r,y)\,(y-y(\theta)). \qquad (86)$$

Using the definition of $\varepsilon_{CL}$ given in (73), (86) can be re-written as

$$y = P(\theta_0, u(\theta)) - \partial P_u(\theta_0, u)\partial C_y(r, y)\, \varepsilon_{CL} \tag{87}$$

Subtract now (71) from (87) and use (73). One gets

$$\varepsilon_{CL} = P(\theta_0, u(\theta)) - P(\theta, u(\theta)) - \partial P_u(\theta_0, u)\partial C_y(r, y)\, \varepsilon_{CL} \tag{88}$$

Using a series expansion around $\theta$, one has

$$\begin{aligned} P(\theta_0, u(\theta)) - P(\theta, u(\theta)) &= P(\theta, u(\theta)) + P'(\theta, u(\theta))\,(\theta_0 - \theta) - P(\theta, u(\theta)) \\ &= P'(\theta, u(\theta))\,(\theta_0 - \theta), \end{aligned} \tag{89}$$

neglecting higher order terms in $(\theta_0 - \theta)$. Here $P'(\theta, u(\theta))$ has to be read as $P'(\theta, u)|_{u=u(\theta)}$. Therefore (88) becomes

$$\varepsilon_{CL} = P'(\theta, u(\theta))\,(\theta_0 - \theta) - \partial P_u(\theta_0, u)\partial C_y(r, y)\, \varepsilon_{CL} \tag{90}$$

from which one obtains

$$[I + \partial P_u(\theta_0, u)\partial C_y(r, y)]\varepsilon_{CL} = P'(\theta, u(\theta))\,(\theta_0 - \theta) \tag{91}$$

from which (82) results using the definition of $P_{CL}$ given in (74).

**Step II:** (Stability proof) With $\phi(t)$ given by (76), (82) together with the P.A.A. given by (75) and (5) define an equivalent feedback system characterized by the following equations:

$$\varepsilon_{CL} = y_1 = P_{CL}^{-1}\left(-P'(\theta, u(\theta))\,\widetilde{\theta}(t)\right) = P_{CL}^{-1} u_1 = -P_{CL}^{-1} y_2 \tag{92}$$

$$\dot{\widetilde{\theta}}(t) = F(t)[P'(\theta, u(\theta))]^T \varepsilon_{CL} = F(t)[P'(\theta, u(\theta))]^T u_2 \tag{93}$$

$$y_2 = P'(\theta, u(\theta))\,\widetilde{\theta}(t) \tag{94}$$

where

$$\widetilde{\theta}(t) = \theta(t) - \theta_0 \tag{95}$$

and $u_j$, $y_j$, $j = 1, 2$ define the inputs and outputs of the equivalent feedforward and feedback blocks, respectively. Refer to Figure 2 for a similar equivalent feedback system ($\varepsilon$ is replaced by $\varepsilon_{CL}$ and $u$ by $u(\theta)$).

Consider the equations (93) and (94) together with (95). Equations (93) and (94) correspond to a state space representation considered in Lemma 3 with

$$\begin{aligned} &A = 0, B = F(t)\phi(t),\ C = \phi^T(t),\ \ D = 0 \\ &x = \widetilde{\theta},\ u = \varepsilon_{CL} = u_2, y = y_2 = P'(\theta, u(\theta))\,\widetilde{\theta}(t). \end{aligned} \tag{96}$$

The system (96) has the same structure as the system (18). Therefore it will also satisfy an inequality of the type (21). It follows then from Definition 3 that the equivalent feedback block belongs to the class $N(\Gamma)$ with $\Gamma = \lambda_2(t)$ (i.e. it falls short of being provably passive).

By hypothesis, $P_{CL}^{-1}$ belongs to class $\Lambda(\lambda(t))$ with $\lambda(t) > \lambda_2(t)$. It now follows by a straightforward application of Theorem 6 that $u_1 \in \mathcal{L}_2$, $x_1 \in \mathcal{L}_\infty$, $\tilde{\theta} \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} x_1(t) = 0$. By hypothesis (see Assumption (iii)), $\phi(t)$ (given by (76)) and all its time derivative are bounded; this implies that $u_1 = -y_2 = -\phi(t)^T\tilde{\theta} \in \mathcal{L}_\infty$. The boundedness of $u_2 = y_1$ follows from the boundedness of $x_1$ and $u_1$ and Equation (163). It is now straightforward to see that $\dot{u}_1 \in \mathcal{L}_\infty$. Indeed,

$$\dot{u}_1 = -[\phi(t)^T\dot{\tilde{\theta}} + \dot{\phi}(t)^T\tilde{\theta}]$$

and both term on the right hand side of the equality sign are individually in $\mathcal{L}_\infty$. By Barbalat's lemma (see [14], Corollary 2.9, pg 86), $u_1 \in \mathcal{L}_2$, $u_1 \in \mathcal{L}_\infty$ and $\dot{u}_1 \in \mathcal{L}_\infty$ imply that $\lim_{t\to\infty} u_1(t) = 0$. ∎

### Relaxation of the strong strictly passive condition

*Algorithm AFNLCLOE*

Neglecting the swapping correction terms which anyway become negligible when one uses decreasing adaptation gains ($\lambda_2(t) > 0$, $\lim_{t\to\infty} \lambda_1(t) = 1$), (82) can be also written as

$$\varepsilon_{CL} = P_{CL}^{-1} P_{CL}(\theta)(P_{CL}^{-1}(\theta)\, P'(\theta, u(\theta))\, [\theta_0 - \theta(t)]) \tag{97}$$

where the time-varying operator $P_{CL}(\theta)$ is defined in (75). In this case, following the same procedure as for the NLCLOE algorithm one has to choose

$$\phi(t) = P_{CL}^{-1}(\theta)\, P'(\theta, u(\theta)). \tag{98}$$

In this case one filters $P'(\theta, u(\theta))$ through a linear time-varying closed-loop system which depends upon the current estimate $\theta$. $\phi(t)$ can also be viewed as an approximation of the gradient of a quadratic criterion in terms of $\varepsilon_{CL}$ around $\theta_0\,(t_0)$.

The corresponding strongly strictly passive condition will become

$$H = P_{CL}^{-1}\, P_{CL}(\theta) - \frac{\lambda(t)}{2} I; \quad \lambda(t) > \lambda_2(t),\ \forall\, t > t_0 \tag{99}$$

should be strongly strictly passive.

Clearly in the vicinity of $\theta_0$, this condition is much more likely to be satisfied, than condition (78) for NLCLOE.

This of course requires that at each instant $P_{CL}^{-1}(\theta)$ derived by (74) is stable. If this is not the case, then as in the identification of linear models (e.g. recursive maximum likelihood, adaptative filtered closed-loop output error) one uses the last stable estimated filter $P_{CL}^{-1}(\theta)$.

### 2.3 Robustness Analysis

The robustness analysis will be done along the same lines as for the open loop case.

The plant will be described by

$$y = P_0(u, v) + \Delta P(u, v) \tag{100}$$

where $P_0(u, v)$ is the "reduced" order plant, $v(t)$ is a zero mean bounded disturbance, and $\Delta P(u, v)$ is a BIBO operator that is due to the unmodeled part of the system. Note that the BIBO assumption might be unnecessarily restrictive.

The estimated model is assumed to be represented by:

$$y(\theta) = P(\theta, u) \tag{101}$$

with the property that $P_0(u, 0) = P(\theta_0, u)$.

The true input $u$ and the estimated input $u(\theta)$ are generated by (69) and (72) respectively.

To start with, we show that the effect of the noise and the unmodeled dynamics upon the closed-loop system can be considered to be additive. Denote by

$$y = P(\theta_0, u) = P_0(u, 0) \tag{102}$$

$$u = -C(y, r) \tag{103}$$

the values of the input and output obtained for the reduced order plant in the absence of noise.

Denote by

$$\bar{y} = P_0(\bar{u}, v) + \Delta P(\bar{u}, v) \tag{104}$$

$$\bar{u} = -C(\bar{y}, r) \tag{105}$$

the values of the plant input and output, i.e. in the presence of noise and with the unmodeled dynamics.

Define

$$\bar{y} = y + y_p \tag{106}$$

$$\bar{u} = u + u_p \tag{107}$$

where $y_p$ and $u_p$ are the perturbations coming from the noise $v$ and the unmodeled plant dynamics.

Then

$$\bar{y} = P(\theta_0, u) + \partial P_u(\theta_0, u)\, u_p + \partial P_{0_v}(u, 0)\, v \tag{108}$$

$$+\Delta P(u, 0) + \partial \Delta P_u(u, 0)\, u_p + \partial \Delta P_v(u, 0)\, v \tag{109}$$

and

$$\bar{u} = -C(y + y_p, r) = -C(y, r) - \partial C_y(r, y)\, y_p. \tag{110}$$

Here, $\partial P_{0_v}(u,0)$ denotes the linearization of $P_0$ in response to a perturbation in $v$ around the trajectory $u$ and $v=0$. Note that terms of order higher than one in the Taylor series expansion have been neglected; these are taken care of subsequently. Also, $\partial \Delta P_u(u,0)$ and $\partial \Delta P_v(u,0)$ denote the linearization of $\Delta P$, respectively, in response to a perturbation in $u$ and $v$ around the trajectory $u$ and $v=0$. Therefore

$$y_p = [\partial P_u(\theta_0,u) + \partial \Delta P_u(u,0)]\, u_p + [\partial P_{0_v}(u,0) + \partial \Delta P_v(u,0)]\, v + \Delta P(u,0) \tag{111}$$

$$u_p = -\partial C_y(r,y)\, y_p \tag{112}$$

and combining (111) and (112) one gets

$$y_p = \widetilde{P}_{CL}^{-1}\, [(\partial P_{0_v}(u,0) + \partial \Delta P_v(u,0))\, v + \Delta P(u,0)]. \tag{113}$$

where $\widetilde{P}_{CL}^{-1} = [I + (\partial P_u(\theta_0,u,0) + \partial \Delta P_u(u,0))\, \partial C_y(r,y)]^{-1}$ is assumed to be a BIBO (asymptotically) stable I/O operator leading to a bounded $y_p$ .

On the other hand the neglected terms in the developments leading to (82) for the closed-loop output error and (113) for the perturbation term have also to be taken into account. Therefore the equation of the closed-loop output error will take the form

$$\varepsilon_{CL} = P_{CL}^{-1}\, \phi(t,\theta)^T\, [\theta_0 - \theta(t)] + w(t) \tag{114}$$

with

$$w(t) = y_p(t) + \mathcal{O}(\theta_0 - \theta)$$

where $y_p$ reflects the perturbation due to the unmodeled part of the plant and the possible bounded output disturbances, and $\mathcal{O}(\theta_0 - \theta)$ reflects the effect of the high order terms in all Taylor series expansions.

One has the following result

**Theorem 5.** *Assume that the closed-loop output error is described by:*

$$\varepsilon_{CL} = H\, \phi^T(t)\, (\theta_0 - \theta(t)) + w(t) \tag{115}$$

*where $w(t)$ represents the combined effect of unmodeled dynamics, bounded disturbances and of the high order term in the Taylor expansions around the nominal trajectories. Here, $H$ and $\phi$ depend on the algorithm used.*

- *Assume that $H$ is a linear time-varying operator.*
- *Assume that the true closed-loop system is stable.*
- *Assume that $C(y,r)$, $\partial C_y(r,y)$, $\Delta P$ and $\widetilde{P}_{CL}^{-1}$ are BIBO operators.*
- *Assume that the P.A.A. of (75), (5) with $\lambda_1(t) \equiv 1$ is used.*
- *Assume that the external excitation $r(t)$ and the equivalent disturbance $w(t)$ are norm bounded, i.e.*

$$\lim_{t\to\infty} \int_{\tau=0}^{t} r^2(\tau)\, d\tau \le \alpha^2; \quad \alpha^2 < \infty, \tag{116}$$

$$\lim_{t\to\infty} \int_{\tau=0}^{t} w^2(\tau)\, d\tau \le \beta^2; \quad \beta^2 < \infty. \tag{117}$$

- *Assume that $\mathcal{O}(\theta_0 - \theta)$ is norm bounded*

*Then the closed-loop output error $\varepsilon_{CL}(t)$, the predicted output $y(\theta, t)$ and the predicted input $u(\theta, t)$ are norm bounded if*

$$\bar{H} = H - \frac{\lambda(t)}{2} I; \quad \lambda(t) > \lambda_2(t) \; \forall \, t \tag{118}$$

*is a strongly strictly passive linear time-varying operator.*

**Corollary 3.** *Under the same condition (118) of Theorem 5, if the equivalent disturbance $w(t)$ and the external excitation $r(t)$ are mean square bounded, i.e.*

$$\lim_{t\to\infty} \int_{\tau=0}^{t} r^2(\tau)\, d\tau \le \alpha^2 \, t + k_r; \quad \alpha^2 < \infty; \quad 0 < k_r < \infty, \tag{119}$$

$$\lim_{t\to\infty} \int_{\tau=0}^{t} w^2(\tau)\, d\tau \le \beta^2 \, t + k_w; \quad \beta^2 < \infty; \quad 0 < k_w < \infty \tag{120}$$

*then $\varepsilon_{CL}$, $y(\theta)$, $u(\theta)$ and $\phi(t)$ are mean square bounded.*

In fact this theorem says that even when one uses simplified nonlinear models, provided that the error between the true plant and a nominal reduced model is small in some sense, the boundedness of the signals is assured by the passivity conditions of Theorem 4, now evaluated for the nominal reduced model. The result is akin to those in Lyapunov stability theory in the presence of nonvanishing disturbances.

**Proof of Theorem 5:**

Defining

$$\widetilde{\theta}(t) = \theta(t) - \theta_0, \tag{121}$$

(115) can be written as

$$\varepsilon_{CL} = -\, H \, \phi^T(t) \, \widetilde{\theta}(t) + w = H \tilde{u} + w \tag{122}$$

where

$$\tilde{u} = -\phi^T(t) \, \widetilde{\theta}(t). \tag{123}$$

Defining

$$\mu(t) = \varepsilon_{CL}(t) + \frac{\lambda_2(t)}{2} \, \phi^T(t) \, \widetilde{\theta}(t) \tag{124}$$

one obtains from (122)

$$\begin{aligned} \mu(t) &= -\left\{ \left[ H - \frac{\lambda(t)}{2} \right] + \frac{1}{2} \left[ \lambda(t) - \lambda_2(t) \right] \right\} \phi^T(t) \, \widetilde{\theta}(t) + w \\ &= -\tilde{H} \phi^T(t) \, \widetilde{\theta}(t) + w = \tilde{H} \, \tilde{u} + w. \end{aligned} \tag{125}$$

The operator $\tilde{H}$ is not only strongly strictly passive but in addition it is input strictly passive since $\lambda(t) - \lambda_2(t) > 0 \quad \forall t$.

¿From the properties of input strictly passive systems one has using (125)

$$\int_{t_0}^{t} \tilde{u}(\tau)\,\mu(\tau)\,d\tau \geq -\gamma_0^2 + \delta \int_{t_0}^{t} \tilde{u}^2(\tau)\,d\tau + \int_{t_0}^{t} w(\tau)\,\tilde{u}(\tau)\,d\tau; \quad \text{for some } \delta > 0, \forall\, t \geq t_0. \tag{126}$$

On the other hand taking into account the input-output properties of the adaptation algorithm (75) and (5), one has from (21) (in closed-loop framework) with $\phi^T(t) = P'(\theta, u(\theta))$ and taking into account (123) and (124)

$$-\int_{t_0}^{t} \tilde{u}(\tau)\,\mu(\tau)\,d\tau = \int_{t_0}^{t} P'(\theta, u(\theta))\,\tilde{\theta}(\tau)\,\epsilon_{CL}(\tau)\,d\tau + \frac{1}{2}\int_{t_0}^{t} \lambda_2(\tau)\,\|\tilde{\theta}^T(\tau)\phi(\tau)\|^2\,d\tau \tag{127}$$

$$\geq -\frac{1}{2}\,\tilde{\theta}^T(t_0)\,F^{-1}(t_0)\,\tilde{\theta}(t_0) \tag{128}$$

and (126) becomes

$$\delta \int_{t_0}^{t} \tilde{u}^2(\tau)\,d\tau \leq \gamma_0^2 - \int_{t_0}^{t} w(\tau)\,\tilde{u}(\tau)\,d\tau + \frac{1}{2}\,\tilde{\theta}^T(t_0)\,F^{-1}(t_0)\,\tilde{\theta}(t_0). \tag{129}$$

Expanding the inequality $-[\rho\,\tilde{u}(t) + w(t)]^2 \leq 0, \quad \rho > 0$, and integrating from $t_0$ to $t$ one gets

$$-\frac{\rho}{2} \int_{t_0}^{t} \tilde{u}^2(\tau)\,d\tau \leq \frac{1}{2\rho} \int_{t_0}^{t} w^2(\tau)\,d\tau + \int_{t_0}^{t} w(\tau)\,\tilde{u}(\tau)\,d\tau. \tag{130}$$

Adding (129) and (130) one obtains

$$\left(\delta - \frac{\rho}{2}\right) \int_{t_0}^{t} \tilde{u}^2(\tau)\,d\tau \leq \gamma_0^2 + \frac{1}{2\rho} \int_{t_0}^{t} w^2(\tau)\,d\tau + \frac{1}{2}\,\tilde{\theta}^T(t_0)\,F^{-1}(t_0)\,\tilde{\theta}(t_0). \tag{131}$$

¿From the norm boundedness of $w(t)$ and with $\rho < 2\delta$ inequality (131) implies that $\tilde{u}(t)$ will also be norm bounded.

The signal

$$\tilde{y} = H\,\tilde{u} \tag{132}$$

is also norm bounded since $H$ is a BIBO operator. It remains to show that $\varepsilon_{CL}(t)$ and $\phi(t)$ are norm bounded.

¿From (122) and (132), $\varepsilon_{CL}(t) = \tilde{y}(t) + w(t)$ from which one concludes that $\varepsilon_{CL}(t)$ is norm bounded since $\tilde{y}(t)$ and $w(t)$ are norm bounded. Since $y(t)$ is norm bounded by stability of the true closed-loop system it results also that $y(\theta, t) = y(t) - \varepsilon_{CL}(t)$ is norm bounded. If the controller $C(y, r)$ and $\partial C_y(r, y)$ are bounded input bounded output stable it results that $u(\theta, t)$ will also be bounded. ■

The proof extends straightforwardly for mean square boundedness.

**Remark II.2:**

- Suppose that $\Delta P(u,v) = 0$ for simplicity, i.e. the system can be modeled exactly. Then (113) reduces to

$$y_p = [I + \partial P_u(\theta_0, u, 0)\partial C_y(r,y)]^{-1}\, \partial P_v(\theta_0, u, 0)\, v. \tag{133}$$

  Note that if the noise is additive, $\partial P_v(\theta_0, u, 0) = 1$ in the equation above.
- It follows from (113) that $w(t)$ depends on $u$ and $y$ and it results that both $w(t)$ and $\phi(t,\theta)$ depend on the reference signal $r$. This shows that $w(t)$ and $\phi(t,\theta)$ are not independent and this causes the NLCLOE algorithm to produce biased estimates.
- The situation is different in the linear case where a consistent estimate is obtained when the system is in the model set and the reference and noise signal are independent; see e.g. [8]. Indeed, it follows that (133) reduces to

$$y_p = (I + PC_y)^{-1}\, v \tag{134}$$

  which is independent of the reference signal $r$. In the linear case and with the system in the model set, $w = y_p$ is therefore independent of $\phi(t,\theta)$.

### 2.4 An Example

Consider the open-loop unstable plant model described by

$$\dot{x} = u + \theta_0\, x^2 \tag{135}$$

$$y = x + v \tag{136}$$

with $x, u, y$ in $\mathbf{R}^1$. It is assumed that $\theta_0 < 0$.

Consider the controller

$$u = -\,(y^3 + by^2) + r = -C(y) + r. \tag{137}$$

It is assumed that $\theta_0$ is unknown but $b$ is known. Several remarks can be made concerning this closed-loop system.

1. For $b = \theta_0$ and $v \equiv 0$, the closed-loop system equation becomes

$$\dot{x} = -x^3 + r$$

   and the closed-loop system is asymptotically stable.
2. For $b \neq \theta_0$ the closed-loop system is BIBO

The estimated plant model will be described by

$$\dot{x}(\theta) = u(\theta) + \theta\, x(\theta)^2 \tag{138}$$

$$y(\theta) = x(\theta) \tag{139}$$

and the estimated control will be given by

$$u(\theta) = -[y(\theta)^3 + by(\theta)^2] + r = -C(y(\theta)) + r \tag{140}$$

To apply and analyze the identification algorithm we need the following quantities(with $p = \frac{d}{dt}$)

$$P'(\theta, u(\theta)) = (p - 2\theta\, y(\theta))^{-1}\, y(\theta)^2, \tag{141}$$
$$\partial P_u(\theta, u(\theta)) = (p - 2\theta\, y(\theta))^{-1}, \tag{142}$$
$$\partial P_u(\theta_0, u, 0) = \partial P_u(\theta_0, u(\theta_0)) = (p - 2\,\theta_0\, y)^{-1}, \tag{143}$$
$$\partial C_y(r, y(\theta)) = 3\,y^2(\theta) + 2b\, y(\theta), \tag{144}$$
$$\partial C_y(r, y) = \partial C_y(r, y(\theta_0)) = 3\,y^2 + 2b\, y. \tag{145}$$

One can express now $P_{CL}(\theta)$ and $P_{CL}^{-1}(\theta)$:

$$P_{CL}(\theta) = \left[1 + \frac{3\,y^2(\theta) + 2b\,y(\theta)}{p - 2\theta\, y(\theta)}\right] = \frac{p + [3\,y^2(\theta) - 2(\theta - b)\, y(\theta)]}{p - 2\theta\, y(\theta)}, \tag{146}$$
$$P_{CL}^{-1}(\theta) = \frac{p - 2\theta\, y(\theta)}{p + [3\,y^2(\theta) - 2(\theta - b)\, y(\theta)]}. \tag{147}$$

For this example the various algorithms will have the following forms:

*NLOLOE*

The observation vector is

$$\phi(t) = (p - 2\theta\, y(\theta))^{-1}\, y(\theta)^2 \tag{148}$$

with the Open-Loop Output Error predictor shown in Figure 1. Note that the open-loop plant is unstable but is maintained in an "open-loop stable region".

*NLCLOE*

The observation vector $\phi(t)$ is as in (148) with the Closed-Loop Output Error predictor shown in Figure 3. The convergence condition requires that $P_{CL}^{-1}(\theta_0) - \frac{\lambda}{2}$ be strongly strictly passive where $P_{CL}^{-1}(\theta)$ is given by (147). In this example one should make the assumption that $-2\theta_0\, y > 0\ \forall\, t$ i.e. with $\theta_0 < 0$ this means $y(t) > 0$ must hold for all $t \geq t_0$, as well as the assumption that $3y^2 - 2(\theta_0 - b)\, y > 0\ \forall\, t \geq t_0$. This can be achieved along a trajectory generated with $r > 0$ (for $\theta_0 > 0$ the sign of r should be changed).

Notice that if $y$ is a constant signal, with $-2\theta_0\, y > 0$ and $3y^2 - 2(\theta_0 - b)\, y > 0$ for all $t > 0$, then $P_{CL}^{-1}(\theta_0) - \frac{\lambda}{2}$ will be strongly strictly passive for small $\lambda$. This suggests that low frequency signals should be used, as $P_{CL}^{-1}(\theta_0) - \frac{\lambda}{2}$ is still likely to be strongly strictly passive along the associated trajectories.

We apply the *NLOLOE* and the *NLCLOE* algorithms using the previous example with $b = -0.4$, $r = 2 + 0.5\sin(0.1\, t)$ and $v$ zero mean white Gaussian noise with variance $\sigma^2$. The parameter which is to be identified recursively is given by

$$\theta_0(t) = \begin{cases} -0.5 & \text{for } t \leq 315 \\ -0.5 + 0.25\sin(0.03\,t) & \text{for } t > 315, \end{cases} \quad (149)$$

i.e. the parameter $\theta_0$ is first held constant and then allowed to vary sinusoidally. We adopt a least squares strategy with forgetting factor ($\lambda_1 = 0.5$, $\lambda_2 = 1$) and the algorithm is initialized with $\theta(0) = 0$.

Figure 4 shows the identification results in a noiseless situation. Both the *NLOLOE* and the *NLCLOE* algorithms allow a consistent identification of $\theta_0$. The tracking results are better with the *NLOLOE* algorithm. Figure 5 shows (as can be expected) the appearance of a systematic bias on the estimate in a noisy situation with the *NLCLOE* algorithm. Note that the noise effect can be be reduced (at the expense of the tracking performance) by increasing the value of $\lambda_1$.

The advantage of the *NLCLOE* algorithm lies in the identification of unstable plants in a closed-loop situation; we refer to [4] for an example with a modified *NLCLOE* algorithm. Another advantage of the *NLCLOE* algorithm lies in the recursive identification of reduced complexity models in a low noise situation.

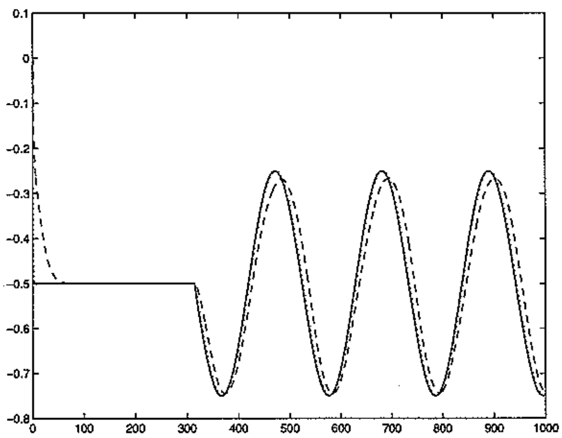


**Fig. 4.** Identification of $\theta_0(t)$ (—) in the noiseless case ($\sigma^2 = 0$), respectively, using the *NLCLOE* (−−) and *NLOLOE* ($\cdots$) algorithms .

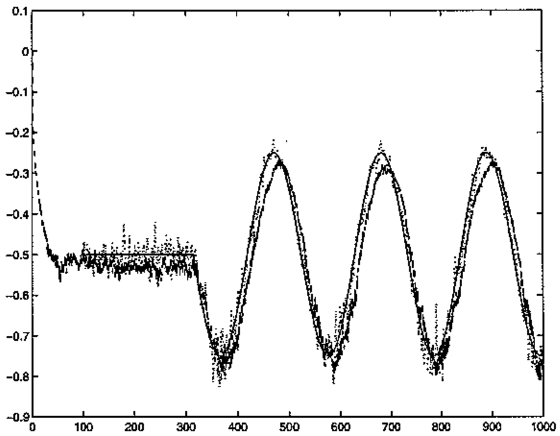


**Fig. 5.** Identification of $\theta_0(t)$ (—) in a noisy situation ($\sigma^2 = 0.01$) using the *NL-CLOE* (-–) and *NLOLOE* ($\cdots$) algorithms with $\lambda_1 = 0.5$ and $\lambda_2 = 1$.

## Conclusion

The key contribution of this paper has been to show that the framework for a number of open loop and closed-loop output error identification algorithms can be pushed out from linear systems to nonlinear systems. Hence our results, not surprisingly, for the most part assume that the high order terms can be neglected in certain Taylor series expansions, or we assume that they are at least small. Other than that, both the noisy and noiseless case are captured, as is the possibility that the true plant may not lie in the model set and that the parameters can be slowly time varying. Possible relationship with Extended Kalman Filters and nonlinear observers deserves to be studied in the future.

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## A Appendix

Consider the system

$$y = Hu \tag{154}$$

and assume that it accepts a state space representation

$$\dot{x} = f(x, u, t) \tag{155}$$

$$y = h(x, t) \tag{156}$$

with $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $u \in \mathbf{R}^m$, $f$, $h$ continuous in $t$ and smooth in $x$. Suppose $f(0,0,t)=0$ and $h(0,t)=0$ for all $t \geq 0$.

**Definition 1.** The system $H$ is said to be **strongly strictly passive** if there exist a positive definite (storage) function $V(x,t)$ which satisfies

$$\gamma_1(|x|) \leq V(x,t) \leq \gamma_2(|x|) \tag{157}$$

$$V(0,t)=0, \quad \forall t \geq 0 \tag{158}$$

where $\gamma_1(|x|)$ and $\gamma_2(|x|)$ are class $\mathcal{K}_\infty$ functions, and there exists a positive definite function (dissipation rate) $\psi(x) \geq \gamma_3(|x|)$; $\gamma_3(\cdot) \in \mathcal{K}_\infty$ such that

$$\int_{t_0}^{t} y^T(\tau)u(\tau)\,d\tau \geq V(x(t),t) - V(x(t_0),t_0) + \int_{t_0}^{t} \psi(x(\tau))\,d\tau \tag{159}$$
$$\forall t, t_0 \text{ with } t \geq t_0.$$

**Definition 2.** A system $S$ with input $u$, output $y$ and state $x$ (see (155) and (156)) is said to belong to the class $L(\Lambda)$ if it is strongly strictly passive and in addition the following strengthened version of (159) holds

$$\int_{t_0}^{t} y^T(\tau)\,u(\tau)\,d\tau \geq V(x(t),t) - V(x_0,t_0) + \int_{t_0}^{t} \psi(x,\tau)\,d\tau$$
$$+\frac{1}{2}\int_{t_0}^{t} u^T(\tau)^T \Lambda(\tau)\,u(\tau)\,d\tau; \quad \Lambda(t) > 0 \quad \forall t \geq t_0. \tag{160}$$

**Remark:** The system $S$ belonging to the class $L(\Lambda)$ has an excess of passivity

**Definition 3.** A system $S$ with input $u$, output $y$ and state $x$ (see (155) and (156)) is said to belong to the class $N(\Gamma)$ if the integral of the input output product satisfies the following modified version of (159)

$$\int_{t_0}^{t} y^T(\tau)\,u(\tau)\,d\tau \geq V(x(t),t) - V(x_0,t_0) + \int_{t_0}^{t} \psi(x,\tau)\,d\tau$$
$$-\frac{1}{2}\int_{t_0}^{t} y^T(\tau)\,\Gamma(\tau)\,y(\tau)\,d\tau; \quad \Gamma(t) \geq 0 \quad \forall t \geq t_0 \tag{161}$$

where $V$ and $\psi$ are non negative functions.

**Remarks:**

1. The system $N(\Gamma)$ has a lack of passivity.¡
2. Note that there is no $K_\infty$ property imposed on $V$ and $\psi$ in contrast to the $L(\Lambda)$, and strong strict passivity does not follow from (161).

We now turn to some generalizations of the Positive Real Lemma [1] to time-varying systems [7,15]. Consider the linear time-varying multivariable system

$$\dot{x} = A(t)\,x(t) + B(t)\,u \tag{162}$$

$$y = C(t)\,x(t) + D(t)\,u \tag{163}$$

with $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $u \in \mathbf{R}^m$ and $A(t)$, $B(t)$, $C(t)$ and $D(t)$ continuous in $t$.

**Lemma 2.** *([15,7]) The system (162), (163) is passive if there exists a symmetric time-varying positive definite matrix function $P(t)$ differentiable with respect to $t$, a symmetric time-varying semi-definite matrix $Q(t)$ and matrices $S(t)$ and $R(t)$ such that*

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \tag{164}$$

$$B^T(t)P(t) - C(t) = S^T(t) \tag{165}$$

$$D(t) + D^T(t) = R(t) \tag{166}$$

$$\begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix} \geq 0 \quad \text{for all} \quad t \geq t_0. \tag{167}$$

The following lemma is trivial to prove.

**Lemma 3.** *If the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ satisfy the set of equations (164), (165) and (166) for some matrices $P(t)$, $Q(t)$, $S(t)$, $R(t)$ with appropriate dimension, the integral of the input-output product can be expressed as*

$$\begin{aligned} \int_{t_0}^{t} y^T(\tau)\, u(\tau)\, d\tau = &\frac{1}{2} x^T(t)\, P(t)\, x(t) - \frac{1}{2} x^T(t_0)\, P(t_0)\, x(t_0) \\ &+\frac{1}{2} \int_{t_0}^{t} \left[ x^T(\tau)\, Q(\tau)\, x(\tau) + 2\, u^T(\tau)\, S(\tau)\, x(\tau) \right. \\ &\left. + \, u^T(\tau)\, R(\tau)\, u(\tau) \right], \; \forall t \geq t_0. \end{aligned} \tag{168}$$

**Theorem 6.** *Consider the feedback connection of two systems $S_1$ and $S_2$ with state space realizations, containing state vectors $x_1$ and $x_2$ respectively. Suppose that $S_1$ is linear time-varying and belongs to the class $L(\Lambda)$ and its storage function $V_1$ and dissipation rate $\psi_1$ are independent of $x_2$. Suppose that the system $S_2$ belongs to the class $N(\Gamma)$ and its storage function $V_2$ and dissipation rate $\psi_2$ are independent of $x_1$. Suppose that $V_1$ and $V_2$ are differentiable. Suppose that no external excitation is acting on this feedback system. Then, if*

$$\Lambda(t) - \Gamma(t) \geq \delta \quad \forall t \geq t_0 \quad \textit{and some } \delta > 0, \tag{169}$$

- *the equilibrium state $x^T = [x_1^T, x_2^T]$ is globally uniformly stable (with $x_1(t)$ and $x_2(t) \in \mathcal{L}_\infty$),*
- *Also,*

$$\lim_{t \to \infty} x_1(t) = 0 \quad \textit{and} \quad u_1 \in \mathcal{L}_2. \tag{170}$$

**Proof:** Follows the lines of [6]. See also [9].

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