Progress in Systems and Control Theory, Vol. 25

Riccati Equations, Network Theory and Brune Synthesis: Old Solutions for Contemporary Problems

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1 Introduction

Riccati equations have a natural connection with network theory. Classical passive network synthesis procedures, employing often frequency domain spectral factorization, can be mirrored by state variable procedures which rely on knowledge of a steady state Riccati equation solution. In contrast to many occurrences of steady state Riccati equations, it is possible (especially in network applications) to encounter equations which have strong, but not stabilizing solutions. Such equations constitute a problem for much software. Classical network synthesis procedures actually dealt with a frequency domain version of this problem using tools such as the Brune synthesis. The Riccati equation equivalent involves a deflation technique which will be exposed.

The aim of this paper is to expose certain connections between network theory and Riccati equations. Our special focus is on Riccati equations where there exist strong but not stabilizing solutions. These concepts are defined later; suffice it to say here that the "closed—loop" system matrix has eigenvalues in the closed but not open left half plane. In terms of spectral factorizations, one obtains spectral factors which have zeros on the imaginary axis.

In [1], a number of connections between Riccati differential equations, steady state Riccati equations and passive networks are explored. Given a positive real transfer function matrix $Z(s)$, it is a result of classical network theory [4, 7, 8] that there exists a spectral factorization

$$Z(s) + Z^T(-s) = W^T(-s)W(s) \tag{1.1}$$

where $W(s)$ is stable and has all zeros in $\mathrm{Re}[s] \leq 0$ and sometimes $\mathrm{Re}[s] < 0$. This result has its parallel in the solution of certain Riccati equations defined using a state—variable realization of $Z(s)$. Moreover, just as the factorizations (1.1) plays a major role in defining synthesis procedures (i.e. procedures for defining an interconnection of resistors, capacitors, inductors etc whose impedance is a prescribed $Z(s)$), so the solution of a Riccati

[1]The authors acknowledge funding of activities of the Cooperative Research Centre for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program, funding of this research by the US Army Research Office, Far East, the Office of Naval Research, Washington.

equation associated with a state variable realization of $Z(s)$ can be used as a basis for synthesis.

Exploring these ideas is not the focus of this paper (though some are reviewed in the next section). Rather, we focus on a contemporary problem in the use of Riccati equations, that of understanding how one might solve an equation where a strong but not stabilizing solution exists. Such equations typically cause problems for software, but can arise in, for example, characterizing a transfer function which just achieves, rather than falls arbitrarily short of, a prescribed gain.

We shall explain how a deflation process can assist in solving such equations. Moreover, we motivate the process by an old procedure of network synthesis—the Brune synthesis—where the equivalent spectral factorization problem (requiring a spectral factor with imaginary axis zeros) is deliberately contrived, in order to allow a deflation type of solution to the synthesis problem.

Section 2 of the paper reviews the connection between Riccati equations and linear passive multiport networks. Section 3 characterizes some situations leading to Riccati equations with strong but not stabilizing solutions, and reviews (in outline form) a state—variable version of the Bruce synthesis. This motivates the development in Section 4 of a deflation process for solving what we term "difficult" Riccati equations. There is of course other literature on such equations. We note especially [10, 11, 12]. Section 5 offers brief concluding remarks.

2 Riccati Equations and Linear Passive Multiport Networks

Consider a p-port network with input current vector $u \in R^p$, and port voltage vector $v \in R^p$, with the network consisting of an interconnection of a finite number of time—invariant passive resistors, capacitors, inductors, transformers and gyrators. Suppose that an impedance function $Z(s)$ is well defined with bounded u leading to bounded y. Under these circumstances

$$Z(s) = H^T(sI - F)^{-1}G + J \tag{2.2}$$

for some minimal quadruple $\{F, G, H, J\}$. In writing (2.2), we make no claim that the implied state vector necessarily has entries comprising inductor currents and/or capacitor voltages.

It is well known that $Z(s)$ satisfies several properties, collectively known as the *positive real property*. These are (with * denoting transpose complex conjugate).

$$Z(s) \text{ is analytic in } \mathrm{Re}[s] > 0 \tag{2.3}$$

$$Z(s) + Z^*(s) \geq 0 \text{ for } \mathrm{Re}[s] > 0 \tag{2.4}$$

Equivalently,

$$Z(s) \text{ is analytic in } \mathrm{Re}[s] > 0 \tag{2.5}$$

$$\text{Any pole of } Z(s) \text{ on } \mathrm{Re}[s] = 0 \text{ is simple, with residue matrix that is nonnegative hermitian} \tag{2.6}$$

$$\begin{aligned} &Z(j\omega) + Z^T(-j\omega) \geq 0 \text{ for all real } \omega \\ &(\text{provided } j\omega \text{ is not a pole of } Z(s)) \end{aligned} \tag{2.7}$$

There are implications of these conditions for the state-variable description of $Z(s)$. For example, (2.7) implies (letting $\omega \to \infty$) that

$$J + J^T \geq 0 \tag{2.8}$$

The passivity of the network means that if the network is initially unexcited, and an arbitrary current is injected starting at time 0, then the energy absorbed by the network over an interval $[0, t_1]$ for arbitrary t_1 is necessarily nonnegative. More precisely, with

$$\dot{x} = Fx + Gu \quad x(0) = 0 \tag{2.9}$$

there holds (with $R = J + J^T$)

$$\int_0^{t_1} u^T y dt = \frac{1}{2} \int_0^{t_1} (u^T R u + 2x^T H u) dt \tag{2.10}$$

for all $u(\cdot)$. Equivalently, if the network is initially storing energy ($x(0) = x_0 \neq 0$), and an arbitrary $u(\cdot)$ is applied starting at time 0, there is an upper bound to the energy which can be extracted from the network over any interval, i.e. with

$$\dot{x} = Fx + Gu \quad x(0) = x_0 \tag{2.11}$$

there holds for all u

$$-\int_0^{t_1} u^T y dt \leq \alpha(x_0) \tag{2.12}$$

Here, $\alpha(x_0)$ is a nonnegative function of x_0, representing the initially stored energy. Equivalently,

$$V(x_0, u(\cdot), t_1) = \int_0^{t_1} u^T y = \frac{1}{2} \int_0^{t_1} (u^T R u + 2x^T H u) dt \tag{2.13}$$

is bounded below, independently of $u(\cdot)$ and t_1, by some function of x_0.

2.1 A Variational Problem

For each x_0, one can ask the question: by manipulating $u(\cdot)$, how can we extract the maximum energy from the network? More precisely, we ask: for the system (2.11) with performance index (2.13), how do we choose $u(\cdot)$ and t_1 to minimize the index?

Before addressing the solution, let us make several observtions.

(a) the problem makes sense precisely because of passivity, i.e. because $V(x_0, u(\cdot), t_1)$ is known to be bounded below independently of $u(\cdot)$ and t_1;

(b) consider two indices, $V(x_0, u(\cdot), t_1)$ and $V(x_0, u(\cdot), t_2)$ where $t_2 > t_1$. By choosing $u(t) = 0$ over $[t_1, t_2]$, the second index can always be made to equal the first. Hence

$$\inf_{u(\cdot)} V(x_0, u(\cdot), t_1)$$

for fixed x_0 is monotone decreasing in t_1.

(c) in the light of (2.12), and since the least value of $V(x_0, u(\cdot), t_1)$ obtainable by choosing $u(\cdot)$ is no greater than the value obtained with $u(\cdot) \equiv 0$, there holds

$$-\alpha(x_0) \leq \inf_{u(\cdot)} V(x_0, u(\cdot), t_1) \leq 0 \tag{2.14}$$

(d) it proves useful to examine both the case of arbitrary but fixed finite t_1, and the case $t_1 = \infty$. Then one can consider optimizing over t_1. In the light of (b), it will be no surprise that the case $t_1 = \infty$ delivers the infimum over all $u(\cdot)$ and t_1 of $V(x_0, u(\cdot), t_1)$.

(e) If $R = \frac{1}{2}(J + J^T)$ is singular, there is in general no smooth $u(\cdot)$ giving the desired minimum; if R is nonsingular, there is a smooth $u(\cdot)$ (these facts are not apriori obvious).

2.2 Solution to the Finite Interval Variational Problem

Let us make the assumption

$$R \text{ is nonsingular} \tag{2.15}$$

Then for fixed t_1, we have

$$V^*(x_0, t_1) = \inf_{u(\cdot)} V(x_0, u(\cdot), t_1) = x_0^T \Pi(0, t_1) x_0 \tag{2.16}$$

where $\Pi(.,t_1)$ is the symmetric matrix satifying the Riccati differential equation

$$-\dot{\Pi} = \Pi(F - GR^{-1}H^T) + (F - GR^{-1}H^T)^T \times \Pi - \Pi GR^{-1}G^T\Pi - HR^{-1}H^T \tag{2.17}$$

with boundary condition $\Pi(t_1, t_1) = 0$. The associated optimum $u(\cdot)$ is given in feedback form as

$$u^*(t) = -R^{-1}[G^T\Pi(t,t_1) + H^T]x(t) \tag{2.18}$$

so that the state trajectories of the network evolve according to

$$\dot{x} = [F - GR^{-1}H^T - GR^{-1}G^T\Pi(t,t_1)]x \tag{2.19}$$

The proof of this result, see eg. [1], uses fairly standard ideas of linear—quadratic optimization. It can also be established by a first principles, completion-of-square argument.

Note that (2.17) could only give trouble if it had a finite escape time. A minor variation on the argument leading to (2.14) shows that

$$-\alpha(x(t)) \leq x^T(t)\Pi(t,t_1)x(t) \leq 0 \tag{2.20}$$

It is not difficult to show that because this holds for all $x(t), \Pi(t,t_1)$ must have entries which are bounded independently of t and t_1, thus negating the possibility of an escape time .

From (2.14) and (2.16), it is clear that $\Pi(0,t_1) \leq 0$. We can in fact establish strict negative definiteness. Linear—quadratic variational theory actually yields that the minimizing control is unique. If there were to hold $x_0^T\Pi(0,t_1)x_0 = 0$ for some $x_0, u(t) \equiv 0$ would give this value for the index, and then the uniqueness argument would imply that

$$[G^T\Pi(t,t_1) + H^T]x(t) = 0 \quad \forall t \in [0,t_1]$$

Then (2.17) would yield (on right multiplication by x)

$$-\dot{\Pi}x = \Pi Fx + F^T\Pi x$$

or, since now $\dot{x} = Fx$ (with $u \equiv 0$),

$$\frac{d}{dx}(\Pi x) = -F^T(\Pi x)$$

At $t = t_1$, $\Pi x = 0$. Hence $\Pi(t,t_1)x(t) = 0 \ \forall t \in [0,t_1]$. Then $H^Tx(t) = 0 \ \forall t \in [0,t_1]$. Together with $\dot{x} = Fx$ on $[0,t_1]$, observability implies $x_0 = 0$.

For later reference, we record the fact that because $\inf_{u(\cdot)} V(x_0, u(\cdot), t_1)$ is monotone decreasing in t_1, the matrix $\Pi(0,t_1)$ is monotone decreasing in t_1, i.e.
$\Pi(0,t_1) \geq \Pi(0,t_2)$ for $t_2 \geq t_1$.

2.3 The Limiting Properties of $\Pi(t, t_1)$

With $\Pi(t, t_1)$ as defined above, it is trivial to see that

$$\Pi(0, t_1 - t) = \Pi(t, t_1) \tag{2.21}$$

which is a kind of shift—invariance property. Also, as commented earlier,

$$\inf_{u(\cdot)} V(x_0, u(\cdot), t_1) = x_0^T \Pi(0, t_1) x_0 \geq -\alpha(x_0) \tag{2.22}$$

is monotone decreasing in t_1.

Suppose $x_0 \in R^n$. By choosing $x_0 = e_1, e_2, \ldots, e_n$, we see that

$$\lim_{t_1 \to \infty} \Pi_{ii}(0, t_1) = \bar{\Pi}_{ii} \tag{2.23}$$

exists, and then by choosing $x_0 = (e_i + e_j)$ for all $i \neq j$ we see that

$$\lim_{t_1 \to \infty} \Pi_{ij}(0, t_1) = \bar{\Pi}_{ij} \tag{2.24}$$

exists. So there exists

$$\begin{aligned}
\bar{\Pi} &= \lim_{t_1 \to \infty} \Pi(0, t_1) && (2.25)\\
&= \lim_{t_1 \to \infty} \Pi(0, t_1 - t) \quad \text{for fixed } t \\
&= \lim_{t \to -\infty} \Pi(0, t_1 - t) \\
&= \lim_{t \to -\infty} \Pi(t, t_1) \text{ by (2.21)} && (2.26)
\end{aligned}$$

Thus $\bar{\Pi}$ is the limit as $t \to -\infty$ of solutions of (2.17).

With a little more work, one can prove that $\bar{\Pi}$ is a steady state solution of (2.17) i.e.

$$\begin{aligned}
0 = \bar{\Pi}(F - GR^{-1}H^T) + (F - GR^{-1}H^T)^T \\
\times \bar{\Pi} - \bar{\Pi}GR^{-1}G^T\bar{\Pi} - HR^{-1}H^T
\end{aligned} \tag{2.27}$$

Obviously, since $\Pi(0, t_1) < 0$ and $\bar{\Pi} \leq \Pi(0, t_1)$, there holds

$$\bar{\Pi} < 0 \tag{2.28}$$

This can also be obtained directly by using the observability of (F, H) and (2.27).

An obvious question now is whether one can simply take limits in the equation for $u^*(\cdot)$ and the associated closed—loop trajectory, to assert for the the infinite time problem that

$$u^*(t) = -R^{-1}[G^T\bar{\Pi} + H^T]x(t) \tag{2.29}$$

and

$$\dot{x} = [F - GR^{-1}(G^T\bar{\Pi} + H^T)]x \tag{2.30}$$

The answer is not always. What can go wrong is the following. The optimal index $x_0^T\Pi(0, t_1)x_0$ is made up of the sum of two terms, viz

$$\frac{1}{2}\int_0^{t_1} u^{*T}Ru^*\,dt$$

and

$$\int_0^{t_1} x^T Hu^*\,dt$$

with $x(\cdot)$ computed along (2.19). The first term is always positive [if $x_0 \neq 0$], and the second necessarily negative, since $x_0^T\Pi(0, t_1)x_0 < 0$. When $t_1 \to \infty$, it may happen that the first term diverges to $+\infty$ and the second term to $-\infty$, even though their sum remains well behaved. If this divergence occurs, (2.29) and (2.30) come into question.

2.4 Spectral Factorization

Suppose we start with a positive real $Z(s)$ and associated realizing quadruple $\{F, G, H, J\}$, where $J + J^T = R > 0$. Let X be *any* solution of the same equation as satisfied by $\bar{\Pi}$, viz

$$\begin{aligned} 0 = X(F - GR^{-1}H^T) + (F - GR^{-1}H^T) \\ \times X - XGR^{-1}G^TX - HR^{-1}H^T \end{aligned} \tag{2.31}$$

and define

$$W(s) = R^{\frac{1}{2}} + R^{-\frac{1}{2}}(XG + H)^T(sI - F)^{-1}G \tag{2.32}$$

Then via algebraic mainpulation, one can show that

$$Z(s) + Z^T(-s) = W^T(-s)W(s) \tag{2.33}$$

Thus steady state solutions of (2.29) correspond to *spectral factors* $W(s)$ of $Z(s) + Z^T(-s)$, with the same matrices F and G occurring in the state—variable realization of $W(s)$ as occur in the state—variable realization of $Z(s)$.

Classical network synthesis of a positive real $Z(s)$ typically exploited spectral factors $W(s)$. State-variable approaches to network synthesis define the synthesising network in terms of F, G, H, J and a solution X to (2.31). [In fact, solutions to an inequality replacing (2.31) also will yield syntheses, with a nonminimal number of resistors, and all the syntheses obtained this way have a minimal number of reactive elements.]

2.5 Isolating the Riccati Equation Solution and Spectral Factor

For those who are well–versed in linear quadratic theory, it will be no surprise that there is interest in characterizing that particular solution $\bar{\Pi}$ of the steady state Riccati equation (2.31) obtained as the limit of the transient Riccati differential equation (2.17), and in particular using a characterization involving the eigenvalues of the limiting "closed—loop" matrix [see (2.30)], which is $F - GR^{-1}(G^T\bar{\Pi} + H^T)$. A key result is the following:

Theorem 2.1. *Let Z(s) with minimal realization $\{F, G, H, J\}$ be positive real, with $J + J^T > 0$. Let $\Pi(t, t_1)$ solve the Riccati differential equation (2.17) and let $\bar{\Pi} = \lim_{t\to-\infty} \Pi(t, t_1)$. Define*

$$\bar{F} = F - GR^{-1}(G^T\bar{\Pi} + H^T) \tag{2.34}$$

and $\bar{W}(s)$ by (2.32) with $X = \bar{\Pi}$, so that (2.33) with $\bar{W}$ replacing W holds. Then the eigenvalues of $\bar{F}$, which are the zeros of $\bar{W}(s)$, satisfy

$$Re\lambda_i(\bar{F}) \le 0 \tag{2.35}$$

We remark that there is a world of difference between the more common results for linear quadratic poblems of the type $\mathrm{Re}\lambda_i(\bar{F}) < 0$ and the result of this theorem. Different proof techniques are needed. That aside, the crucial conclusion is that $\bar{\Pi}$ defines that particular spectal factor in (2.33) which is minimum phase (but not necessarily strictly so), and is known to be unique to within constant orthogonal matrix multiplication on the left, [2].

Proof. Define $V(t) = \Pi(t, t_1) - \bar{\Pi}$. From (2.17) and (2.27), a little manipulation yields

$$\dot{V} = -V\bar{F} - \bar{F}^T V + VGR^{-1}G^T V \tag{2.36}$$

$$V(t_1) = -\bar{\Pi} \tag{2.37}$$

Further, $V(t) \to 0$ as $t \to -\infty$, and since
$\Pi(t, t_1) = \Pi(0, t_1 - t)$ is monotone increasing in $t, V(t)$ is nonnegative definite in t and monotone increasing as $t \to t_1$. Set $Y(t) = V(t_1 - t)$ for $t \ge 0$. Then

$$\dot{Y} = Y\bar{F} + \bar{F}^T Y - YGR^{-1}G^T Y \tag{2.38}$$

$$Y(0) = -\bar{\Pi} \tag{2.39}$$

with $Y(t)$ nonnegative and monotone decreasing to 0 as $t \to \infty$. Now set $Z^{-1}(t) = Y(t)$. Obviously Z is defined near $t = 0$, since $Y(0) = -\bar{\Pi}$ is nonsingular. Also,

$$\dot{Z} = -\bar{F}Z - Z\bar{F}^T + GR^{-1}G^T \tag{2.40}$$

$$Z(0) = -(\bar{\Pi})^{-1} \tag{2.41}$$

□

The linearity of the equation for $Z(t)$ ensures $Z(t)$ is defined for all $t \geq 0$. Further, $Z(t)$ is monotone increasing, with $Z(t)$ tending to infinity in the sense that $\lambda_{\min}[Z(t)] \to \infty$ as $t \to \infty$. [Note that $\lambda_{\min}(Z) = [\lambda_{\max}(Y)]^{-1}$ and the convergence of Y to zero ensures $\lambda_{\max}(Y) \to 0$]

To establish a contradiction, suppose that $\bar{F}$ has an eigenvalue with positive real part. For convenience, suppose it is real and $\bar{F}^T m = \lambda m$ for $m \neq 0$, $\lambda > 0$. Let $z(t) = m^T Z(t) m$. Then, with $n = m^T GR^{-1}G^T m$

$$\dot{z} = -2\lambda z + n \tag{2.42}$$

The solution is bounded for all $t \geq 0$, which contradicts

$$z(t) \geq \lambda_{\min}[Z(t)] m^T m.$$

Hence Re $\lambda_i(\bar{F}) \leq 0$, as required.

In studying the convergence of solution of Riccati differential equations to the steady state solution, it is common to link the rate of convergence (which is usually exponential) to the eigenvalues of the steady state closed—loop matrix.

If indeed, Re$\lambda_i(\bar{F}) < 0$, then one can argue that $\Pi(t, t_1) - \bar{\Pi}$ as $t \to \infty$ converges to zero exponentially fast [the difference being expressible in terms of $\exp(\bar{F}t)$]. What happens if Re$\lambda_i(\bar{F}) = 0$? It is easiest to work with $Y(t) = V(t_1 - t) = \Pi(0, t) - \bar{\Pi}$.

Corollary 2.1. *Adopt the same hypothesis as for Theorem 2.1, and let*

$$\dot{Y} = Y\bar{F} + \bar{F}^T Y - YGR^{-1}G^T Y \tag{2.43}$$

$$Y(0) = -\bar{\Pi} \tag{2.44}$$

Suppose that for some eigenvalue of $\bar{F}$, there holds Re$\lambda_i(\bar{F}) = 0$. Then $Y(t) \to 0$ at rate $0(t^{-1})$.

Proof. For convenience, suppose that the eigenvalue with zero real part is real, i.e. zero. Thus suppose that for $m \neq 0$, there holds $\bar{F}m = 0$. Let

$Z^{-1}(t) = Y(t)$, and $z(t) = m^T Z(t) m$. Then the proof of the theorem (see (2.42)) yields that

$$z(t) = -m^T \bar{\Pi}^{-1} m + tn$$

Now

$$m^T Y(t) m m^T Z(t) m \geq (m^T m)^2$$

so that

$$m^T Y(t) m \geq \frac{(m^T m)^2}{nt - m^T \bar{\Pi}^{-1} m}$$

This shows that $Y(t)$ can converge no faster than at rate $O(t^{-1})$.

□

The fact that $Y(t)$ converges no slower than this rate relies on a lengthy calculation explicitly computing $Y(t)$, when $\bar{F}$ is in Jordan form and will be omitted here.

2.6 Summary to this point

1. Transient Riccati equations can be associated with a minimal realization of a positive real transfer function matrix $Z(s)$ for which $Z(\infty) + Z^T(\infty)$ is nonsingular.

2. The solution of the transient equation approaches a steady state solution of the equation.

3. Steady state solutions to the Riccati equation immediately yield a synthesis of $Z(s)$.

4. The rate of approach of transient equation solution to a steady state is exponential when

$$Z(j\omega) + Z^*(j\omega) > 0$$

for all real ω (apart from $j\omega$ which are poles of $Z(s)$. Otherwise it is $O(1/t)$.

5. The limiting solution of the Riccati equation defines a "closed—loop" system matrix with eigenvalues in the closed left half plane, and a spectral factor $W(s)$ of $Z(s)$ satisfying

$$Z(s) + Z^T(-s) = W^T(-s) W(s)$$

which is minimum phase (and so may have zeros on the imaginary axis).

3 Difficult Riccati Equations

What does the section title refer to? It refers to Riccati equations of the form

$$XA + A^T X - XBB^T X + Q = 0 \tag{3.45}$$

in which the associated Hamiltanian matrix

$$\mathcal{H} = \begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \tag{3.46}$$

has eigenvalues which are pure imaginary. As is well known, [3] if (3.45) has a stabilizing solution, call it $\bar{X}$, in the sense that

$$\text{Re}\lambda_{\ i}(A - BB^T \bar{X}) < 0 \tag{3.47}$$

the the eigenvalues of H must split into the disjoint sets $\{\text{Re}\lambda_i(A - BB^T\bar{X}\}$ and $\{-\text{Re}\lambda_i(A - BB^T\bar{X})\}$. and $\bar{X}$ is in fact unique. In these circumstances, much attention has been given to the construction of effective numerical procedures for finding $\bar{X}$, often using eigenspaces associated with these sets.

There are known situations where such an eigenvalue split is not possible.

(a) Recall the ideas of the last section. Suppose that $Z(s)$ defined by a minimal $\{F, G, H, J\}$ with
$J + J^T > 0$ is positive real with

$$Z(j\omega) + Z^T(-j\omega)$$

singular for some real ω. Then the Riccati equation

$$\begin{aligned} X(F - GR^{-1}H^T) + (F - GR^{-1}H^T)^T X - X \\ \times GR^{-1}G^T X - HR^{-1}H^T = 0 \end{aligned} \tag{3.48}$$

has a solution $\bar{\Pi}$ for which

$$\text{Re}\lambda_i(F - GR^{-1}H^T - GR^{-1}G^T\bar{\Pi}) \leq 0 \tag{3.49}$$

and strict inequality is not valid. The Hamiltonian matrix here is

$$\mathcal{H} = \begin{bmatrix} F - GR^{-1}H^T & -GR^{-1}G^T \\ -HR^{-1}H^T & -F^T + HR^{-1}G^T \end{bmatrix} \tag{3.50}$$

(b) Let $S(s)$ be a stable strictly proper transfer function matrix with minimal realization $\{F, G, H\}$ and $S(\infty) = 0$. When does $S(s)$ obey

$$\|S(j\omega)\| \leq \gamma$$

for all real ω? When $\gamma = 1$, this is simply asking the question, when is $S(s)$ a scattering matrix, an old network theory construct [4]. For arbitrary γ, the question has come into much prominence in connection with H_∞ theory. (The question can of course also be considered for $S(\infty) \neq 0$, and the ideas are effectively the same, but algebraically more complicated to explain.) The following is by now a well known result, see eg [5].

Theorem 3.1. *Let $S(s) = H^T(sI - F)^{-1}G$, with the $Re(F) < 0$. Then $\|S(j\omega)\| \leq \gamma$ for all ω if and only if the following equation is solvable:*

$$XF + F^TX + \gamma^{-2}XGG^TX + HH^T = 0 \tag{3.51}$$

This is a Riccati equation. When $\|S(j\omega\| < \gamma$, then and only then there exist a solution $\bar{X}$ which is stabilizing, i.e.

$$\text{Re}\lambda_i\left[F + \gamma^{-2}GG^T\bar{X}\right] < 0 \tag{3.52}$$

Else, we can simply require

$$\text{Re}\lambda_i\left[F + \gamma^{-2}GG^T\bar{X}\right] \leq 0 \tag{3.53}$$

In the first case, $\bar{X}$ is the limit approached exponentially fast by the solution of a Riccati differential equation; also, it can be obtained by any one of a number of numerical algorithms. In the second case, $\bar{X}$ is still the limit of a differential equation solution (but the approach rate is not exponential), and most numerical algorithms give trouble.

(c) In H_∞ theory in order to find a stabilizing controller yielding closed—loop gain less than some prescribed γ, two Riccati equations must have stabilizing solutions, call then $\bar{X}$ and $\bar{Y}$, and the product $\bar{X}\bar{Y}$ must also satisfy a strict inequality on its singular values in terms of γ. If the H_∞ problem is solvable for some γ, then there is an infimum of such values γ. At this infimum, it may be that one of the Riccati equations has a strong but not stabilizing solution.

3.1 Network Theory Approach to Difficult Riccati Equations

There is an old approach to network synthesis which provides one approach to dealing with difficult Riccati equations, the Brune synthesis procedure, [6, 8].

From a classical point of view, what happens is the following. A positive real $Z(s)$ is prescribed, and a synthesis is required, (i.e. one seeks a description of a network of passive elements whose impedance is $Z(s)$). In outline the procedure is as follows.

1. A positive scalar ρ is found so that
$$Z(j\omega) + Z^T(-j\omega) - \rho I \geq 0 \tag{3.54}$$
for all ω, with singularity of the object on the left for some ω_0

2. An orthogonal constant T is found so that with Z replaced by $\bar{Z} = T^T ZT$, there holds
$$\bar{Z}_{11}(j\omega_0) + \bar{Z}_{11}(-j\omega_0) - \rho = 0 \tag{3.55}$$

3. $\bar{Z}(s) - (\rho/2)I$ is now regarded as the entity to synthesise (and from this a synthesis of $Z(s)$ will follow). Note that at $s = j\omega_0$, the real part of the 1-1 term is zero.

4. $\bar{Z}(s) - (\rho/2)I$ is expressed as the impedance of a low complexity (degree 2 in general) $2n$-port network, whose elements are readily computable, cascaded with an n-port of complexity less than that of $\bar{Z}(s) - (\rho/2)I$. (Complexity here means McMillan degree, or order of a minimal state—variable realization).

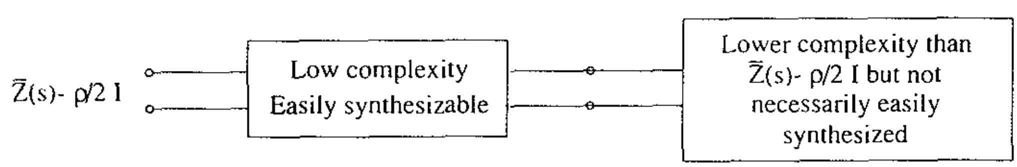

Figure 1: Cascade (Brune) synthesis

One now has the basis for a recursion—what remains to be synthesised after each step is always less complicated than what was there before. Repeating the steps eventually leads to termination of the procedure.

There is a state—variable view of this procedure, set out in [9]. The key to obtaining a synthesis is to solve a Riccati equation. Steps 1, 2 and 3 are the same. In step 4, a coordinate basis change is obtained and in the new coordinate basis, the state vector of $\bar{Z}(s) - (\rho/2)I$ can be partitioned into two subvectors, one corresponding to the easily synthesised, low complexity network, and one to the "lower" complexity but not-necessarily-easily-synthesised network. In the changed coordinate basis, the Riccati equation solution, call it P, is of the form
$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{3.56}$$
with P_1 a readily computable matrix (in general 2×2,) and P_2 satisfying a lower dimmesion Riccati equation than P. The eigenvalues of the Hamiltonian matrix associated with the P equation are the union of $\pm j\omega_0$, perhaps

repeated, and the eigenvalues of the Hamiltonian matrix associated with the P_2 equation.

Thus in terms of the Riccati equation, *one step of the Brune synthesis procedure corresponds to changing the coordinate basis of the Riccati equation so that part of the solution can be explicitly calculated and pulled out, while the remaining part of the solution satisfies a reduced order Riccati equation, which will be "less" difficult, in that the associated Hamiltonian will have at least two fewer imaginary eigenvalues.*

For scalar positive real transfer functions, some more detail is as follows. Suppose that
$z(s) = \frac{1}{2}R + h_a^T(sI - F_a)^{-1}g_a$ with $\mathrm{Re}z(j\omega_0) = 0$. Let T_a be a nonsingular matrix with last two columns
$(\omega_0^2 I + F_a^2)^{-1}g_a, -F_a(\omega_0^2 I + F_a^2)^{-1}g_a$. Set
$F_b = T_aF_aT_a^{-1}, g_b = T_ag_a, h_b^T = h_a^TT_a^{-1}$. Compute

$$\begin{bmatrix} h_b^T(\omega_0^2 I + F_b^2)^{-1} \\ h_b^T(\omega_0^2 I + F_b^2)^{-1}F_b \end{bmatrix} = \begin{bmatrix} K_{12} & K_{22} \end{bmatrix} \tag{3.57}$$

where K_{22} is 2×2. Set

$$T_b^{-1} = \begin{bmatrix} I & 0 \\ -K_{22}^{-1}K_{12} & I \end{bmatrix} \tag{3.58}$$

where the $2-2$ block entry of T_b^{-1} is 2×2. Also set

$$F_c = T_bF_bT_b^{-1}, \; g_c = T_bg_b, \; h_c^T = h_b^TT_b^{-1}.$$

This ensures that for any $z(s)$, whether or not $\mathrm{Re}z(j\omega_0) = 0$, that

$$\begin{bmatrix} (\omega_0^2 I + F_c^2)^{-1}g_c & -F_c(\omega_0^2 I + F_c^2)^{-1}g_c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.59}$$

and

$$\begin{bmatrix} h_c^T(\omega_0^2 I + F_c^2)^{-1} \\ h_c^T(\omega_0^2 I + F_c^2)^{-1}F_c \end{bmatrix} = \begin{bmatrix} 0\cdots 0 & \\ & K_{22} \\ 0\cdots 0 & \end{bmatrix} \tag{3.60}$$

and if $\mathrm{Re}\; z(j\omega_0) = 0$ with $d/ds[\mathrm{Re}Z(s)] = 0$ at $s = j\omega_0$, there also holds

$$K_{22} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} \tag{3.61}$$

for some $\alpha, \beta \neq 0$. All solutions of the Riccati equation

$$P(F - gR^{-1}h^T) + (F - gR^{-1}h^T)^T P - PgR^{-1}g^T \times P - hR^{-1}h^T = 0 \tag{3.62}$$

are of the form

$$P = P_1 \dot{+} \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} \tag{3.63}$$

where P_1 satisfies an equation of smaller dimension:

$$P_1(F_1 - g_1 R^{-1}h_1^T) + (F_1 - g_1 R^{-1}h_1^T)^T P_1 - P_1 g_1 \times R^{-1}g_1^T P_1 - h_1 R^{-1}h_1^T = 0$$

and $z_1(s) = \frac{1}{2}R + h_1^T(sI - F_1)^{-1}g_1$ is a positive real transfer function, of lesser degree than $z(s)$.

4 Deflation and Difficult Riccati Equations

We return to (3.45) and (3.46) repeated for convenience with trivial change of notation as

$$PA + A^T P - PBB^T P + Q = 0 \tag{4.64}$$

$$\mathcal{H} = \begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \tag{4.65}$$

We shall also use the function

$$\Phi(j\omega) = I + B^T(-j\omega I - A^T)Q(j\omega I - A)^{-1}B \tag{4.66}$$

In the sequel, we shall have occasion to use some tranformations on (4.64) through (4.66). The transformations relate to replacing an optimization problem for $\dot{x} = Ax + Bu$ by one for $\dot{x} = (A - BB^T L)x + Bu$ for some symmetric L. Replacing (4.64) we have

$$\begin{aligned} P_L(A - BB^T L) + (A - BB^T L)^T P_L - P_L BB^T P_L \\ + (Q - LA - A^T L + LBB^T L) = 0 \end{aligned} \tag{4.67}$$

It is trivial to observe that P satisifies (4.64) if and only if $P_L = P - L$ satisfies (4.67). The "closed—loop" matrix in both cases is $A - BB^T P$ and so P is stabilizing or strong for (4.64) if and only if P_L is stabilizing or strong for (4.67). The Hamiltonian matrix associated with (4.67) is

$$\mathcal{H}_L = \begin{bmatrix} A - BB^T L & -BB^T \\ Q - LA - A^T L - LBB^T L & -A^T + BB^T L \end{bmatrix} \tag{4.68}$$

Observe that

$$\mathcal{H}_L = \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} \mathcal{H} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \tag{4.69}$$

and so $\mathcal{H}$ and $\mathcal{H}_L$ have the same eigenvalues. Next, we have

$$\begin{aligned}\Phi_L(j\omega) &= I + B^T(-j\omega I - A^T - LBB^T)^{-1} \\ &\quad \times (Q - LA - AA^T L + LBB^T L) \\ &\quad \times (j\omega I - A - BB^T L)^{-1} B && (4.70) \\ &= [I + B^T(-j\omega I - A^T)^{-1} LB]\Phi(j\omega) \\ &\quad \times [I + B^T L(j\omega I - A)^{-1} B] && (4.71)\end{aligned}$$

Thus at points ω_0 which are not poles or zeros of Φ and Φ_L we conclude that $\Phi(j\omega_0)$ is positive (nonnegative) definite if and only if $\Phi_L(\omega_0)$ is positive (nonnegative)definite.

If (A, B) is stabilizable, it is clear that without loss of generality, we can assume there exists L ensuring that $\Phi_L(j\omega)$ has no poles on the imaginary axis.

In this section, we shall restrict attention to Riccati equation (4.64) *for which* (A, B) *is stabilizable.* Without loss of generality therefore, in the remainder of this section, we shall make the assumption:

Assumption 4.1. *In the Riccati equation* (4.64), *the pair* (A, B) *is stabilizable and* A *has no eigenvalues on the imaginary axis.*

In this case, the following result is well known, see [3].

Theorem 4.1. *Let* A, B, Q *be matrices of compatible dimensions such that* $Q = Q^T$, *and assumption 1 holds. Define* Φ *as in* (4.66). *Then*

1. *The following statements are equivalent*

   (a) $\Phi(j\omega) > 0$ *for all* $0 \le \omega \le \infty$

   (b) There exists a unique stabilizing real solution $P_s = P_s^T$ *to* (4.64) *i.e.* $Re\lambda_i(A - BB^T P_s) < 0$

   (c) The Hamiltonian matrix $\mathcal{H}$ *has no* $j\omega$ *-axis eigenvalue*

2. *The following statements are equivalent*

   (a) $\Phi(j\omega) \ge 0$ *for all* $0 \le \omega \le \infty$

   (b) There exists a unique strong real solution $P_s = P_s^T$ *to* (4.64), *i.e.* $Re\lambda_i[A - BB^T P_s] \le 0$

Our principal concern in this section is of course with case 2 in the above theorem.

We now need to record several more results, which are an amalgam and development of ideas of [10] and [11].

Theorem 4.2. *Adopt the hypothesis of Theorem 4.1. Suppose that*

$$\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \Lambda \tag{4.72}$$

where the matrices are of compatible dimensions, Λ has dimension no greater than that of A and $\begin{bmatrix} X^T & Y^T \end{bmatrix}^T$ has full column rank. Suppose also that eigenvalues of $-\Lambda^*$ are controllable modes of (A, B), and that $X^*Y = Y^*X$. Then X has full column rank.

Proof. Let real ω_0 be chosen such that
$(j\omega_0 I - A)$, $(j\omega_0 I - \Lambda)$ and $\Phi(j\omega_0)$ are all nonsingular and $\Phi(j\omega_0) > 0$. (This is possible since $\lim_{\omega\to\infty} \Phi(j\omega) = I$, and does not rely on an assumption that
$\Phi(j\omega) \geq 0$ for all ω.) Suppose in order to obtain a contradiction that X does not have full column rank. Then there exists a nonzero α such that $X(j\omega_0 I - \Lambda)\alpha = 0$. From (4.72), we derive

$$(j\omega_0 I - A)X\alpha + BB^T Y\alpha = 0 \tag{4.73}$$

or

$$X\alpha = -(j\omega_0 I - A)^{-1} BB^T Y\alpha \tag{4.74}$$

□

Then from (4.72) we obtain

$$Q(j\omega_0 I - A)^{-1} BB^T Y\alpha - (j\omega_0 I + A^T)Y\alpha = Y(\Lambda - j\omega_0 I)\alpha$$

whence

$$\Phi(j\omega_0)B^T Y\alpha = B^T(-j\omega_0 I - A^T)^{-1} Y(\Lambda - j\omega_0 I)\alpha$$

Premultiplying by $\alpha^* Y^* B^T$ and using (4.73) again gives

$$\begin{aligned} \alpha^* Y^* B^T \Phi(j\omega_0) B^T Y\alpha &= \alpha^* X^* Y(\Lambda - j\omega_0 I)\alpha \\ &= \alpha^* Y^* X(\Lambda - j\omega_0 I)\alpha = 0 \end{aligned}$$

By the positive definiteness of $\Phi(j\omega_0)$, this yields

$$B^T Y\alpha = 0$$

and by (4.74)

$$X\alpha = 0$$

Hence the null space of X is $(j\omega_0 I - \Lambda)^{-1}$ invariant, and so there exists λ and $\beta \neq 0$ such that $X\beta = 0$ and

$$(j\omega_0 I - \Lambda)^{-1}\beta = (j\omega_0 - \lambda)^{-1}\beta$$

whence $\Lambda\beta = \lambda\beta$ and $X(j\omega_0 I - \Lambda)\beta = 0$. Repeating the above argument starting not from $X(j\omega_0 I - \Lambda)\alpha = 0$ but $X(j\omega_0 I - \Lambda)\beta = 0$ yields $B^T Y\beta = 0$. From $-QX - A^T Y = Y\Lambda$ and $\Lambda\beta = \lambda\beta$ follows $-A^T(Y\beta) = \lambda(Y\beta)$. Since $X\beta = 0, \beta \neq 0$ and $\begin{bmatrix} X^T & Y^T \end{bmatrix}^T$ is full rank, $Y\beta \neq 0$. The controllability hypothesis is then contradicted.

In reference to Theorem 4.1, our principal concern in this section is with case 2. Notice that in contrast to case 1, there is no statement (c). This deficiency is in fact remedied in the following theorem, which uses Theorem 4.2 in its proof.

Theorem 4.3. *Adopt the same hypotheses as Theorem 4.1. Then condition 2a and 2b of Theorem 4.1 are equivalent to*

1. (4.72) *holds with square* Λ

2.

$$Re\lambda_i(\Lambda) \leq 0 \tag{4.75}$$

3.

$$\begin{bmatrix} X^T & Y^T \end{bmatrix}^T \text{ is of full column rank} \tag{4.76}$$

4.

$$X^*Y = Y^*X \text{ with } YX^{-1} \text{ real if } X \text{ is nonsingular} \tag{4.77}$$

Proof. Suppose a strong real solution $P_s = P_s^T$ exists to (4.64). Choose $X = I$ and $Y = P_s$ and $\Lambda = A - BB^T P_s$. Then properties (1)–(4) are immediately verified. Conversely, suppose properties (1)–(4) hold. By stabilizability, eigenvalues of $-\Lambda^*$ are controllable modes of (A, B). By Theorem 4.2, X is nonsingular. Define

$$P_s = YX^{-1} \tag{4.78}$$

Then the first row of (4.72) yields $A - BB^T P_s = X\Lambda X^{-1}$ and by (4.75) we obtain that $\mathrm{Re}\lambda_i(A - BB^T P_s) \leq 0$.
Since
$A - BB^T P_s = X\Lambda X^{-1}, P_s A - P_s BB^T P_s = Y\Lambda X^{-1}$. The second row of (4.72) yields $-Q - A^T P_s = Y\Lambda X^{-1}$. The Riccati equation is immediate. From (4.77) and (4.78), we obtain that P_s is real and symmetric. □

The next result is more subtle. The motivation for it is as follows. One can regard solution procedures which obtain P_s as picking out eigenspaces of $\mathcal{H}$. If $\mathcal{H}$ has
$j\omega$-axis eigenvalues and $\Phi(j\omega) \geq 0$ with singularity at $\omega = \omega_0$, such eigenvalues are always of even multiplicity. If one eigenvector corresponding to $j\omega_0$ is picked out, it is then not a priori clear what eigenvector of the (at least two dimensional eigenspace) corresponding to eigenvalue $-j\omega_0$ should be picked out. (Though it is not obvious, a particular selection does need to be made.)

Theorem 4.4. *Adopt the hypotheses of Theorem 4.1. Suppose that $\Phi(j\omega) \geq 0$ and that*

$$\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = j\omega_0 \begin{bmatrix} x \\ y \end{bmatrix} \tag{4.79}$$

for some nonzero vector $\begin{bmatrix} x^T & y^T \end{bmatrix}^T$, ω_0 real and nonzero. Then

$$x^* y - y^* x = 0 \tag{4.80}$$

If (A, B) is stabilizable, Rex and Imx are independent vectors.

By hypothesis, A has no pure imaginary eigenvalue so $j\omega_0$ is not an eigenvalue of A. Then from (4.79)

$$x = -(j\omega_0 I - A)^{-1} BB^T y \tag{4.81}$$

and

$$y + (-j\omega_0 I - A^T)^{-1} Q (j\omega_0 I - A)^{-1} BB^T y = 0 \tag{4.82}$$

or

$$\Phi(j\omega_0) B^T y = 0$$

Since $\Phi(j\omega_0)$ is nonnegative, it follows that
$f(j\omega) = y^* B \Phi(j\omega) B^T y$ has a minimum at $\omega = \omega_0$. Setting the derivative $f'(j\omega_0)$ to zero gives

$$\begin{aligned} &y^* BB^T (-j\omega_0 I - A^T)^{-1} Q (j\omega_0 I - A)^{-2} j BB^T y \\ &+ y^* BB^T (-j\omega_0 I - A^T)^{-2} Q (j\omega_0 I - A)^{-1} \\ &\times (-j) BB^T y = 0 \end{aligned}$$

Simplifying with (4.82) gives

$$\begin{aligned} &- y^* (j\omega_0 I - A)^{-1} j BB^T y + y^* BB^T \\ &\times (-j\omega_0 I - A^T)^{-1} j y = 0 \end{aligned}$$

and from (4.81) there follows

$$x^*y - y^*x = 0$$

as required.

Obviously, (4.79) also implies

$$\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = -j\omega_0 \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

and consequently with $x = u + jv, y = m + jn, (u, v, m, n \text{ real})$

$$\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} u & v \\ m & n \end{bmatrix} = \begin{bmatrix} u & v \\ m & n \end{bmatrix} \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \tag{4.83}$$

Observe that since $\begin{bmatrix} x^T & y^T \end{bmatrix}^T$ and $\begin{bmatrix} \bar{x}^T & \bar{y}^T \end{bmatrix}^T$ correspond to different eigenvalues, they are independent vectors. Hence the matrix $\left[\begin{smallmatrix} u & v \\ m & n \end{smallmatrix}\right]$ has full column rank. Also, Theorem 4.4 implies that

$$(u^T - jv^T)(m + jn) - (m^T - jn^T)(u + jv) = 0$$

or

$$u^T n = v^T m$$

i.e. we have the following symmetry:

$$\begin{bmatrix} u & v \end{bmatrix}^T \begin{bmatrix} m & n \end{bmatrix} = \begin{bmatrix} m & n \end{bmatrix}^T \begin{bmatrix} u & v \end{bmatrix} \tag{4.84}$$

By Theorem 4.2, and using the stabilizability, it follows that $\begin{bmatrix} u & v \end{bmatrix}$ has full column rank.

We now have the basis for deflation of the problem of finding the strong solution P to (4.64), under a stabilizability assumption and nonnegativity of $\Phi(j\omega)$ assumption.

We are of course interested in deflating out the pure imaginary eigenvalues of $\mathcal{H}$.

Our starting point is the matrix of two columns:

$$\begin{bmatrix} X \\ \cdots \\ Y \end{bmatrix} = \begin{bmatrix} u & v \\ \cdots & \cdots \\ m & n \end{bmatrix} \tag{4.85}$$

Let S be a nonsingular matrix such that

$$SX = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.86}$$

Replace (4.64), (4.65) and (4.83) by

$$\begin{aligned}(S^{-T}PS^{-1})SAS^{-1} + (SAS^{-1})^T(S^{-T}PS^{-1})\\ - (S^{-T}PS^{-1})SBB^TS^T(S^{-T}PS^{-1})\\ + S^{-T}QS^{-1} = 0\end{aligned}$$

$$\begin{aligned}\mathcal{H}_s &= \begin{bmatrix} SAS^{-1} & -SBB^TS^T \\ -S^{-T}QS^{-1} & -(SAS^{-1})^T\end{bmatrix}\\ &= \begin{bmatrix} S & 0 \\ 0 & S^{-T}\end{bmatrix}\mathcal{H}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^T\end{bmatrix}\\ H_s\begin{bmatrix} SX \\ S^{-T}Y\end{bmatrix} &= \begin{bmatrix} SX \\ S^{-T}Y\end{bmatrix}\Omega\end{aligned} \tag{4.87}$$

with obvious definition of Ω.

The expression for Φ, if computed for $\bar{A} = SAS^{-1}$, $\bar{B} = SB$ etc remains unchanged. Obviously, all that is involved here is a change of basis.

It is crucial to observe at this point that (4.84) is equivalent to

$$X^TY = Y^TX$$

and accordingly after transformation, where $\bar{X} = SX, \bar{Y} = S^{-T}Y$,

$$\bar{X}^T\bar{Y} = \bar{Y}^T\bar{X}$$

or, with $\bar{Y} = [\,\bar{Y}_1^T \;\; \bar{Y}_2^T\,]^T$,

$$\bar{Y}_1 = \bar{Y}_1^T$$

Hence there exists a *symmetric* L such that

$$L\begin{bmatrix} I \\ 0\end{bmatrix} + \bar{Y} = 0 \tag{4.88}$$

with L computable from the original vectors u, v, m, n associated with the eigenvectors of $\mathcal{H}$. If $\bar{Y}_1$ is nonsingular, one possibility is

$$L = -\begin{bmatrix} I \\ \bar{Y}_2\bar{Y}_1^{-1}\end{bmatrix}\bar{Y}_1\,[I \quad \bar{Y}_1^{-1}\bar{Y}_2]$$

In light of (4.86) through (4.88), we now have

$$\begin{bmatrix} \bar{A} - \bar{B}\bar{B}^TL & -\bar{B}\bar{B}^T \\ -\bar{Q} + L\bar{A} + \bar{A}^TL - L\bar{B}\bar{B}^TL & -\bar{A}^T - L\bar{B}\bar{B}^T\end{bmatrix}\begin{bmatrix}\begin{bmatrix} I \\ 0\end{bmatrix} \\ 0\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix} I \\ 0\end{bmatrix} \\ 0\end{bmatrix}\begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0\end{bmatrix}$$

With $\tilde{A} = \bar{A} - \bar{B}\bar{B}^T L$ etc,

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & -\bar{B}_1\bar{B}_1^T & -\bar{B}_1\bar{B}_2^T \\ \tilde{A}_{21} & \tilde{A}_{22} & -\bar{B}_2\bar{B}_1^T & -\bar{B}_2\bar{B}_2^T \\ -\tilde{Q}_{11} & -\tilde{Q}_{21}^T & -\tilde{A}_{11}^T & -\tilde{A}_{21}^T \\ -\tilde{Q}_{21} & -\tilde{Q}_{22} & -\tilde{A}_{12}^T & -\tilde{A}_{22}^T \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Omega \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.89}$$

At once, we see that $\tilde{A}_{21} = 0, \tilde{Q}_{11} = 0$ and $\tilde{Q}_{21} = 0$.

Accordingly,

$$\Phi_L(j\omega) = I + \bar{B}_2^T(-j\omega I - \tilde{A}_{22}^T)^{-1}\tilde{Q}_{22}(j\omega I - \tilde{A}_{22})^{-1}\bar{B}_2 \tag{4.90}$$

and $(\tilde{A}_{22}, \bar{B}_2)$ is stabilizable because $(\bar{A}, \bar{B})$ is stabilizable.

Evidently, we have achieved a degree reduction. It is trivial to check that the Hamiltonian matrix in (4.89) after row and column permutation is

$$\begin{bmatrix} \tilde{A}_{11} & * & * & * \\ 0 & -\tilde{A}_{11}^T & * & * \\ 0 & 0 & \tilde{A}_{22} & -\bar{B}_2\bar{B}_2^T \\ 0 & 0 & -\tilde{Q}_{22} & -\tilde{A}_{22}^T \end{bmatrix}$$

which shows that the Hamiltonian matrix associated with the new form of Φ i.e. with $\tilde{A}_{22}, \bar{B}_2$ and $\tilde{Q}_{22}$ has the same eigenvalues as the original one, *excluding* two eigenvalues at $j\omega_0$ and $-j\omega_0$.

We also have the following key result.

Theorem 4.5. *With notation as above, there exists a strong solution $\tilde{P}_{s22}$ for the Riccati equation of the reduced order problem defined by $\tilde{A}_{22}, \bar{B}_2, \tilde{Q}_{22}$ with $\tilde{A}_{22}, \bar{B}_2$ stabilizable if and only if there exists a strong solution P_s of the Riccati equation for the original problem defined by A, B, Q. The solutions are related by*

$$P_s = S\left(L + \begin{bmatrix} 0_{2\times 2} & 0 \\ 0 & \tilde{P}_{s22} \end{bmatrix}\right) S^T \tag{4.91}$$

Proof. Suppose first that P_s exists and is strong. Then $\bar{P}_s = S^{-1}P_sS^{-T}$ is a strong solution for the Riccati equation for $\bar{A}, \bar{B}, \bar{Q}$. Then

$$\tilde{P}_s = S^{-1}P_sS^{-T} + L \tag{4.92}$$

is the strong solution of the Riccati equation for $\tilde{A}, \bar{B}$ and $\tilde{Q}$. Write this as

$$\tilde{P}_s \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{A}_{11}^T & 0 \\ \tilde{A}_{12}^T & \tilde{A}_{22}^T \end{bmatrix} \tilde{P}_s - \tilde{P}_s \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \times \begin{bmatrix} \bar{B}_1^T & \bar{B}_2^T \end{bmatrix} \tilde{P}_s + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix} = 0 \tag{4.93}$$

Because P_s exists and is strong, by Theorem 4.1 $\Phi(j\omega) \geq 0$ (with Φ computed as in (4.66)). Then $\Phi_L(j\omega) \geq 0$ with $(\tilde{A}_{22}, \bar{B}_2)$ stabilizable. Consequently, again by Theorem 4.1 there exists a strong solution to

$$\tilde{P}s_{22}\tilde{A}_{22} + \tilde{A}_{22}^T\tilde{P}s_{22} - \tilde{P}s_{22}\bar{B}_2\bar{B}_2^T\tilde{P}s_{22} + \tilde{Q}_{22} = 0 \tag{4.94}$$

It is trivial to see then that

$$\tilde{P}_s = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}s_{22} \end{bmatrix} \tag{4.95}$$

is a strong solution to (4.93). This solution is unique. Hence (4.91) follows for (4.92) and (4.95). The argument is clearly reversible. □

Let us now try to sum up how we deflate pure imaginary eigenvalues.

(a) We start with (4.64) and (4.65) in which (A, B) is stabilizable and $\Phi(j\omega)$ is nonnegative, but may have $j\omega$-axis poles

(b) to avoid possible confusion of $j\omega$ axis poles and zeros of $\Phi(j\omega)$, we find a symmetric L such that $A - BB^TL$ has no $j\omega$ - axis eigenvalues, and transform along the lines of (4.67) through (4.71). We now drop this L.

(c) Let $[x^T \quad y^T]$ be an eigenvector of $\mathcal{H}$ corresponding to eigenvalue $j\omega_0, 0 < \omega_0 < \infty$. Let $x = u + jv$, $y = m + jn$, with u, v, m, n real. With

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} u & v \\ m & n \end{bmatrix}$$

find S so that $SX = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and L symmetric so that

$$L\begin{bmatrix} I \\ 0 \end{bmatrix} + S^{-T}Y = 0$$

(d) Replace A by $\bar{A} = SAS^{-1}$, B by $\bar{B} = SB, Q$ by $\bar{Q} = S^{-T}QS^{-1}$ and then $\bar{A}$ by $\tilde{A} = \bar{A} - \bar{B}\bar{B}^T\bar{L}$, and $\bar{Q}$ by $\tilde{Q} = \bar{Q} - L\bar{A} - \bar{A}^TL + L\bar{B}\bar{B}^TL$

(e) The new Hamiltonian matrix can be replaced by a smaller one associated with $\tilde{A}_{22}, \bar{B}_2, \tilde{Q}_{22}$ and

$$P_s = S\left(L + \begin{bmatrix} 0_{2\times 2} & 0 \\ 0 & \tilde{P}_{s22} \end{bmatrix}\right)S^T$$

relates a strong solution for the associated smaller dimension Riccati equation to the strong solution of the original Riccati equation.

Above, we have described how to deal with a pure imaginary $j\omega$ axis eigenvalue. Dealing with an eigenvalue at the origin is easier.

Suppose that for real x and y,

$$\begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

We change the coordinate basis so that $\bar{x} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ and choose a symmetric L such that

$$L \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \bar{y} = 0$$

(where $\bar{y}$ is the result of transforming y through the coordinate basis change). Then as a replacement to (4.89), we have

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & -\bar{B}_1\bar{B}_1^T & -\bar{B}_1\bar{B}_2^T \\ \tilde{A}_{21} & \tilde{A}_{22} & -\bar{B}_2\bar{B}_1^T & -\bar{B}_2\bar{B}_2^T \\ -\tilde{Q}_{11} & -\tilde{Q}_{12} & -\tilde{A}_{11}^T & -\tilde{A}_{21}^T \\ -\tilde{Q}_{12}^T & -\tilde{Q}_{22} & -\tilde{A}_{12}^T & -\tilde{A}_{22}^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Evidently the first column is zero, Φ_L is now given by (4.90), the effective Hamiltonian matrix has dimension 2 less than the original one, with the same eigenvalues apart from two eigenvalues at the origin.

5 Concluding Remark

Let us note several directions in which these ideas could be extended. First, suppose that the Hamiltonian matrix has an eigenvalue of $j\omega_0$, ω_0 real, of multiplicity $2r$ for $r > 1$. Is there some straightforward procedure for deflating not in r steps, but in one? Second, any $j\omega$-axis eigenvalue of the Hamiltonian matrix is multiple. This means that numerical problems can arise in finding the eigenvalue. How best may these be addressed? Finally, in H_∞ problems, it may be the case that the infimum closed—loop gain gives rise to a Riccati equation with indefinite quadratic term and Hamiltonian matrix with pure imaginary eigenvalues. Variation on the deflation procedure to deal with this situation would be welcome.

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