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Open Problems in Mathematical Systems and Control Theory

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Positive system realizations

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1 Description of the problem

Consider a rational discrete-time transfer function $H(z)$ with the property that the associated causal impulse response $h(k)$ is nonnegative for all k. A positive realization is a quadruple A, b, c, d such that

$$H(z) = d + c(zI - A)^{-1}b$$

and all entries of A, b, c, d are nonnegative.
Easily checkable necessary and sufficient conditions for the existence of such A, b, c, d are known, see [1, 2, 3]. The conditions require checking of the poles of $H(z)$, and a sufficient condition is that $H(z)$ have a single pole of maximum modulus. There are rational $H(z)$ with nonnegative $h(k)$ which have no positive realization, but they are in a sense nongeneric. A number of open problems relating to realization however remain:

1. How may one straightforwardly determine the least dimension of any positive realization? In particular, when is this dimension equal to the McMillan degree of $H(z)$?

2. How are positive realizations of least dimension related? In particular, apart from reordering of the states, is it true or not that the set of least dimension positive realizations is a connected set?

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3. For an $n-th$ order $H(z)$ for which the least dimension of positive realizations is N, what is the maximum number (generically) of zeros entries that can be included in $\{A, b, c\}$? In particular, can canonical forms for positive realizations be found?

4. How can one approximate a high order positive realization by a low order positive realization where the measure of approximation reflects either transfer function approximation error on the unit circle, or impulse response approximation error.

2 History of the problem

The problem of positive realization has been around for some time; significant progress towards a solution was made in [4], where the existence of a positive realization was cast in terms of the existence of a polyhedral cone with certain properties. Knowledge of a cone almost immediately yields a solution of the construction problem.
References [1, 2, 3], gives conditions on the poles of $H(z)$ for the existence of the cone. Two examples in the survey [5] are:

$$\begin{aligned} H(z) &= -\frac{1}{z+0.8} + \frac{1}{z-0.8} + \frac{1}{z+0.5} \\ h(2k) &= 2[(0.8)^{2k-1}] - (0.5)^{2k-1} \quad k = 1, 2\ldots \\ h(2k+1) &= (0.5)^{2k} \quad k = 1, 2\ldots \end{aligned}$$

No 3rd order positive realization exists, but a 4th order one does. Second, for the transfer function/impulse response pair

$$H(z) = \frac{1}{2} = \left[\frac{\frac{1}{2}}{z-\frac{1}{2}} - \frac{z(\frac{1}{2}\cos 2) - \frac{1}{4}}{z^2 - z\cos 2 + \frac{1}{4}}\right]$$

$$h(k) = \left(\frac{1}{2}\right)^k sin^2 k$$

no (finite dimensional) positive realization exists.
Reference [6] has the example

$$H(z) = \frac{1}{z-1} - 25\frac{(0.4)^{4-N}}{z-0.4} + 75\frac{0.2^{4-N}}{z-0.2}$$

for which the least dimension of a positive realization is N, assuming $N \geq 4$. These examples perhaps give insight into the subtle nature of the problem.

3 Motivations

Questions 1 and 2 are clearly of a basic systems theory nature, and are of interest in their own right. Additionally, there is a technology termed "charge-routing networks" [7] which is available to realize a positive realization of a digital filter transfer function. Questions 1 and 2 are relevant in selecting a way to realize a positive realization. Lest it be thought that the technology is of too limited applicability (as digital transfer functions are perhaps unlikely to have nonnegative impulse responses), we comment that any transfer function $H(z)$ can be expressed as

$$H(z) = H_1(z) - H_2(z)$$

where H_1 and H_2 have nonnegative impulse responses, H_1 has degree one more than H and H_2 has degree 1.

Questions 2 and 3 are relevant in considering problems of identification, especially of compartmental systems, which are associated with positive realizations, [8].

Question 4 is motivated by a related but even harder problem of great practical significance: how may one approximate a hidden Markov model having a large number of states by a hidden Markov model with a small number of states?

The connection between hidden Markov models and positive systems is explained in, for example, [5].

4 References

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