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a tribute to Thomas Kailath

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BUILDING SPECIAL LINEAR SYSTEM REALIZATIONS OF SPECIAL TRANSFER FUNCTIONS

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This paper is dedicated to Professor Thomas Kailath, in thanks for the inspiration provided from his papers and books over three decades.

ABSTRACT

Linear system realization problems require the construction of a linear system realization, or a quadruple of real matrices F, G, H, J, of a given real rational $W(s)$, so that $W(s) = J + H'(sI - F)^{-1}G$. When $W(s)$ has certain properties, corresponding to symmetry, passivity, or losslessness, it is possible to select a realization that closely (but not necessarily exactly) reflects these properties. We review such possibilities, then consider a new one, initially for discrete time systems: if the impulse response associated with W is nonnegative, can the entries of the realizing matrices be chosen to be nonnegative also? Necessary and barely differing sufficient conditions are presented for this to be the case.

1 INTRODUCTION

Linear system theory results of two or three decades ago are often focussed on the question of relating rational transfer function matrices, and state-variable realizations of these transfer function matrices. Thus with $W(s)$ a real rational matrix in s, normally with $W(\infty)$ finite, a quadruple of real matrices $\{F, G, H, J\}$ is a realization of $W(s)$ when

$$W(s) = J + H'(sI - F)^{-1}G \tag{18.1}$$

Of course, the underlying state-variable equations are

$$\dot{x} = Fx + Gu \qquad y = H'x + Ju \tag{18.2}$$

and the Laplace transforms $U(s)$ and $Y(s)$ of $u(.)$ and $y(.)$ [with zero initial conditions] are related by $Y(s) = W(s)U(s)$.

Given F, G, H, and J, it is obvious that $W(s)$ is uniquely defined. Given $W(s)$, there are an infinity of possible quadruples $\{F, G, H, J\}$. The investigation of the one-to-many mapping $W(s) \rightarrow \{F, G, H, J\}$ has generated a wealth of results. One group of results focuses on minimal dimension realizations, viz those for which F, which is square, has least dimension. Here, there are elegant answers to questions like: how can the minimal dimension be characterized in terms of $W(s)$? How are minimal dimension realizations related? How can minimal dimension realizations be found? What minimal parametrizations for minimal dimension realizations can be found? These sorts of questions are extensively treated in [1].

Other questions stem from trying to carry over certain properties which $W(s)$ may have to similar properties of the associated realization. For example, we can ask what can be said about the realization, or existence of special realizations, in case W has symmetry, passivity or losslessness properties, eg

$$W(s) = W'(s) \tag{18.3}$$

$$W(s) + W'(-s) \geq 0 \quad \text{or} \quad I - W'(-s)W(s) \geq 0 \quad \text{for } Re[s] \geq 0 \tag{18.4}$$

$$W(s) + W'(-s) = 0 \quad \text{or} \quad W'(-s)W(s) = I \tag{18.5}$$

The properties of the transfer function matrix exhibited in (18.3) through (18.5) often arise in physical systems, for reasons explained by physics. Thus the symmetry condition (18.3) has its roots in observations of Maxwell on the behaviour of electromagnetic systems, and of structural systems [2,3]. The condition (18.4) can be linked to results of Boltzmann, Johnson and Nyquist on noise in electrical networks [4] (more generally encapsulated by physicists in the fluctuation-dissipation theorem [5]). Physical systems typically have internal structure, and realizations have to deal with internal structure. Properties of special realizations of $W(s)$ satisfying one of (18.3) through (18.5) will often reflect what physics has to say about the internal structure of physical systems. In the next section, we will reveal the properties of realizations $\{F, G, H, J\}$ that are securable in the light of one of (18.3) through (18.5). The third section considers a further input-output property and the question of how it can be reflected in realizations. In particular, we shall suppose that $W(s)$ has a nonnegative impulse response. We shall report on recent results for this problem. They should not be viewed in isolation, but as further results along the lines of those summarized in section 2.

2 SYMMETRY AND LIKE PROPERTIES

2.1 Symmetric $W(s)$

Let $W(s) = W'(s)$. Suppose that $\{F, G, H, J\}$ is a minimal realization. From (18.1) it follows that $\{F', G, H, J'\}$ is a minimal realization. Then from a standard theorem connecting minimal realizations [1] there exists a unique nonsingular T such that

$$TFT^{-1} = F' \qquad TG = H \qquad H'T^{-1} = G \qquad J = J' \tag{18.6}$$

It is easily checked that these equations imply the same equations with T replaced by T' and thus $T = T'$. Writing $T = V'\sum V$ where V is nonsingular and $\sum = diag[I_m, -I_p]$ and setting $F_o = VFV^{-1}$, $G_o = VG$, $H_o' = H'V^{-1}$ gives a minimal realization $\{F_o, G_o, H_o, J\}$ where

$$\begin{bmatrix} \sum & O \\ O & I \end{bmatrix} \begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix} = \begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix}' \begin{bmatrix} \sum & O \\ O & I \end{bmatrix} \tag{18.7}$$

This is the nearest one can get to a symmetric realization. The integer $m-p$ is the Cauchy index of $W(s)$, [6]. When a network of inductors, capacitors, transformers and resistors is modelled with state-vector corresponding to inductor currents and capacitor voltages, the associated realization obeys (18.7), with m, p corresponding to the numbers of the two types of energy storage elements.

2.2 Passivity

Equations (18.4) are two ways of describing passivity: consider a network with m ports, and with port voltage and current vectors v and i. Let $W(s)$ be the transfer function taking currents to voltages. If for all $i(.)$,

$$\int_{t_o}^{\infty} v'idt \geq 0 \tag{18.8}$$

whenever $i(t) = 0$ for $t \leq t_o$ and the network is unexcited at the time $t = t_o$, the network is passive and the first of (18.4) holds, [7]. The other inequality follows if excitation and responses are defined a little differently, using incident and reflected waves, an idea which is very common in microwave circuits [8] and in digital filtering where the transfer function is a discrete time entity (z-transform) [9]. The main result [10] is that a minimal realization $\{F_o, G_o, H_o, J\}$ of a transfer function matrix satisfying the first (18.4) can always be found so that

$$\begin{bmatrix} -F_o & G_o \\ -H_o' & J \end{bmatrix} + \begin{bmatrix} -F_o & G_o \\ -H_o' & J \end{bmatrix}' \geq 0 \tag{18.9}$$

The reflecting of the second (18.4) in a realization is more transparent when the corresponding discrete-time problem is considered. In discrete-time, one is concerned with underlying state-variable difference equations

$$x_{k+1} = Fx_k + Gu_k \qquad y_k = H'x_k + Ju_k \tag{18.10}$$

and when z-transforms $U(z)$ and $Y(z)$ are given by $U(z) = \sum u_k z^{-k}$, $Y(z) = \sum y_k z^{-k}$,there holds $Y(z) = W(z)U(z)$ where

$$W(z) = J + H'(zI - F)^{-1}G \tag{18.11}$$

Replacing the second of (18.4) there is

$$I - W'(z^{-1})W(z) \geq 0 \qquad \text{for } |z| \geq 1$$

and the main result [10] is that there exists a minimal realization

$$\{F_o, G_o, H_o, J_o\}$$

such that

$$I - \begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix}' \begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix} \geq 0 \tag{18.12}$$

2.3 Losslessness

In lossless systems inequality (18.8), under the constraint that $i(.) \in L_2$, is replaced by equality, and the first equality of (18.5) holds. Then (18.9) holds with equality. In contrast to the proof of (18.9), which is difficult, one can proceed by noting that if $\{F, G, H, J\}$ is a minimal realization, $W'(-s) = W(s)$ implies that $\{-F', H, -G, -J'\}$ is also a minimal realization. This yields existence of $\{F_o, G_o, H_o, J_o\}$ such that

$$\begin{bmatrix} -F_o & G_o \\ -H_o' & J \end{bmatrix} + \begin{bmatrix} -F_o & G_o \\ -H_o' & J \end{bmatrix}' = 0 \tag{18.13}$$

Likewise, if $W'(z^{-1})W(z) = I$, there exists minimal $\{F_0, G_0, H_0, J\}$ with

$$\begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix}' \begin{bmatrix} F_o & G_o \\ H_o' & J \end{bmatrix} = I \tag{18.14}$$

3 NONNEGATIVE REALIZATION

Nonnegative impulse responses arise in a number of applications area, including (linear) compartmental systems [11,12] and charge-routing networks, [13]. A type of generalization arises in the area of hidden Markov models [14]. We pose the basic question in discrete time and for simplicity, but with no loss of generality, for scalar transfer functions. Let $H(z)$ be a rational transfer function with $H(z) = \sum_{k\geq 1} h_k z^{-k}$. Suppose further that the impulse response sequence h_k is nonnegative for all k. Realizations of $H(z)$ are triples $\{F, g, h\}$ for which $H(z) = h'(zI - F)^{-1}g$, and h_k is related to F, g, h by $h_k = h'F^{k-1}g$. We would like the nonnegativity of the impulse response to be reflected in the state-variable realization. Specifically we ask: **when does there exist a nonnegative realization, ie a triple** $A \in R_+^{N\times N}, b \in R_+^N, c \in R_+^N$ **for which**

$$H(z) = c'(zI - A)^{-1}b \tag{18.15}$$

[Here $R_+^N, R_+^{N\times N}$ denote N-vectors and $N \times N$ matrices with all entries nonnegative]. In a number of ways, this is a more difficult question than the ones considered in section 2. It is for example known that: 1. if $H(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are coprime polynominals and $q(z)$ has degree n, there may exist no nonnegative realizations with $N = n$, ie no minimal (in the linear systems sense) realization with the nonnegativity property, but there may exist a nonnegative realization with $N > n$, [15]. 2. Some rational $H(z)$ with nonnegative impulse response have no nonnegative realizations of any dimension, [16], eg

$$H(z) = \frac{1}{2}\left[\frac{\frac{1}{2}}{z - \frac{1}{2}} - \frac{z\left(\frac{1}{2}\cos 2\right) - \frac{1}{4}}{z^2 - z\cos 2 + \frac{1}{4}}\right] \tag{18.16}$$

for which

$$h_k = \left(\tfrac{1}{2}\right)^k sin^2 k. \tag{18.17}$$

Evidently, playing around with minimal dimension realizations alone (useful as they are with all their special properties) is unlikely to yield a solution to the problem. Nevertheless, it is possible to reformulate the problem using a minimal realization, [17]

Theorem 18.1 *Let $H(z)$ be a rational transfer function with minimal realization F, g, h of dimension n. Let $\mathcal{R}$ denote the cone spanned by g, Fg, F^2g, ie. the set $\{\sum_{i=0}^{\infty} \alpha_i F^i g | \alpha_i \geq 0\}$. If $H(z)$ has a nonnegative realization, there exists a (finite dimensional) matrix P such that [with $\mathcal{P}$ the cone generated by the columns of P and $\mathcal{P}^*$ the dual cone]*[1]

$$\mathcal{R} \subset \mathcal{P} \quad \text{and} \quad \mathcal{F}\mathcal{P} \subset \mathcal{P} \quad \text{and} \quad h \in \mathcal{P}^* \tag{18.18}$$

Moreover if for some $n \times N$ matrix P, with $\mathcal{P}$ = cone P, the equations (18.18) hold, there exists a nonnegative realization $\{A, b, c\}$ of $H(z)$ with $A \in R_+^{N\times N}$, and b, $c \in R_+^N$.

Theorem 18.1 in effect replaces the original problem by one concerning the existence of a cone.

[1]Let $\mathcal{X}$ be a cone. Then the dual cone $\mathcal{X}^*$ is defined by $\{y | x'y \geq 0 \ \forall x \in \mathcal{X}\}$

3.1 Simplifying Observations

To make progress in solving the reformulated problem of existence and computation of P, several simplifying observations are helpful. The first is that without loss of generality, one can work with a special minimal realization of $H(z)$. Suppose that with coprime numerator and denominator

$$H(z) = \frac{p_1 z^{n-1} + \cdots + p_n}{z^n + q_1 z^{n-1} + \cdots + q_n}. \tag{18.19}$$

Then one can take

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_n & -q_{n-1} & -q_{n-2} & \cdots & -q_1 \end{bmatrix} \quad g = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \quad h' = \begin{bmatrix} 1\,0 \cdots 0 \end{bmatrix}. \tag{18.20}$$

With $h_i = h'F^{i-1}g$ the i-th impulse response coefficient of $H(z)$, and

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ h_n & h_{n+1} & h_{n+2} & \cdots \end{bmatrix}. \tag{18.21}$$

it follows that $H(z)$ has a nonnegative realization if and only for some N there exists an $n \times N$ matrix P for which

$$\text{Cone}(H_n) \subset \mathcal{P} \tag{18.22a}$$

$$F\mathcal{P} \subset \mathcal{P} \tag{18.22b}$$

$$\textit{The first row of } P \textit{ has nonnegative entries.} \tag{18.22c}$$

The second observation is

Lemma 18.1 *Let $H(z)$ be a rational n-th order transfer function with nonnegative impulse response. Then $H(z)$ has a nonnegative realization if and only if $H_\alpha(z) = H(\alpha z)$ has a nonnegative realization for any positive α.*

The third observation is linked to the Perron-Frobenius theorem [18]. Any nonnegative matrix has an eigenvalue of maximum modulus which is positive. So if $H(z) = c'(zI - A)^{-1}b$, it is clearly possible for this eigenvalue of A to be a pole of $H(z)$. Because the dimension of A in general may exceed the degree of the denominator of $H(z)$, the poles of $H(z)$ in general will be a subset of the eigenvalue set of A. A nontrivial and highly important result is

Theorem 18.2 *Let $H(z)$ be a rational n-th order transfer function with nonnegative impulse response and suppose it possesses a nonnegative realization $\{A, b, c\}$. Then it also possesses a nonnegative realization of $\{\bar{A}, \bar{b}, \bar{c}\}$ of lesser or equal dimension such that among the eigenvalues of $\bar{A}$ of maximum modulus, which necessarily contain a positive real eigenvalue $\bar{\lambda}, \bar{\lambda}$ is a pole of $H(z)$.*

There is an immediate new necessary condition on H(z) with nonnegative impulse response for it to have a nonnegative realization.

Corollary 18.1 *Let $H(z)$ be a rational transfer function with nonnegative impulse response. A necessary condition for $H(z)$ to have a nonnegative realization is that the poles of $H(z)$ of maximum modulus be a subset of those which are the allowed eigenvalues of maximum modulus of a nonnegative matrix.*

This explains the earlier example with $h_k = (\frac{1}{2})^k sin^2 k$. The poles of $H(z)$ are $\frac{1}{2}$, $\frac{1}{2}exp(\pm 2j)$. If there were a nonnegative realization A, b, c, there would be one with $\frac{1}{2}$ as the maximum modulus of the eigenvalue of A. Then results on nonnegative matrices [18] rule out A having an eigenvalue of $\frac{1}{2}exp(\pm 2j)$, no matter what its dimension.

3.2 Main Result

It turns out that the necessary condition of Corollary 18.1 is almost a sufficient condition for nonnegative realizability [16].

Theorem 18.3 *Let $H(z)$ as given by (18.19) have a nonnegative impulse response. Suppose that*

a) $H(z)$ has one single pole of maximum modulus, which is positive real and may be multiple; or

b) all poles of $H(z)$ of maximum modulus are simple, one is positive real, and all are located at angles corresponding to one or more integer roots of unity; further, if $\bar{\lambda}$ is the magnitude of the maximum modulus poles, $\lim_{k\to\infty} \inf \bar{\lambda}^{-k} h_k > 0$.

Then there exists an $n \times N$ matrix P such that (3.8) holds, and accordingly $H(z)$ has a nonnegative realization.

3.3 Continuous-Time Systems

Consider a transfer function $H(s) = h'(sI - F)^{-1}g$ for which the impulse response $h(t) = h'e^{Ft}g \; for \; t \geq 0$ is nonnegative. The interest is in finding triples A, b, c for which $h(t) = c'e^{At}b$ where all entries of b and c and **off-diagonal** entries of A are nonnegative. The absence of restriction on the diagonal entries of A is standard and motivated in applications. The main result is as follows

Theorem 18.4 *Let* $H(z) = \frac{\sum_{i=1}^{n} p_i s^{n-i}}{s^n + \sum_{i=1}^{n} q_i s^{n-i}}$ *be a rational n-th order transfer function corresponding to an impulse response* $h(\cdot)$, *with* $h(t) \geq 0$ *for all* $t \geq 0$. *Necessary and sufficient conditions for* $H(s)$ *to have a nonnegative realization are that there is a unique (possibly multiple) pole of* $H(s)$ *with maximal real part, and the pole is real; and with* $H(s) = h'(sI - F)^{-1}g$, *there exists* $\lambda > 0$ *such that* $H_{d\lambda}(z) = h'(zI - F - \lambda I)^{-1}g$ *has a nonnegative (discrete-time) impulse response.*

4 CONCLUSION

The main result of this paper is concerned with giving a construction of a nonnegative realization of a transfer function with nonnegative impulse response. As such, it is one result of a number concerned with carrying over an input-output property (ie property of a transfer function matrix or impulse response) to a property of an associated state-variable realization. In contrast to most such results however, realizations which are minimal in the linear systems theory sense may not be relevant.

A number of questions are left hanging. For example, how may the minimal dimension under the nonnegativity constraint, call it N_+, be determined? What is the relation among two nonnegative realizations of dimension N_+? How may the set of all nonnegative realizations of dimension N_+ be characterized?

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