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Multiplier Theory and Operator Square Roots: Application to Robust and Time-Varying Stability

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Abstract. This paper considers the extension of a number of passive multiplier theory based results, previously known only for linear time invariant scalar systems, to time varying and multivariable settings. The extensions obtained here have important applications to the stability of both adaptive systems and linear systems in general. We demonstrate in this paper that at the heart of the extensions carried out here lies the result that if a stable multivariable and/or linear time varying system is stable under all scalar constant, positive feedback gains, then it has a well defined square root. The existence of this square root is demonstrated through a constructive Newton-Raphson based algorithm. The extensions provided here (dealing with robust stability and introduction of time-varying gains) though different in form from their linear time invariant scalar counterparts, do recover these as a special case.

1. Introduction and Problem Motivation. This paper is concerned with finding time-varying, multivariable generalizations of some multiplier theory results involving Strictly Positive Real (SPR) functions.

The following is a well known result in linear systems theory [1]. Consider an asymptotically stable linear time invariant (LTI), single input single output (SISO) system with a strictly proper transfer function $H(s)$. Then the system in Fig. 1 is asymptotically stable for all

$$0 \leq k \leq 1, \tag{1.1}$$

if, and only if, there exists a SPR scalar operator Z(s), such that

$$Z(s)(1 + H(s)) \tag{1.2}$$

is SPR. The concept of a SPR operator is defined as follows.

Definition 1.1. *A real, square matrix transfer function Z(s) is Positive Real (PR) if:*

1. *Z(s) is analytic in the right half plane; and*
2. *for all* $Re[s] \geq 0$, $Z(s) + Z^H(s) \geq 0$ *where the superscript H denotes the Hermitian transpose.*

We say Z(s) is SPR if for some $\alpha > 0$, $Z(s - \alpha)$ *is PR.*

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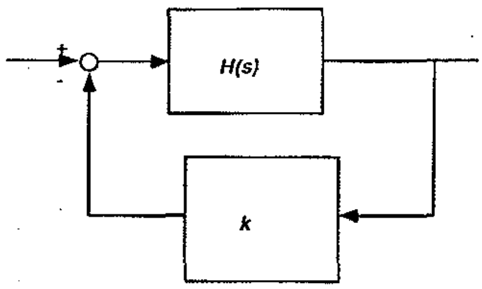

FIG. 1. *A closed-loop configuration*

We note that strictly speaking this result as presented in [1] allows for the presence of simple purely imaginary poles in both $H(s)$ and the closed loop above. In such a case all appearances of SPR operators in the statement of the result of [1], must be replaced by PR operators. For the purposes of this paper, however, only the case concerning asymptotic stability is relevant, and imaginary axis poles are precluded.

From this result spring a number of other important results of which two are cited below: The first states that two scalar polynomials of equal degree $p_1(s)$ and $p_2(s)$, have the property that $p_1(s) + kp_2(s)$ is Hurwitz (i.e. has roots in the open left half plane) for all k as in (1.1) iff there exists an asymptotically stable minimum phase $G(s)$, such that $G(s)(p_1(s) + kp_2(s))$ is Strictly Positive Real (SPR) for all k as in (1.1); in turn, this holds iff there exists an asymptotically stable minimum phase $G(s)$ such that $G(s)p_1(s)$ and $G(s)(p_1(s)+p_2(s))$ are SPR. As will be evident in a later section of this paper, this has an important application in certain adaptive systems problems involving a single unknown parameter. An alternative way of viewing this result is that every convex combination of two monic polynomials with the same degree is Hurwitz iff there exists a *single* stable minimum phase operator whose product with every such convex combination is SPR (see [2] which in fact considers the more general case of convex combinations of more than two polynomials).

The second result concerns the stability of a class of linear time varying (LTV) systems. Specifically, suppose that the configuration in Fig. 1 is stable with *a degree of stability* α for all k as in (1.1). Now consider the LTV systems obtained in Fig. 1, when the feedback gain $k(t)$ is allowed to be time varying while obeying

$$0 < k(t) < 1. \tag{1.3}$$

Then it has been shown in [3, 4] that the closed loop retains stability whenever, there exist T and $\delta \in (0, \alpha)$ for which

$$\sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \left[\frac{d}{d\tau} ln \frac{k(\tau)}{1-k(\tau)} \right]^+ d\tau < 2(\alpha - \delta) \tag{1.4}$$

where

$$[a]^+ = \begin{cases} a; & a \geq 0 \\ 0; & a < 0 \end{cases}$$

See [5] for an association between the result of [1] and that of [3, 4], using tools that include the Popov-Kalman-Yakubovic (PKY) Lemma.

The question addressed in this paper is: *to what extent do these results extend to systems that are MIMO LTI or for that matter LTV*. The ability to answer this question depends critically on the existence of the square root of certain MIMO and/or LTV systems. This can be understood by noting that the result of [1] can itself be viewed in the following terms. The stability of the closed loop of Fig. 1 for all $k \in [0, 1]$ is equivalent to the transfer function $1 + H(s)$ having a phase function that lies in $(-\pi, \pi)$, [6]. Accordingly, there is a well defined square root of $1 + H(s)$. Then a $Z(s)$ chosen as a suitable approximant of the inverse of this square root will be SPR, as indeed will be the product in (1.2).

Having dispensed with some preliminaries in Section 2, the first question we ask concerns square, MIMO and/or LTV, strictly causal continuous operators H such that both H and $[I + kH]^{-1}$ are stable for all k as in (1.1). Observe, that this corresponds to a stable closed loop of the form of Fig. 1, with the MIMO or LTV operator H occupying the position of $H(s)$. Then, using a Newton-Raphson technique, we explain in Section 3, that in such a case $I + H$ does indeed have a square root. Of course H is presumed to be the operator relating inputs and outputs of a strictly causal system. Stability corresponds to the boundedness of the operator given suitable input and output norms.

Sections 4 and 5 respectively provide the analogs of the result of [1] and its first consequence mentioned in the foregoing. Both these results assume that H is finite dimensional, i.e. has a finite dimensional state variable description. Section 5 also discusses the application of this latter result to certain types of adaptive identification algorithms involving MIMO, LTI systems. The results of this Section also resolve an open problem presented in [7]. Section 6 states the analog of the [3, 4] result. Each of the results in Sections 4 through 6, though different from their SISO, LTI counterparts, capture these as special cases. Section 7 contains conclusions.

2. Preliminaries. In this Section we make precise the general framework of this paper by presenting some definitions and assumptions. All systems in this paper will be represented by square, possibly MIMO or LTV, real, continuous operators mapping L_2 to L_2. Consider such an operator G. Then possibly MIMO or G^a will denote the *Adjoint* of G, i.e. if G has impulse response $g(t, \tau)$ then G^a has the impulse response $g'(\tau, t)$. For an input signal $x(t)$, Gx will denote the corresponding output, i.e. if $g(t, \tau)$ is the impulse response of G then

$$[Gx](t) = \int_{-\infty}^{\infty} g(t, \tau)x(\tau)d\tau. \tag{2.1}$$

This operator is *causal* if $g(t,\tau) = 0 \;\; \forall t < \tau$. In this case the upper limit in the integral of (2.1) can be replaced by t. The *inner product* between two signals $x(t)$ and $y(t)$ will be

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x'(t)y(t)dt \tag{2.2}$$

and the *norm* of a signal $x(t)$ will denote the L_2-norm

$$\|x\| = \sqrt{\langle x, x\rangle}. \tag{2.3}$$

The norm of G will be the induced L_2 operator norm

$$\|G\| = \sup_{\|x\|=1} Gx. \tag{2.4}$$

In the sequel we will use the terms bounded and stable interchangeably to signify operators that have a finite norm. Moreover, the operator G^n for a positive integer n will designate the combined operator obtained by a cascade of n operators G. A bounded operator $R : L_2 \to L_2$ will be called the inverse of G if $GR = RG = I$. In such a case we denote $R = G^{-1}$ and note that the existence of G^{-1} automatically signifies its stability. Further, G will be called *symmetric* or *self adjoint* if

$$G^a = G. \tag{2.5}$$

Every symmetric operator G can in turn be expressed as:

$$G = G_c + G_{ac} \tag{2.6}$$

where G_c is causal and called the *causal part* of G; G_{ac} is anticausal and called the *anticausal part* of G; and together they obey

$$G_c^a = G_{ac}. \tag{2.7}$$

If a term such as αI appears in G then it will be shared equally between G_c and G_{ac}; i.e. each of G_c and G_{ac} will get $0.5\alpha I$.

While the results of Section 3 rely only on an operator based description of the underlying systems, those of the subsequent Sections do require that some of these operators have a State Variable Description. In either case certain stability and existence assumptions are made. The first is on the operator based description of the open loop system H mentioned in the Introduction.

ASSUMPTION 2.1. *The operator $H : L_2 \to L_2$ is causal and $[I + kH]^{-1} : L_2 \to L_2$ is invertible and causal for all $k \in [0,1]$. Further, there exist numbers M_1 and M_2 such that:*

$$\|[I + kH]^{-1}\| \leq M_1 \qquad \forall k \in [0,1]; \tag{2.8}$$

and

$$\|H\| \leq M_2. \tag{2.9}$$

Moreover, the impulse response $h(t,\tau)$ of H is finite for all finite t and τ.
REMARK 2.2. *The boundedness assumption on $h(t,\tau)$ precludes the presence of impulse functions in $h(t,\tau)$, much like the strict properness assumption in the LTI case. In actual fact all the results derived here remain valid, if one permits the presence of terms like $\delta(t-\tau)$ in $h(t,\tau)$, as long as $[I+kH]^{-1}$ exists and is causal $\forall k \in [0,1]$. This boundedness assumption is made as it considerably simplifies the notation in Section 6.*
REMARK 2.3. *Under these conditions both H and the feedback loop of Fig. 1 are Bounded Input, Bounded Output (BIBO) stable.*

To provide the assumption on the state variable realization (SVR) of H we introduce the following notation. For a given continuous square matrix function $A(t)$ we designate

$$A_\alpha(t) = \alpha I + A(t). \tag{2.10}$$

We will be concerned with the notion of degree of stability of operators such as H. To this end we introduce H_α having SVR:

$$\{A_\alpha(t), B(t), C(t)\}, \tag{2.11}$$

where each of $A_\alpha(t), B(t), C(t)$ is a bounded, continuous function of time. We also make the following definition.
DEFINITION 2.4. *The matrix $A(t)$ is* exponentially asymptotically stable with degree of stability $\alpha > 0$ *(α-eas) if for the LTV system*

$$\dot{x}(t) = A(t)x(t) \tag{2.12}$$

$\exists c, \gamma > 0$ *such that for all $x(t_0)$ and $t \geq t_0$,*

$$\|x(t)\|e^{\alpha(t-t_0)} \leq c\|x(t_0)\|e^{-\gamma(t-t_0)}. \tag{2.13}$$

If $\alpha = 0$, we simply say that $A(t)$ is eas. Further, we will call a system with SVR, $\{A(t), B(t), C(t), D(t)\}$, all matrices bounded and continuous, α-eas (resp. eas) if $A(t)$ is α-eas (resp. eas).
REMARK 2.5. *In (2.10) $A_\alpha(t)$ is eas iff $A(t)$ is α-eas.*

Then we have the following assumption.
ASSUMPTION 2.6. *The system H_α has an SVR*

$$\{A_\alpha(t), B(t), C(t)\},$$

such that $[A_\alpha(t), B(t)]$ is uniformly completely controllable (u.c.c.), *[8],* $[A_\alpha(t), C(t)]$ *is* uniformly completely observable (u.c.o.), *[8], and both the systems H_α and*

$$\{A_\alpha(t) - kB(t)C'(t), B(t), -kC(t), I\}$$

are eas, for all $k \in [0,1]$.

REMARK 2.7. *It should be noted that any H_α that satisfies Assumption 2.6 also satisfies Assumption 2.1 as $\{A_\alpha(t) - kB(t)C'(t), B(t), -kC(t), I\}$ is precisely the SVR of $[I + kH_\alpha]^{-1}$. Further under the given uco and ucc conditions, BIBO stability, as exemplified by (2.8,2.9), is equivalent to eas, [8].*

REMARK 2.8. *Observe that a system having an SVR is necessarily causal. Since in this paper eas as a property has been defined in terms of SVR's, in any statement to the effect that a given system is eas will implicitly be indicating the causality of that system.*

REMARK 2.9. *Part of the utility of working with H_α stems from the easily proved facts that $\|H_\alpha\| > M_2$ implies $\|H\| < M_2$, and $\|[I + kH_\alpha]^{-1}\| < M_1$, implies $\|[I+kH]^{-1}\| < M_1$.* As is clear from the Introduction, the motivating result for this paper, namely that of [1], concerns SPR operators. As such, SPR as a concept does not apply to LTV systems. Instead the more appropriate concept is that of Strict Passivity [4]. Note that for LTI systems SPR and Strict Passivity are equivalent properties. In operator theoretic terms, Strict Passivity in turn is equivalent to the concept of a positive operator defined in Definition 2.10 below. Thus it is this concept of positivity that will replace the SPR property that underlies the LTI, SISO results.

DEFINITION 2.10. *An operator $P : L_2 \to L_2$ is called Strictly Positive ($P \geq \epsilon I > 0$) if for all x in L_2*

$$\langle x, Px \rangle \quad \geq \quad \epsilon \langle x, x \rangle. \tag{2.14}$$

In Section 4 we will need the concept of *Spectrum* of a LTV operator.

DEFINITION 2.11. *The resolvent set $\rho(H)$ of an operator $H : L_2 \to L_2$ is the set of all complex numbers λ such that $[\lambda I - H]^{-1} : L_2 \to L_2$ exists. The complement of all $\rho(H)$ in the complex plane is called the spectrum of H and is denoted $\sigma(H)$.*

REMARK 2.12. *The set $\sigma(H)$ is a bounded closed set, [9]. Further, [10] its elements vary continuously with continuous variations in H.*

3. Existence of the Square Root. The principal contribution of this Section is:

1. to argue that subject to Assumption 2.1, I+H has a square root and
2. to give an algorithm for constructing this square root.

In the sequel, we say that $G : L_2 \to L_2$ is the square root of $I + F$ with $F : L_2 \to L_2$ if

$$G^2 = I + F. \tag{3.1}$$

To compute the square root we propose a Newton-Raphson algorithm obtained as follows. Suppose the current estimate of the square root, should of course the square root exist, is G_i and that the true square root is $G_i + \Delta G$. Then

$$[G_i + \Delta G]^2 = I + F,$$

whence neglecting the second order term, and assuming that G_i is invertible and that G_i and ΔG commute, one obtains

$$\Delta G = \frac{1}{2}[(I + F)G_i^{-1} - G_i]$$

or

$$G_{i+1} = \frac{1}{2}[(I+F)G_i^{-1} + G_i]. \tag{3.2}$$

It is possible to establish the following result.

THEOREM 3.1. *Let $F : L_2 \to L_2$ be bounded and causal, $[I + kF]^{-1}$ exist and be causal for all $k \in [0,1]$ and let G_i, $i = 0, 1, \cdots$, be the sequence of operators defined by (3.2). Then, there exists an $\epsilon > 0$ such that whenever $\|F\| < \epsilon$ holds, so do the following for all $k \in [0,1]$: (i) there exists bounded $G(kF) = \lim_{i\to\infty} G_i(kF) : L_2 \to L_2$; (ii) $G(kF)^{-1} : L_2 \to L_2$ exists and both $G(kF)$ and $G(kF)^{-1}$ are causal; (iii) $G(kF)$ and $G(kF)^{-1}$ commute with any operator that commutes with F; (iv) $G(0) = I$; (v) $G(kF)$ varies continuously with k; and (vi) $G^2(kF) = I + kF$.*

This above theorem has a serious restriction, viz that $\|F\| < \epsilon$. The operator H may not have this property. It is possible to adopt a nested strategy for determining the square root. Let $NR(I + F, I)$ describe the quantity obtained via (3.1) and (3.2), with $G_0 = I$. With M_1 and M_2 as defined in Assumption 2.1, we choose δ_1, δ_2 such that

$$\delta_1 M_2 \le \min\{\epsilon, 1\} \qquad \delta_2 M_1 M_2 \le \min\{\epsilon, 1\} \quad N = \frac{1-\delta_1}{\delta_2} \text{ is integer} \tag{3.3}$$

Then a finite number of square root operators are defined by

$$U_o = NR(I + \delta_1 H, I) \tag{3.4a}$$
$$V_m = NR(I + \delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}, I) \tag{3.4b}$$
$$U_m = U_{m-1} V_m \tag{3.4c}$$

and there results

$$U_N^2 = I + H \tag{3.5}$$

In this procedure, $\|\delta_2 U_{m-1}^{-1} H U_{m-1}^{-1}\| \le \epsilon$ holds at each step.

A number of further points should be made:

- each U_i commutes with H
- an arbitrarily accurate approximation to U_N can be obtained when the Newton-Raphson iteration sequences are truncated; such approximations are rational in H
- there exists a square root $X(kH)$ of $I + kH$ for $k \in [0,1]$ which varies continuously with $k, X(H) = U_N$
- if H has a state variable realization with degree of stability α, and $\widehat{X}$ is an approximation to $X(H)$ obtained by truncation of the Newton-Raphson sequence, then $\widehat{X}$ has degree of stability α.

4. Existence of Passive Multipliers. Having demonstrated the existence of the square root of $I + H$, we now generalize the result of [1] and its first implication discussed in the Introduction. Instead of focussing on PR type properties, we will consider SPR type (or strict passivity) properties. This is simply a matter of minor technicality in an attempt to avoid having to deal with singular situations.

In the spirit of [1] the principal result we derive takes the following form: Under Assumption 2.6 there exist operators, $X_{1\alpha}$ and $X_{2\alpha}$, both eas and having eas inverses, for which:

1. $X_{1\alpha}X_{2\alpha}$ is Strictly Positive.
2. $X_{1\alpha}[I + H_\alpha]X_{2\alpha}$ is Strictly Positive.

Observe that, since in the LTI, SISO case all operators in question are mutually commutative, this directly reduces to one direction of the Brockett and Willems result. The other direction will be discussed later. In keeping with the requirements of the next Section, in our discussion here, we will pay special attention to degree of stability considerations.

In view of the results of Section 3, the starting point of our development here will be that there exists an X_α which is causal stable and has a causal stable inverse such that

$$X_\alpha^2 = I + H_\alpha. \tag{4.1}$$

Hence,

$$X_\alpha = [I + H_\alpha]X_\alpha^{-1} = X_\alpha^{-1}[I + H_\alpha]. \tag{4.2}$$

Because $[I + kH_\alpha]^{-1}$ exists for all $k \in [0, 1]$, it turns out that the spectrum of $I + H_\alpha$ avoids the negative real axis, and the spectrum of X_α is then necessarily confined to $Re(\lambda) > \epsilon$ for some $\epsilon > 0$. From this follows the existence of a self adjoint operator P guaranteeing that $X_\alpha^a P + PX_\alpha > 0$. Moreover P is positive definite and has a factorization $W^a W$ for a causal and causally invertible W.

We then have the following Theorem that captures one direction of the results of [1]

THEOREM 4.1. *Under Assumption 2.6, there exist finite dimensional eas and eas invertible operators* X_α^{-1} *and* W, *such that*

$$[W(I + H_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I + H_\alpha)X_\alpha^{-1}W^{-1} > 0. \tag{4.3}$$

and

$$[WX_\alpha^{-1}W^{-1}]^a + [WX_\alpha^{-1}W^{-1}] > 0. \tag{4.4}$$

In the Theorem statement, X_α and W can be taken to be finite dimensional approximations of the operators X_α, W described in the preamble to the theorem.

REMARK 4.2. *As an alternative to the interpretation provided at the beginning of this Section, this theorem also says that there is a causal operator* $WX_\alpha^{-1}W^{-1}$ *that is strictly positive (i.e. (4.4) holds) and such that the product of this operator with* $W(I+H_\alpha)W^{-1}$ *is also positive (i.e. 4.3) holds). Notice that Assumption 2.1 holds with* H *replaced by* WHW^{-1}. *So the difference with the original time-invariant scalar result lies in the introduction of* W. *Because in the time-invariant scalar case the various operators commute,* W *drops out of the picture. This difference reappears in the next section when we generalize the [3] result.*

Before discussing the second direction of the [1] result we turn now to the following Corollary.

COROLLARY 4.3. *Under Assumption 2.6, there exist finite dimensional eas and eas invertible operators* X_α^{-1} *and* W, *such that for all* $k \in [0,1]$

$$[W(I+kH_\alpha)X_\alpha^{-1}W^{-1}]^a + W(I+kH_\alpha)X_\alpha^{-1}W^{-1} > 0.$$

Proof. Follows from the fact that the above equation holds for $k=0$ and $k=1$ and the fact that positivity is a convex property. □

Observe that eas Strictly Positive operators have an inverse that is eas. Thus as long as H_α is eas and one can find eas and eas invertible operators W, X_α such that (4.3) and (4.4) hold, then the operator $(I+H_\alpha)^{-1}$ must be eas for all $k \in [0,1]$. Thus the analog of the reverse direction of the [1] result also holds.

5. Solution to a Problem Posed in [7]. Motivated by adaptive systems problems, [7] had posed the following question: Suppose the following set of square Matrix Polynomials:

$$\{A_1(s) + kA_2(s) | k \in [0,1]\} \tag{5.1}$$

has all its members Hurwitz (i.e. the determinant is Hurwitz). Does there exist a single LTI operator $Z(s)$ such that all members of the set

$$\{[A_1(s) + kA_2(s)]Z(s) | k \in [0,1]\}$$

are SPR? The next Theorem shows that such construction of SPR products is possible provided one allows multiplication from both sides.

THEOREM 5.1. *There exist, square, stable minimum phase matrix transfer functions* $Z_1(s)$ *and* $Z_2(s)$ *with the former strictly proper and the latter biproper, such that with* $A_1(s)$ *and* $A_2(s)$ *two square matrix polynomials, and* $A_1^{-1}A_2$ *strictly proper, all members of the set*

$$\{Z_1(s)[A_1(s) + kA_2(s)]Z_2(s) | k \in [0,1]\}$$

are biproper and SPR, iff all members of the set (5.1) are Hurwitz.

Proof. For necessity, note (i) that the inverse of an SPR matrix is SPR, and thus stable and (ii) the stability of Z_i prevents unstable zeros of $A_1 + kA_2$ from being cancelled in forming the product $Z_1[A_1 + kA_2]Z_2$. For sufficiency, note that the operator corresponding to the biproper transfer function

$$I + k[A_1(s)]^{-1}A_2(s) \tag{5.2}$$

satisfies Assumption 2.6. Thus, from Corollary 4.3, there exist biproper (as can be seen from their SVR), operators $W(s)$, and $X(s)$, each eas and having eas inverse, such that:

$$W(s)[I + k[A_1(s)]^{-1}A_2(s)]X^{-1}(s)W^{-1}(s)$$

is SPR for all $k \in [0,1]$. Then, choosing $Z_1(s) = W(s)[A_1(s)]^{-1}$ and $Z_2(s) = X(s)W^{-1}(s)$, the result follows. □

The main application of this result is in output error adaptive identification [11]. Consider the identification of the proper MIMO plant:

$$[A_1(s) + kA_2(s)]Y(s) = [B_1(s) + kB_2(s)]U(s) \tag{5.3}$$

with k a scalar unknown parameter and $u(t)$ and $y(t)$, the input and output of the plant. To identify the plant generally, one performs state variable filtering to avoid explicit differentiation of the various signals. This requires rewriting of the model as

$$Z_1(s)[A_1(s) + kA_2(s)]Y(s) = Z_1(s)[B_1(s) + kB_2(s)]U(s) \tag{5.4}$$

such that $Z_1(s)[A_1(s) + kA_2(s)]$ is biproper. Then, for exponential convergence of the underlying identification algorithm, one requires that $Z_1(s)[A_1(s)+kA_2(s)]$ be SPR. This can be seen readily from the result of [12] which treats the SISO case. As k is unknown the underlying SPR condition is difficult to ensure. However, suppose *a priori* bounds are available for k. In fact without sacrificing generality, assume that $k \in [0,1]$. Then as long as $[A_1(s)+kA_2(s)]$ is Hurwitz for all $k \in [0,1]$, one can choose square, stable, minimum phase matrix transfer functions $Z_1(s)$ and $Z_2(s)$ such that the requirements of Theorem 5.1 are satisfied. Then, noting that $Z_2(s)$ is biproper, one can rexpress the plant as

$$Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)\bar{Y}(s) = Z_1(s)[B_1(s) + kB_2(s)]U(s)$$

where

$$\bar{Y}(s) = Z_2^{-1}(s)Y(s)$$

acts as the converted output. Observe it can be constructed from $Y(s)$ without any explicit differentiation. Further, as $Z_1(s)[A_1(s) + kA_2(s)]Z_2(s)$ is SPR, the output error identification algorithm for this redefined system will be exponentially convergent.

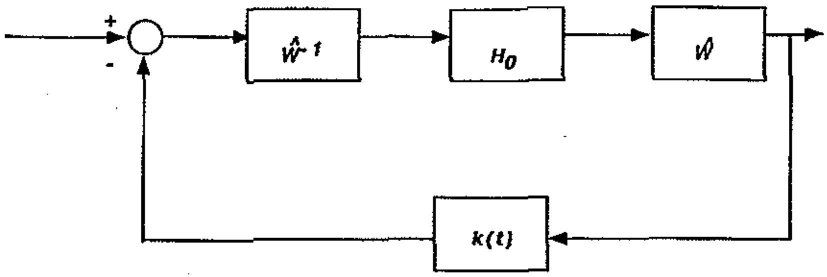

FIG. 2. *Closed-loop under time varying feedback.*

6. Generalization of the Freedman Zames Result. In this Section we consider the second consequence of the result of [1] namely that of [3]. To this end the principal result to be derived is as follows:

THEOREM 6.1. *Suppose Assumption 2.6 holds. Then there exists an eas and eas invertible operator $\hat{W}$ (independent of k) such that the operator $[I+k(t)\hat{W}H_0\hat{W}^{-1}]^{-1}$ is eas provided (1.3) and (1.4) hold.*

Essentially, this result states that provided the closed loop of Fig. 1 (with $H(s)$ replaced by H_0) is α-eas for all time invariant feedback gains in the open interval [0,1], then under a logarithmic time variation bound as in [3], by suitable pre and post filtering of H_0, the closed loop in Fig. 2 is also stable. Observe, if $H(s)$ is scalar LTI, then the underlying commutativity recovers the result of [3]. Moreover, the fact that the pre and post filters $\hat{W}^{-1}$ and $\hat{W}$ are independent of the particular trajectory that the time-varying feedback gain follows simplifies their selection.

7. Conclusion. In this paper we have a generalized a number of results concerning passive multiplier theory to MIMO LTV systems. Between them these results cover important issues spanning a number of areas. Though the focus here is on continuous time systems extension to discrete time systems is straightforward. To what extent generalizations to nonlinear time varying systems problems are possible remains however an open problem. It would also be interesting to establish a result on the stability of the determinant of convex combinations of more than two matrix polynominals.

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