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On the Robust Stability of Time-Varying Linear Systems

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Abstract. This paper considers the robust stability of time-varying linear systems described by a linear differential equation whose coefficients vary inside given intervals and with restricted magnitudes of the rates of change of the coefficients. This problem can be considered as a generalization of the Kharitonov problem, which is in turn a generalization of the Hurwitz problem, and it was formulated as an open problem in [7].
To solve this problem Lyapunov theory is used where a Lyapunov function is obtained (using characteristics of positive real functions [2], [3]) which is multiaffine in the polynomial coefficients. With this Lyapunov function extreme point results are obtained. The structure of the Lyapunov matrix as well as the structure of the conditions for the solution of a robust positive real function problem are characterized. A second approach based on the critical stability conditions is also suggested but the Lyapunov matrix thus obtained is no longer in general multiaffine in the parameters. Examples of low order systems are given. The resulting stability conditions are only sufficient.

1. Introduction. Robust stability of time-invariant systems with parameter uncertainty has received much attention since the seminal paper of Kharitonov [7]. Also, robust stability of time-varying systems was investigated in several publications e.g. [2-6]. In [6] the bounds on the variation of the parameters of discrete time systems were determined which depend on predetermined parameter regions in the parameter space. In [7] the following problem was formulated as an open problem:

Given a time-varying differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \text{ with} \tag{1.1}$$

$$\underline{a}_i \leq a_i(t) \leq \bar{a}_i \tag{1.2}$$

$$-\alpha_i \leq \frac{da_i}{dt} \leq \alpha_i \tag{1.3}$$

find conditions for exponential asymptotic stability of (1.1) subject to (1.2) and (1.3). A discrete version of this problem with no restriction on the time-variation

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rate is solved in [4]. This problem is a generalization of the time-invariant problem of Kharitonov, which is in turn a generalization of the Hurwitz problem. In order to derive sufficient conditions for stability of the time-variable system it is natural to use Lyapunov theory and indeed quadratic Lyapunov functions. Equations (1.4) through (1.7) define the state variable representation of (1.1), Lyapunov function and its derivative:

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) \tag{1.4}$$

$$V(\underline{x},t) = \underline{x}^T P(t)\underline{x}(t) \tag{1.5}$$

$$\dot{V}(\underline{x}) = -\underline{x}^T(Q-\dot{P})\underline{x} \tag{1.6}$$

$$A^T P + PA = -Q. \tag{1.7}$$

It is well known that if $V(\underline{x},t)$ is a positive definite decrescent function with negative definite derivative then the equilibrium is exponentially stable, [8].

Now if P and Q as solutions of the Lyapunov equation (1.7) are multiaffine in the uncertain coefficients a_i then convexity implies the positive definiteness of $Q-\dot{P}$ is determined by the positive definiteness at the extreme values the coefficients (the vertices), which can be checked easily.

The multiaffine Lyapunov function and derivative are obtained using positive real functions and the Kalman Yakubovic-Popov Lemma [9], [10].

In Section 2 some preliminary results are given. In Section 3 we show how to construct the Lyapunov function using the characteristics of positive real functions and how a modified Hurwitz matrix appears in the procedure. Solutions for low order systems are given.

In Section 4 the solution of the Lyapunov equation is given using the determinant of the Hurwitz matrix. In this latter case P and Q are no longer multiaffine in the parameters and therefore no vertex results are obtained except for $n=2$.

Section 5 gives examples solved by different Lyapunov functions, and Section 6 gives the conclusions.

2. Preliminaries.

2.1. Let $p(s) = s^n + a_1 s^{n-1} + \cdots \quad + a_n$ where $\underline{a}_i \leq a_i \leq \bar{a}_i$ i.e. $p(s) \in \mathcal{P}$ where $\mathcal{P}$ is a box in the coefficient space. If there exists $B(s)$ such that $\frac{p_i(s)}{B(s)}, i = 1,2,3,4$ is positive real (SPR) where $p_i(s)$ are the four Kharitonov polynomials, then $\frac{p(s)}{B(s)}$ is SPR for all $p \in \mathcal{P}$ [11].

2.2. In [9] the following result was proved (Corollary 3.1): Consider a convex polytope $\mathcal{P}$ of $\ell-th$ degree polynomials in s. There exists a non-negative integer M and a polynomial $B(s)$ of degree $\ell+M$ such that $\frac{p(s)(1+s)^M}{B(s)}$ is strict positive real (SPR) for all $p(s) \in \mathcal{P}$ if and only if $\mathcal{P}$ is Hurwitz invariant, i.e. for all $p(s) \in \mathcal{P}, p(s_0) = 0$ implies $Re[s_0] < 0$.

2.3. Given a scalar transfer function $T(s) = 1 + \underline{c}^T(sI - A)^{-1}\underline{b}$ with $[A, \underline{c}]$ completely observable and $[A, \underline{b}]$ completely reachable, $T(s)$ is strictly positive real (i.e. poles are in $Res < 0$ and $ReT(j\omega) > 0 \quad \forall\omega$ if and only if there exist symmetric positive definite matrices P and Q and a vector $\underline{q}$ such that

$$A^T P + PA + \underline{q}\underline{q}^T = -Q \text{ and} \tag{2.1}$$

$$P\underline{b} = \underline{c} + \sqrt{2}\underline{q}. \tag{2.2}$$

This is a version of the celebrated Kalman-Yakubovic-Popov Lemma.

2.4. Let

$$K = \{\underline{k} = [k_1 \cdots k_m]^T : \underline{k}_i \le k_i \le \overline{k}_i\}\ , \tag{2.3}$$

let $h(k) \in R^n$ be affine in the elements of k, let $g \in R^n$ and $F \in R^{n\times n}$. Define

$$\Omega = \{A(k) = F + \underline{g}h'(k) \in R^{n\times n} : k \in K\}. \tag{2.4}$$

It is shown in [10] that Ω is Hurwitz invariant if and only if there exists a square matrix with stable characteristic polynomial, a compatibly dimensioned vector $\underline{\omega}$ and a Lyapunov pair $P(k), Q(k)$ depending multiaffinely on the elements of $\underline{k}$ which satisfies the Lyapunov equation (2.5) for all $k \in K$:

$$\Pi^T(k)P(k) + P(k)\Pi(k) < -Q(k) \tag{2.5}$$

where

$$\Pi(k) = \begin{bmatrix} \triangle & \underline{\omega}\,\underline{h}^T(k) \\ \bigcirc & A(k) \end{bmatrix} . \tag{2.6}$$

2.5. We prove the following lemma:

LEMMA 2.1. *Let $z(s)$ satisfy*

1. *all poles of $z(s)$ lie in $Re[s] \le -\sigma < 0$*
2. *$Rez(j\omega) \ge \delta > 0 \quad \forall\omega$*
3. $\lim_{\omega\to\infty} z(j\omega) = 1$.

Then for suitably large a, there holds

(a) *all poles of $\frac{z(s)}{s+a}$ lie in $Re[s] \le -\sigma < 0$*

(b) *$Re\frac{z(j\omega)}{j\omega+a} > 0 \quad \forall\omega < \infty$.*

Proof. The first condition holds if $a > \sigma$. For the second condition

$$Re\frac{z(j\omega)}{j\omega + a} = \frac{z(j\omega)}{j\omega + a} + \frac{z(-j\omega)}{-j\omega + a} = 2\frac{aRez(j\omega) - \omega Imz(j\omega)}{\omega^2 + a^2}.$$

Because $\lim_{\omega\to\infty} Imz(j\omega) = 0$ and $Imz(j\omega)$ is rational in ω

$$\left|\lim_{\omega\to\infty} \omega \cdot Imz(j\omega)\right| < \infty .$$

Also, $Imz(j\omega)$ is bounded in ω, so that for some K,

$$|\omega Imz(j\omega)| < K \quad \forall \text{ real } \omega.$$

Now choose $a > \frac{K}{\delta}$ to secure $Re\frac{z(j\omega)}{j\omega+a} > 0$. □

COROLLARY 2.2. *Let $\mathcal{P}$ be a convex family of stable monic polynomials*

$$p(s) = s^n + a_1 s^{n-1} + \cdots a_n \tag{2.7}$$

where $[a_1 a_2 \cdots a_n] \in \ell$, a convex stable set.

Then there exists a transfer function $w(s)$ such that
(i) $w(s)p^{-1}(s)$ is stable $\quad \forall p \in \mathcal{P}$
(ii) $Rew(j\omega)p^{-1}(j\omega) > 0 \quad \forall$ finite real ω
(iii) $\lim_{s\to\infty} w(s)p^{-1}(s) = 0$.

An application of this result to low degree polynomials is given in Section 3.

3. The Affine Lyapunov Function. As mentioned in the introduction, solutions P and Q of the Lyapunov equation (1.7) which are affine or multiaffine in the coefficients of the differential equation (1.1), yields a vertex-type result for stability of the time variable system. The Lyapunov pair can be determined using SPR functions results of Section 2 together with the Kalman-Yakubovic-Popov Lemma (KYP). The following subsection illustrates the approach.

3.1. $n = 2$:

$$p(s) = s^2 + a_1 s + a_2, \quad \underline{a}_i \le a_i \le \overline{a}_i\,. \tag{3.1}$$

For stability of $p(s)$

$$\underline{a}_i > 0, \quad i = 1, 2. \tag{3.2}$$

For $z(s) = \frac{p(s)}{B(s)}$ to be SPR we choose $B(s)$ to be of degree 1 or 2.

Case (a):

$$B(s) = b_1 s + b_2, \quad b_1 \text{ and } b_2 > 0, \tag{3.3}$$

b_1 or b_2 can be chosen arbitrarily. Then

$$Rez(j\omega) = \frac{(b_1 a_1 - b_2)\omega^2 + b_2 a_2}{\omega^2 b_1^2 + b_2^2} \tag{3.4}$$

is positive if

$$b_1 \underline{a}_1 - b_2 > 0 \tag{3.5}$$

which can always be satisfied by appropriate choice of b_1 and b_2.

A realization of $\frac{1}{z(s)} = \underline{c}^T(sI - A)^{-1}\underline{b}$ in controllable normal form gives

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{c}^T = [b_2 \quad b_1]\,. \tag{3.6}$$

Then the KYP lemma for strictly proper transfer functions has

$$A^T P_1 + P_1 A = -Q_1, \quad P_1 \underline{b} = c. \tag{3.7}$$

Hence,

$$P_1 = \begin{bmatrix} b_1 a_2 + b_2 a_1 & b_2 \\ b_2 & b_1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} & b_2 \\ b_2 & b_1 \end{bmatrix} \tag{3.8}$$

and

$$Q_1 = 2 \begin{bmatrix} b_2 a_2 & 0 \\ 0 & b_1 a_1 - b_2 \end{bmatrix}.$$

The result in (3.8) can also be obtained if we solve the Lyapunov equation for P_1 assuming Q_1 diagonal, and assuming the last column of P_1 to be independent of a_1, a_2; this ensures P_1 and Q_1 are affine in the parameters a_1 and a_2.

We come now to Case (b):

$$B(s) = s^2 + b_1 s + b_2, \quad b_1 \text{ and } b_2 > 0. \tag{3.9}$$

Then

$$Re z(j\omega) = \frac{\omega^4 + (b_1 a_1 - b_2 - a_2)\,\omega^2 + a_2 b_2}{(-\omega^2 + b_2)^2 + b_1^2} \tag{3.10}$$

is positive if

$$b_1 \underline{a}_1 - b_2 - \overline{a}_2 > 0 \tag{3.11}$$

which can always be satisfied by appropriate choice of b_1 and b_2.

Now writing $z^{-1}(s) = 1 + \underline{c}^T (sI - A)^{-1} \underline{b}$ with

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{c}^T = [b_2 - a_2 \quad b_1 - a_1] \tag{3.12}$$

and using the KYP lemma, we get

$$\begin{aligned} P_2 &= \begin{bmatrix} b_1 a_2 + b_2 a_1 & b_2 + a_2 \\ b_2 + a_2 & b_1 + a_1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} & b_2 + a_2 \\ b_2 + a_2 & b_1 + a_1 \end{bmatrix} \\ &= P_1 + \begin{bmatrix} 0 & a_2 \\ a_2 & a_1 \end{bmatrix}, \\ Q_2 &= 2 \begin{bmatrix} b_2 a_2 & 0 \\ 0 & b_1 a_1 - b_2 - a_2 \end{bmatrix} \quad \text{and} \quad \underline{q} = \sqrt{2} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}. \end{aligned} \tag{3.13}$$

3.2. $n = 3$:

$$p(s) = s^3 + a_1 s^2 + a_2 s + a_3, \quad \underline{a}_i \le a_i \le \overline{a}_i, \quad a_i > 0. \tag{3.14}$$

For stability of all $p(s)$ we require

$$\underline{a}_1 \underline{a}_2 - \overline{a}_3 > 0. \tag{3.15}$$

For $z(s) = \frac{p(s)}{B(s)}$ to be SPR, we choose $B(s)$ to be of degree 2 or 3.

(3.16) $$\text{(a)} \quad B(s) = b_1 s^2 + b_2 s + b_3; \quad b_i > 0.$$

Any one of b_1 or b_2 or b_3 can be chosen arbitrarily. It follows that $Re z(j\omega)$ is positive if its numerator coefficients are positive, i.e. if

(3.17) $$b_2\underline{a}_2 - b_1\bar{a}_3 - b_3\bar{a}_1 > 0, \quad b_1\underline{a}_1 - b_2 > 0.$$

A realization of $\frac{1}{z(s)} = \underline{c}^T(sI - A)^{-1}\underline{b}$ in controllable normal form gives

(3.18) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{c}^T = [b_3 \quad b_2 \quad b_1].$$

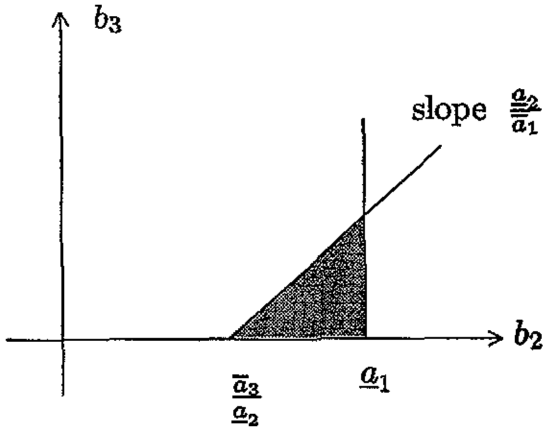

Fig. 1: Graphical interpretation of stability conditions

Then (3.7) gives

(3.19) $$P_1 = \begin{bmatrix} \begin{vmatrix} b_2 & b_3 \\ -a_2 & a_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} & b_3 \\ \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} - b_3 & b_2 \\ b_3 & b_2 & b_1 \end{bmatrix}$$

$$\text{and} \quad Q_1 = 2\begin{bmatrix} b_3 a_3 & 0 & 0 \\ 0 & b_2 a_2 - b_1 a_3 - b_3 a_1 & 0 \\ 0 & 0 & b_1 a_1 - b_2 \end{bmatrix}.$$

The same P_1 and Q_1 follow if one simply solves the Lypunov equation assuming affine P and Q, diagonal Q and last column of P constant.

If we choose $b_1 = 1$, the two conditions in (3.17) which guarantee the positive definiteness of Q_1 can always be satisfied due to the stability condition (3.15). See Figure 1, where the shaded region corresponds to (3.17).

(3.20) $$\text{(b)} \quad B(s) = s^3 + b_1 s^2 + b_2 s + b_3, \quad b_i > 0.$$

Then $Rez(j\omega)$ is positive if its numerator coefficients are positive, i.e. if

$$b_2\underline{a}_2 - b_1\overline{a}_3 - b_3\overline{a}_1, > 0, \quad b_1\underline{a}_1 - b_2 - \overline{a}_2 > 0. \tag{3.21}$$

Let $\frac{1}{z(s)} = 1 + \underline{c}^T(sI - A)^{-1}\underline{b}$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \underline{c}^T = [b_3 - a_3 \ \ b_2 - a_2 \ \ b_1 - a_1]. \tag{3.22}$$

Then using the KYP lemma we get

$$\begin{aligned} P_2 &= \begin{bmatrix} \begin{vmatrix} b_2 & b_3 \\ -a_2 & a_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} & b_3 + a_3 \\ \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} - b_3 & b_2 + a_2 \\ b_3 + a_3 & b_2 + a_2 & b_1 + a_1 \end{bmatrix} \\ &= P_1 + \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & a_2 \\ a_3 & a_2 & a_1 \end{bmatrix}, \end{aligned} \tag{3.23}$$

$$Q_2 = \begin{bmatrix} b_3a_3 & 0 & 0 \\ 0 & b_2a_2 - b_1a_3 - b_3a_1 & 0 \\ 0 & 0 & b_1a_1 - b_2 - a_2 \end{bmatrix}, \underline{q} = \sqrt{2}\begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix}.$$

It is shown in [9] that (3.15) ensures it is always possible to find b_i, such that (3.21) holds.

3.3. $n = 4$:

$$P(s) = s^4 + a_1s^3 + a_2s^2 + a_3s + a_4, \quad \underline{a}_i \le a_i \le \overline{a}_i, a_i > 0. \tag{3.24}$$

For stability of $p(s)$ we require [12]

$$\begin{aligned} &\underline{a}_1\underline{a}_2 - \overline{a}_3 > 0 \\ &\underline{a}_1\underline{a}_2\overline{a}_3 - \underline{a}_1^2\overline{a}_4 - \overline{a}_3^2 > 0 \\ &\overline{a}_1\underline{a}_2\underline{a}_3 - \overline{a}_1^2\overline{a}_4 - \underline{a}_3^2 > 0. \end{aligned} \tag{3.25}$$

For $z(s) = \frac{p(s)}{B(s)}$ to be SPR, we choose $B(s)$ to be of degree 3 or 4.

$$\text{(a)} \quad B(s) = b_1s^3 + b_2s^2 + b_3s + b_4, \quad b_i > 0. \tag{3.26}$$

Any one of $b_1 \cdots b_4$ can be chosen arbitrarily. Also $Rez(j\omega)$ is positive if its numerator coefficients are positive, i.e. if

$$\begin{aligned} &b_1\underline{a}_1 - b_2 > 0, \quad b_2\underline{a}_2 - b_1\overline{a}_3 - b_3\overline{a}_1 + b_4 > 0 \ \text{and} \\ &b_3\underline{a}_3 - b_2\overline{a}_4 - b_4\overline{a}_2 > 0. \end{aligned} \tag{3.27}$$

A realization of $\frac{1}{z(s)} = \underset{\sim}{c}^T(sI - A)^{-1}\underset{\sim}{b}$ in controllable normal form gives

$$
(3.28)\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underset{\sim}{c}^T = [b_4 \;\; b_3 \;\; b_2 \;\; b_1].
$$

Then (3.7) gives

$$
P_1 = \begin{bmatrix}
\begin{vmatrix} b_3 & b_4 \\ -a_3 & a_4 \end{vmatrix} & \begin{vmatrix} b_2 & b_4 \\ -a_2 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & b_4 \\[2ex]
\begin{vmatrix} b_2 & b_4 \\ -a_2 & a_4 \end{vmatrix} & \begin{vmatrix} b_2 & b_3 \\ -a_2 & a_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} - b_4 & b_3 \\[2ex]
\begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} - b_4 & \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} - b_3 & b_2 \\[2ex]
b_4 & b_3 & b_2 & b_1
\end{bmatrix}
$$

(3.29)

$$
Q_1 = 2 \begin{bmatrix} b_4 a_4 & & & \\ & -b_4 a_2 + b_3 a_3 - b_2 a_4 & & \\ & & b_4 - b_3 a_1 + b_2 a_2 - b_1 a_3 & \\ & & & b_1 a_1 - b_2 \end{bmatrix}.
$$

The same P_1 and Q_1 can be obtained by solving the Lyapunov equation assuming affine P_1 and Q_1 with diagonal Q_1 and the last column of P_1 independent of a_i.

Without loss of generality let $b_2 = 1$. Then from (3.27) $b_1 > \frac{1}{\underline{a}_1}$ which means $b_1 = \frac{\alpha}{\underline{a}_1}$ where $\alpha > 1$.

The other two conditions of (3.27) can be rewritten as

$$
(3.30)\qquad \begin{aligned} &\underline{a}_2 - \alpha\frac{\overline{a}_3}{\underline{a}_1} - b_3\overline{a}_1 + b_4 > 0 \\ &b_3\underline{a}_3 - \overline{a}_4 - b_4\overline{a}_2 > 0. \end{aligned}
$$

Figure 2 shows that a solution exists if and only if

$$
(3.31)\qquad \underline{a}_1\underline{a}_2\underline{a}_3 - \overline{a}_4\underline{a}_1\overline{a}_1 - \alpha\underline{a}_3\overline{a}_3 > 0
$$

$$
(3.32)\qquad \text{(b)}\quad B(s) = s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4, b_i > 0.
$$

Then $Rez(j\omega)$ is positive if its numerator coefficients are positive, or if

$$
(3.33)\qquad \begin{aligned} &b_1\underline{a}_1 - b_2 - \overline{a}_2 > 0 \\ &b_4 - b_3\overline{a}_1 + b_2\underline{a}_2 - b_1\overline{a}_3 + \underline{a}_4 > 0 \\ &-b_4\overline{a}_2 + b_3\underline{a}_3 - b_2\overline{a}_4 > 0. \end{aligned}
$$

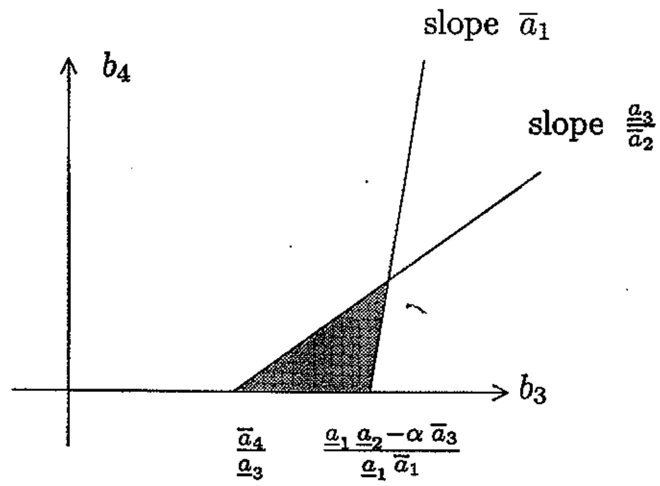

Fig. 2: Graphical interpretation of the stability conditions

Let $\frac{1}{z(s)} = 1 + \underline{c}^T(sI-A)^{-1}\underline{b}$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3.34}$$

$$\underline{c}^T = [b_4 - a_4 \;\; b_3 - a_3 \;\; b_2 - a_2 \;\; b_1 - a_1]\,.$$

Then using the KYP lemma we get

$$P_2 = \begin{bmatrix} \begin{vmatrix} b_3 & b_4 \\ -a_3 & a_4 \end{vmatrix} & \begin{vmatrix} b_2 & b_4 \\ -a_2 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & b_4 + a_4 \\ \begin{vmatrix} b_2 & b_4 \\ -a_2 & a_4 \end{vmatrix} & \begin{vmatrix} b_2 & b_3 \\ -a_2 & a_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} - b_4 & b_3 + a_3 \\ \begin{vmatrix} b_1 & b_4 \\ -a_1 & a_4 \end{vmatrix} & \begin{vmatrix} b_1 & b_3 \\ -a_1 & a_3 \end{vmatrix} - b_4 & \begin{vmatrix} b_1 & b_2 \\ -a_1 & a_2 \end{vmatrix} - b_3 & b_2 + a_2 \\ b_4 + a_4 & b_3 + a_3 & b_2 + a_2 & b_1 + a_1 \end{bmatrix}$$

$$= P_1 + \begin{bmatrix} 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \tag{3.35}$$

(3.35 cont.)

$$Q_2 = \begin{bmatrix} b_4a_4 & & & \\ & -b_4a_2+b_3a_3-b_2a_4 & & \\ & & b_4-b_3a_1+b_2a_2-b_1a_3+a_4 & \\ & & & b_1a_1-b_2-a_2 \end{bmatrix},$$

$$\underline{q} = \sqrt{2}\begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix}.$$

In [9] it was wrongly asserted using an argument involving a dual linear programming problem that $b_1 \cdots b_4$ always exist to satisfy the SPR conditions (3.35).

REMARK 3.1. *The SPR sufficiency conditions (3.27) can be written as*

$$(3.36)\quad \begin{bmatrix} \underline{a}_1 & -1 & 0 & 0 \\ -\overline{a}_3 & \underline{a}_2 & -\overline{a}_1 & 1 \\ 0 & -\overline{a}_4 & \underline{a}_3 & -\overline{a}_2 \\ 0 & 0 & 0 & \underline{a}_4 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } H^*\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix

$$(3.37)\qquad H^* = I^*HI^*$$

where $I^* = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ *and H^* is a modified Hurwitz matrix.*

The inequality of condition (3.36) involves just two extreme points (two corners) in coefficient space, namely $\underline{a}_1, \overline{a}_2, \underline{a}_3, \overline{a}_4$ and $\underline{a}_1, \overline{a}_2, \underline{a}_3, \overline{a}_4$. It can be easily shown that this idea generalises for any degree n.

REMARK 3.2. *In general it can be shown that $P_1 = (p_{ij})$ with*

$$(3.38)\qquad p_{ij} = \begin{vmatrix} b_{n-j} & b_{n+1-i} \\ -a_{n-j} & a_{n+1-i} \end{vmatrix} - p_{i-1,j+1}$$

where $i, j = 1, 2, \ldots, n \quad j \geq i, \quad b_0 = 0, a_0 = 1, p_{0,j} = 0, \quad p_{i,n+1} = 0$ and

$$(3.39)\qquad P_2 = P_1 + \begin{bmatrix} 0 & \cdots & 0 & a_n \\ \vdots & & \vdots & a_{n-1} \\ 0 & \cdots & 0 & \vdots \\ a_n & & a_2 & a_1 \end{bmatrix}.$$

4. Alternative analytic solution of the Lyapunov equations; second order problems.

Consider the Lyapunov equation

$$A^T P + PA = -Q \tag{4.1}$$

with A in companion form. Arranging the diagonal and sub-diagonal entries of P and Q into vectors leads to

$$B\text{vec}P = \text{vec}Q. \tag{4.2}$$

It can be shown by construction and using elementary row operations that B has the following form

$$B = \left[\begin{array}{c|c} I & B^* \\ \hline O & B_{n,n} \end{array}\right] \tag{4.3}$$

where

$$B_{n,n} = (-1)^{n+1} J^* H J^*, \quad J^* = \begin{bmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ -1 & & & & \\ \cdot^{\cdot^{\cdot}} & & & & 0 \end{bmatrix} \tag{4.4}$$

and

$$B^* = \begin{bmatrix} B_{1,2} & C_{1,1} & 0_{1,n-3} \\ B_{2,3} & C_{2,1} & 0_{1,n-4} \\ \vdots & \vdots & \vdots \\ B_{n-3,n-2} & C_{n-3,1} & 0_{n-3,1} \\ B_{n-2,n-1} & C_{n-2,1} & \\ B_{n-1,n} & & \end{bmatrix} \tag{4.5}$$

$$B_{n-r,n-r+1} = \begin{bmatrix} -a_r & & & -a_n \\ & \ddots & 0 & \vdots \\ -a_{r-1} & & & \vdots \\ & \ddots & \ddots & \vdots \\ \cdot & -a_{r-1} & -a_r & -a_{r+1} \end{bmatrix}, \quad r = 1, 2, \ldots n-1 \tag{4.6}$$

$$C_{n-r,1} = \begin{bmatrix} 0 \\ a_n \\ \vdots \\ a_{r+2} \end{bmatrix}, \quad r = 2, \ldots, n-1 \tag{4.7}$$

$0_{n-r,r-2}$ is a zero matrix.

We have

$$\det B = (-1)^{n+1}\det H. \tag{4.8}$$

But $\det H$ is the product of the two critical stability conditions. Therefore, if we choose

$$Q = 2(\det H)I \tag{4.9}$$

then P will have elements which are polynomials in $a_1, \ldots, a_n$.

Now if $n = 2$, $\det H = a_1 a_2$. Then we have

$$Q = 2a_1a_2I, \quad P = \begin{bmatrix} a_1^2 + a_2^2 + a_2 & a_1 \\ a_1 & 1 + a_2 \end{bmatrix} \tag{4.10}$$

so that P is quadratic in a_1, a_2.

From (1.6), if $Q - \dot{P} > 0$ we conclude stability, i.e. for stability

$$\begin{bmatrix} 2a_1a_2 - 2a_1\dot{a}_1 - 2a_2\dot{a}_2 - \dot{a}_2 & \dot{a}_1 \\ \dot{a}_1 & 2a_1a_2 - \dot{a}_2 \end{bmatrix} > 0. \tag{4.11}$$

As (4.11) is multi affine in $a_1, a_2, \dot{a}_1, \dot{a}_2$, we need only to check the corner values.

For $n > 2$ the procedure leads to P with some elements of higher degree than quadratic in the coefficients; therefore corner conditions are no longer sufficient.

REMARK 4.1. *One can take a positive diagonal matrix instead of the unity matrix in Q to improve on the results.*

5. Examples for second order systems. Let $\bar{\dot{a}}_1 = \bar{\dot{a}}_2 = \alpha$, $\underline{a}_1 = \underline{a}_2 = \beta$, $\bar{a}_1 = \bar{a}_2 = \gamma$. We consider the stability problem with the different Lyapunov functions derived in the previous sections.

5.1.

$$\begin{aligned} Q &= 2\begin{bmatrix} b_2a_2 & 0 \\ 0 & b_1a_1 - b_2 \end{bmatrix}, \quad P = \begin{bmatrix} b_1a_2 + b_2a_1 & b_2 \\ b_2 & b_1 \end{bmatrix} \\ \dot{P} &= \begin{bmatrix} b_1\dot{a}_2 + b_2\dot{a}_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{5.1}$$

For $Q > 0$ $\; b_1\underline{a}_1 - b_2 > 0$, i.e. $b_1 > \frac{b_2}{\beta}$.

For $Q - \dot{P} > 0$ we require $\begin{bmatrix} 2b_2a_2 - b_1\dot{a}_2 - b_2\dot{a}_1 & 0 \\ 0 & 2b_1a_1 - 2b_2 \end{bmatrix} > 0$ which is ensured by

$$\alpha < \frac{2b_2\beta}{b_1 + b_2}. \tag{5.2}$$

With $b_2 = 1$ we obtain $b_1 > \frac{1}{\beta}$ and

$$\alpha < \frac{2\beta}{1+\frac{1}{\beta}} = \frac{2\beta^2}{\beta+1}. \tag{5.3}$$

Hence if $\beta = 2$ then $\alpha < 2.66$, and if $\beta = 4$ then $\alpha < 6.4$ These results are independent of γ.

5.2.

$$\begin{aligned} Q &= 2\begin{bmatrix} b_2 a_2 & 0 \\ 0 & b_1 a_1 - b_2 \end{bmatrix} + \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_1^2 \end{bmatrix} = Q_1 + Q_2 \\ P &= \begin{bmatrix} b_1 a_2 + b_2 a_1 & b_2 + a_2 \\ b_2 + a_2 & b_1 + a_1 \end{bmatrix}. \end{aligned}$$

Therefore,

$$Q_1 - \dot{P} = \begin{bmatrix} 2b_2 a_2 - b_1 \dot{a}_2 - b_2 \dot{a}_1 & -\dot{a}_2 \\ -\dot{a}_2 & 2b_1 a_1 - 2b_2 a_2 - 2a_2 - \dot{a}_1 \end{bmatrix} > 0. \tag{5.4}$$

To ensure $Q_1 > 0$ we have $b_1 > \frac{b_2+\gamma}{\beta}$.

Because of the affine character of $Q - \dot{P}$ with respect to a_1 and a_2 we consider only the two vertices $(\underline{a}_1, \underline{a}_2)$ and $(\underline{a}_1, \overline{a}_2)$. From $Q_1 - \dot{P} > 0$ the positivity of the diagonal elements at the two vertices gives

$$\begin{aligned} \alpha &< \frac{2b_2\beta}{b_1+b_2}, \quad \alpha < 2(b_1-1)\beta - 2b_2, \quad \alpha < \frac{2b_2\gamma}{b_1+b_2}, \\ \alpha &< 2b_1\beta - 2\gamma - 2b_2. \end{aligned} \tag{5.5}$$

From (5.4) and (5.5)

$$\alpha < \frac{2b_2\beta^2}{b_2(1+\beta)+\gamma+\frac{\alpha}{2}} \approx \frac{2\beta^2}{\beta+1} \tag{5.6}$$

for large b_2.

The parameters b_1 and b_2 can be chosen large enough to have (5.6) valid and $\det(Q_1 - \dot{P})$ remains positive.

Hence for $\beta = 2$ we get $\alpha < 2.66$. This is independent of γ and is a rough estimate.

5.3.

$$Q = 2a_1 a_2 I, \quad P = \begin{bmatrix} a_1^2 + a_2^2 + a_2 & a_1 \\ a_1 & 1 + a_2 \end{bmatrix},$$

$$Q - \dot{P} = \begin{bmatrix} 2a_1 a_2 - 2a_1\dot{a}_1 - 2a_2\dot{a}_2 - \dot{a}_2 & -\dot{a}_1 \\ -\dot{a}_1 & 2a_1 a_2 - \dot{a}_2 \end{bmatrix} > 0.$$

For positivity of $Q - \dot{P}$ at the four vertices $(\overline{a}_1, \overline{a}_2)$, $(\underline{a}_1, \overline{a}_2)$, $(\overline{a}_1, \underline{a}_2)$, $(\underline{a}_1, \underline{a}_2)$ we get $\alpha < 0.877$ if we assume $\beta = 2, \gamma = 5$.

5.4.

$$Q = 2a_1a_2 \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad P = \begin{bmatrix} a_1^2c + a_2^2d + a_2c & a_1c \\ a_1c & a_2d + c \end{bmatrix}.$$

When we demand the positivity of $(Q - \dot{P})$ at the four vertices and $d \to 0$ we get $\alpha < 1.54$, if $\beta = 2, \gamma = 5$.

5.5.

$$Q = 2a_1a_2, \quad P = \begin{bmatrix} a_1^2 + a_2^2 + a_2 & a_1 \\ a_1 & 1 + a_2 \end{bmatrix},$$

$$\dot{P} = \begin{bmatrix} 2a_1 & 1 \\ 1 & 0 \end{bmatrix} \dot{a}_1 + \begin{bmatrix} 2a_{2+1} & 0 \\ 0 & 1 \end{bmatrix} \dot{a}_2.$$

Now $Q - \dot{P} > 0$ will follow if $2\underline{a}_1\underline{a}_2 - \left\| \begin{matrix} 2\overline{a}_1 & 1 \\ 1 & 0 \end{matrix} \right\|_\infty \alpha - \left\| \begin{matrix} 2\overline{a}_2 + 1 & 0 \\ 0 & 1 \end{matrix} \right\|_\infty \alpha > 0.$

This inequality will follow if $\alpha < \frac{\beta^2}{2\gamma+1} = \frac{4}{11} = 0.363$ for $\beta = 2, \gamma = 5$.

5.6. Q and P as in 5.5.

Now $Q - \dot{P} > 0$ is implied by $2a_1a_2 - \left\| \begin{matrix} 2a_1 & 1 \\ 1 & 0 \end{matrix} \right\|_\infty \alpha - \left\| \begin{matrix} 2a_2 + 1 & 0 \\ 0 & 1 \end{matrix} \right\|_\infty$ $\alpha > 0$.

Because of affine occurence of the parameters we get after evaluation at the four vertices $\alpha < 0.8$ for $\beta = 2$ and $\gamma = 5$.

REMARK 5.1.

The approach of 5.1 is easier to use than that of 5.2 as we deal with less parameters.

5.4 gives better results than 5.3 due to the additional free parameter.

5.5 is more conservative than 5.6

The methods of 5.1 and 5.2 can be used for $n > 2$ as they give vertex results.

6. Conclusion. Results are presented which solve at least partially the open problem in [7] , where the coefficients of a linear differential equation lie in given intervals, the rates of variations of these coefficients are restricted, and exponential stability is required.

If affine or multiaffine P, Q or the Lyapunov equation are obtained, then vertex conditions are obtained which are sufficient for stability. With the Lyapunov function obtained using the critical stability conditions, vertex results are obtained only for $n = 2$. For higher order systems a complex optimization problem has to be solved. It is possible to extend the above results to discrete systems.

References

[1] V.L. Kharitonov: Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. Differential' nye Uravneniya, vol. 14, pp. 1483–1485, 1979.

[2] J.-H. Su and I.-K. Fong: New robust stability bounds of linear discrete-time systems with time varying uncertainties. Int.J.Control, vol. 58, pp. 1461–1467, 1993.

[3] P.H. Bauer and K. Premaratne: Robust stability of time-variant interval matrices. Proc.29th IEEE Conf.Decision and Contro, Honolulu, HI, Dec.1990, pp. 334–335.

[4] P.H. Bauer, M. Mansour and J. Duran: Stability of polynomials with time-variant coefficients, IEEE Trans.Circuits Syst.-I, vol.40 pp. 423–426, June 1993.

[5] P.H. Bauer, K. Premanatne and J. Duran: A necessary and sufficient condition for robust stability of time-variant discrete systems. IEE Trans. Aut. Contro,, vol 38, pp. 1427–1430, Sept. 1993.

[6] K. Premaratne and M. Mansour: Robust stability of time-variant descrete-time systems with bounded parameter perturbations. IEE Trans.Circuits Syst.-I vol 42, pp. 40–45, Jan 1995.

[7] M. Mansour, S. Balemi and W. Truol, Eds.: Robustness of dynamic systems with parameter uncertainties. Birkhäuser, 1992

[8] W. Hahn: Stability of motion. Springer, 1967.

[9] B.D.O. Anderson, S. Dasgupta, P Khargonekar, F.J Kraus and M. Mansour: Robust strict positive realness: characterization and construction. IEEE Trans. Circuits, Systems, vol.37, pp. 869–876, July 1990.

[10] S. Dasgupta, G. Chockalingam, B.D.O. Anderson and M. Fu: Lyapunov functions for uncertain systems with applications to the stability of time varying systems. IEEE Trans. Circuits Syst.-I vol.41, pp. 93–106, Feb 1994.

[11] S. Dasgupta and A.S. Bhagwat: Conditions for designing strictly positive real transfer functions for adaptive output error identification. IEEE Trans. Circuits Syst, vol.34, pp. 731–737, 1987.

[12] B.D.O. Anderson, E.I. Jury and M. Mansour: On robust Hurwitz polynomials. IEEE Trans. Aut. Control, vol.32, pp. 909–913, 1987.