I. INTRODUCTION

The purpose of this chapter is to introduce a technique, and illustrate its application, in two problems of sampled-data control. These two applications, which do not exhaust the range of potential use of the technique, are controller discretization, and sampled-data $H_{\infty}$-controller design. We shall describe each of these problems in turn.

The problem of controller discretization arises in designing digital controllers for use on continuous-time plants. [The motivation for using a discrete-time controller in such a situation hardly needs spelling out]. There are several standard techniques for discrete-time controller design.

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They are usually tied to one of the following two broad approaches.

- one can replace the continuous time plant by a discrete time model (derived by sample and hold considerations) and design a discrete-time controller by any one of a number of methods [1-3].

- one can design a continuous controller, and then one can discretize it. Many methods are offered in textbooks for executing the discretizing step [1-2].

The main potential disadvantage with the first approach is that consideration of intersample behaviour is suppressed at the start of the design step. There are exceptions, such as the LQG design method set out in e.g. [2]. A secondary disadvantage is that discretization of a plant where physical parameters appear prominently in the continuous-time model generally leads to those parameters being smeared all over the matrices of an associated discrete-time state variable realization, so that insight derived from physical considerations is lost.

The major potential disadvantage of the second approach is the failure of all the textbook methods of controller discretizations to respect a certain dictum, indicated momentarily; the consequence is that sampling intervals may have to be taken much shorter than is really necessary, in order that the particular method being used leads to satisfactory closed-loop behaviour.

Controller discretization, in the sense of replacing a continuous-time controller by a discrete-time controller with sample and hold, is a form of approximation. This being so, one should have in mind some sort of error for measuring the quality of approximation. This error reasonably should reflect closed-loop properties, and since the plant as well as the controller forms part of the closed loop, it follows that the controller discretization task should in some way reflect the plant. Yet none of the many schemes of [1-2] for controller discretization involve the plant: the discrete-time controller transfer function is obtained from the continuous-time controller transfer function only. Such schemes are therefore flawed, just like any scheme for order reduction of a high-order
controller is flawed if the scheme does not introduce somehow consideration of the plant, [4]. Of course, the discretization schemes of [1-2] are not fatally flawed. They will work, at least if a suitably small sampling interval is taken; but with larger sampling intervals, they can produce unattractive or even unstable behaviour of the sampled-data system, when a scheme based on sounder premises can behave well.

In this chapter, we will reformulate the controller discretization problem, in a way which reflects closed-loop objectives and therefore involves the plant. Then we shall show how, using a certain tool, the problem can be solved.

The second problem we consider is the generalization of the ideas of $H_{\infty}$ design to the sampled-data case. We work with the scheme of figure 1, in which $w, z, u, y$ are to be thought of as disturbance, to-be-controlled output, feedback input, and measurement of a continuous-time time-invariant plant. The controller should stabilize the closed loop, and minimize or ensure a bound on the gain from $w$ to $z$; the gain has to be measured somehow, and with $w$ and $z$ being measured by their $L_2$ (time) norm, the gain is the induced operator norm. Now if the controller is (like the plant), a linear, time-invariant, continuous-time entity, the (closed-loop) operator from $w$ to $z$ has a transfer function representation, $H(s)$ say, and the induced operator norm is given (via a standard result) by $||H||_{\infty}$.

![Fig. 1. Arrangement for $H_{\infty}$ design problem. The controller is to stabilise and minimize or ensure a bound on the gain from $w$ to $z$](image-url)

The sampled-data version of this problem replaces the controller by a
sampler, discrete-time controller, and hold. (An anti-aliasing filter may also be introduced). Again, the controller must stabilize the closed-loop, and minimize (or at least achieve a bound on) the gain from $w$ to $z$. The closed-loop operator is no longer representable by a transfer function, but it is linear and periodically time-varying. It certainly has an induced norm.

Rather forbidding calculations are available to provide an exact solution to this problem, [5-7]. We illustrate here a conceptually far simpler approach, based on using the same tool as used for the discretization problem. This tool gives an approximate answer, but one for which the approximation error can be made arbitrarily small.

The rather forbidding calculations of [5-7] will not be quickly executable using standard commercial software. In contrast, the methods developed here for the sampled-data $H_\infty$ problem and controller discretization problem are very easily implemented using commercial packages.

A modification of the $H_\infty$ problem (continuous-time or sampled-data version) replaces the $L_2$ norm of $w$ and $z$ by their $L_\infty$ norms. There is of course still an induced operator norm for the closed-loop system. For the sampled-data version, the approach based on the tool described here is the only one so far known.

The tool to which we have been referring to this point is a procedure whereby the induced $L_2$ norm of a linear periodically time-varying operator can be evaluated. The operator results from the interconnection of a number of separate blocks, viz sample elements, hold elements, and linear systems which are time-invariant and operate either in continuous-time or discrete time. Other possible interconnecting components, eg periodic gains, could also be accommodated very easily, but are not needed for this chapter. As will be explained later (in Section 3), a two step procedure can be used for the evaluation of the norm. The first step involves replacement of the continuous-time elements by fast sampled equivalents, so that a multirate discrete-time system results. The second step (involving "blocking" or "lifting") replaces the multi
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rate system by single rate, time-invariant, discrete-time system. The originally sought norm can be obtained (as an approximation but with arbitrarily small error) as the norm of an operator associated with the last system, and this last norm is very easily computable.

The tool is used for both the controller discretization problem, and the sampled-data $H_\infty$ problem. [In fact, as is shown in Section III, the controller discretization problem can be set up as a sampled-data $H_\infty$ problem, so it is not very surprising that the same tool is applicable to both problems].

Now let us explain the structure of the chapter. Section II and III are both concerned with problem formulation, for the controller discretization and sampled-data $H_\infty$ problems respectively. Section IV exposes the tool for solving these problems. Section V contains additional remarks and Section VI examples.

Two important appendices are included, which are essentially self-contained. The first, largely based on [10], is concerned with setting out rigorously the ideas of state, stabilizability, Lyapunov and input-output stability for sampled-data systems. The second is concerned with relating the gain of systems with and without sampling, establishing the crucial theoretical result underpinning the method of Section IV.

II. REFORMULATION OF THE CONTROLLER DISCRETIZATION PROBLEM

In this section, we shall show how the controller discretization problem can be reformulated as the problem of minimizing the induced norm of a linear, periodically time-varying operator. Actually, motivated by two different aspects of closed-loop behaviour, we will present two different operators whose norm is to be minimized.

A. STABILITY MARGIN APPROACH

Let $P(s), C(s)$ and $\tilde{C}(z)$ denote respectively the plant transfer function, the already-designed continuous-time controller transfer function,
and the to-be-found discrete-time controller transfer function. The
discrete-time controller is assumed to have a prescribed sampling rate $T$,
and to be used in conjunction with a zero order hold $H(s)$, and sampler $S$
and antialiasing filter of transfer function $F_a(s)$. See Figures 2 and 3.

Fig. 2. The continuous time loop

Fig. 3. The sampled-data loop; $F_a(s)$ and $H(s)$ are the antialias
and hold transfer functions.

We need to define formally the operation of the sampler and hold. The
sampler is assumed to operate on signals which are piecewise continuous
on $[0, \infty)$, bounded on compact subsets of $[0, \infty)$ and continuous from
the left at every point except the origin. Call this set $B$. Let $B_T$ denote
the subset of $B$ where discontinuities are confined to points $0, T, 2T, \ldots$.

Then if $y = Su$ there holds

\[
\begin{align*}
y(0) &= 0 \\
y(k) &= u(kT) \quad k \geq 1
\end{align*}
\]
The output of the hold element is in the set $B$, in fact $B_T$. If $w = Hv$, 

$$
w(0) = v(0) \\
w(t) = v(k) \quad kT < t \leq (k + 1)T$$

A number of other assumptions on the set-up of Figure 3 also need to be made.

**Assumption A1**

(i) The closed-loop using the continuous-time controller $C(s)$ is stable

(ii) $F_a(s)$ is open-loop stable, with $F_a(\infty) = 0$

(iii) $P(s), C(s)$ and $F_a(s)$ have finite-dimensional representations

(iv) $P(\infty) = 0$

(v) Whenever $\mu$ is a pole of $P(s)$ with nonnegative real part, none of the points $\mu + j2\pi k/T, k \neq 0$, is a pole of $P(s)$; and none of the points $j2\pi k/T, k \neq 0$, is a pole of $P(s)$

All these assumptions are of cause unexceptionable if not completely reasonable. The finite-dimensionality assumption is needed primarily to ensure ease of computation. Assumption A1(v) ensures that the discretization of the loop does not introduce unstabilizable or undetectable modes, [10]. See also Appendix A.

In addition, in the Stability Margin approach to discretization, we require a further assumption which is clearly somewhat restrictive. We comment later on its removability.
Assumption A2

The controller $C(s)$ is open-loop stable.

The word stable used above in several places is meant to exclude poles on $Re[s] = 0$ as well as $Re[s] > 0$, to exclude unstable pole-zero cancellations, and to exclude the possibility that for a closed-loop the closed-loop transfer function from any intermediate input point to any point on the loop can be unstable.

There is no requirement that we restrict attention to scalar systems. In the multivariable case, the sampler $S$ and anti-aliasing filter $F_a(s)$ become diagonal transfer functions or operators.

Now let us redraw the sampled-data loop to emphasis how it differs from the continuous-time loop, see Figure 4. We can think of $\Delta$, the operator defined by the dashed box, as a perturbation to the continuous-time controller $C(s)$ which turns it into the discrete-time controller $C_d(z)$. The operator $\Delta$ is linear, and periodically time-varying.

![Fig. 4. The sampled-data loop represented as the continuous-time loop with perturbation $\Delta$](image-url)
We need to establish that $\Delta$ is a bounded operator. For this, we need first:

**Assumption A3**

$C_d(z)$ is stable.

(This assumption will later be verified as being fulfilled in an algorithm to be presented) Notice that to this point, we have not said how $C_d(z)$ is to be found. We have assumed however that $C(s)$ is stable. It is not surprising therefore that a good algorithm should carry this property of $C(s)$ over to the $C_d(z)$ involved in its approximation. That this indeed happens will later be verified.

In the sequel, the word "stable" will be used in describing (closed) sampled-data loops, and some care must be taken to interpret it. Appendix A surveys a number of issues pertaining to sampled data systems, including their stability.

The discussion of Appendix A is for plant-controller interconnections with the structure of Figure 1, where the controller comprises a strictly proper anti-aliasing filter, sampler, discrete controller and zero order hold. It is obvious that the set up of Figure 3 can be recovered as a special case. Key issues dealt with in Appendix A are the following:

(a) a hybrid state can be defined, such that knowledge of the hybrid state and subsequent external input allows the system's subsequent behaviour to be determined.

(b) if the plant is strictly proper, and the restriction of $\nu$ to $[0, T]$ for every finite $T$ is $L_p$-integrable, then $\xi \in B_T$ irrespective of the closed-loop stability.

(c) if the continuous time plant is stabilizable and detectable from $u$ and $y$, (in Figure 1), if $P(s)$ and $F_d(s)$ are rational with $F_d(s)$ strictly proper, and if condition A1.(v) holds, a stabilizing discrete compensator can be found, so that a nonzero hybrid state decays exponentially fast to zero under zero input conditions.
(d) stability in the above sense means that the closed-loop system defines a bounded operator from \( w \in L_p[0, \infty) \) to \( z \in L_p[0, \infty) \) \( \forall p \in [1, \infty] \).

When we use the word stable for a sampled-data system, we mean that the two properties mentioned in (c) and (d) are present: exponentially fast decay of the hybrid state under zero input conditions, and \( L_p[0, \infty] \) bounded-input, bounded-output stability.

A further observation is also needed in order to conclude that \( \Delta(s) \) is bounded. This is established in Appendix A as Lemma A.7, and is due to [11].

**FACT II.1**

Considered as a mapping from\(^2\) \( L_2[0, \infty) \) to \( l_2(Z_+) \), the operator \( SW(s) \) for any finite-dimensional stable \( W(s) \) with \( W(\infty) = 0 \) is bounded, though \( S \) is not a bounded operator from \( L_2[0, \infty) \) to \( l_2(Z_+) \).

Since \( H : l_2(Z_+) \rightarrow L_2[0, \infty) \), \( C_d(z) : l_2(Z_+) \rightarrow l_2(Z_+) \), \( SF_a : L_2[0, \infty) \rightarrow l_2(Z_+) \) and \( C(s) : L_2[0, \infty) \rightarrow L_2[0, \infty] \) are all bounded, on account of Assumptions A1 - A3 and the above Fact II.1, it follows that

\[
\Delta = H C_d S F_a - C
\]

is a bounded operator for \( L_2[0, \infty) \) to \( L_2[0, \infty) \).

---

\(^2\) \( L_2[0, \infty) \) and \( l_2(Z_+) \) will be taken to include vectors functions each of whose entries are \( L_2 \) or \( l_2 \).
Now consider the redrawing of Figure 4 depicted in Figure 5. Notice that Assumption A1 guarantees the boundedness of the operator defined by \((I + PC)^{-1}P\), while as we have just argued, \(\Delta\) is also bounded. It follows that the operator

\[ J_c \equiv \Delta(I + PC)^{-1}P \quad (2) \]

is also bounded. By a variant on the small-gain theorem, [12], the loop in Figure 5 is bounded-input, bounded-output stable if

\[ ||J_c|| < 1 \quad (3) \]

As an aside we comment that in using here the phrase “bounded-input, bounded-output stable”, we mean that the output norm is \(L_2[0, \infty)\) if the input norm has this property; however, as explained in Appendix A, the above fact concerning \(SW(s)\) is a subresult of a result that says \(SW(s)\) is bounded \(L_p[0, \infty)\) to \(l_p(Z^+), \) for \(1 \leq p \leq \infty\), and the BIBO property will actually apply for any \(L_p[0, \infty)\). Particularly because of our interest in solving a sampled-data version of the \(H_\infty\) problem later (to which we connect the controller reduction problem), it is relevant to focus on \(L_2[0, \infty)\).

Using the above definition of \(J_c\), we can now relate controller approximation to stability (just as has been done for the problem of approximating a high order \(s\)-domain controller transfer function by a low order \(s\)-domain controller transfer function, see [4]).

To guarantee stability when \(C\) is replaced by \(C_d\) (more accurately \(HC_dSF_a\)), it is sufficient that \(||J_c|| < 1\). More than this however, we see that in some sense, to maximize the stability robustness after replacement of \(C\) by \(C_d\), it is reasonable to seek \(J_c\) not only so that the inequality holds but also so that \(||J_c||\) is minimized. Hence, from the point of view of stability robustness criterion, we have:

**FACT II.2**

With \(P(s), C(s), C_d(z), H(s), S\) and \(F_a(s)\) denoting respectively the plant transfer function, continuous-time controller transfer function, discrete-time controller transfer function,
hold element transfer function, sampler of period $T$ and anti-
aliasing filter-transfer function, and under Assumption A1
through A3 the optimum discretization of $C(s)$ is achieved
by choosing a stable $C_d(z)$ to minimize $||J_e||$, where the lin-
ear, periodically time-varying operator $J_e$ is given by

$$J_e = (HC_dSF_e - C)(1 + PC)^{-1}P$$  (4)

As commented earlier, the ideas leading to (4) have also been used in
studying the controller order reduction problem. There, it is possible to
treat open-loop unstable $C(s)$ via two approaches. First, one can write
$C(s)$ as $C_+(s) + C_-(s)$, where $C_+(s)$ has all poles in $Re[s] \geq 0$ and $C_-(s)$
all poles in $Re[s] < 0$; then one approximates $C_-(s)$, and copies $C_+(s)$
additively into the reduced order controller transfer function. There
seems no analog of this possibility here. Second, in seeking to reduce
an unstable controller, one can work with a fractional representation
of $C(s)$ as $N_C(s)D_C^{-1}(s)$, in which $N_C,D_C$ are both proper and stable.
One can consider stability robustness as before, arriving at a problem of
minimizing for low order stable $\hat{N}_C(s),\hat{D}_C(s)$ an index

$$J_{uc} = \left\| \begin{bmatrix} N_C - \hat{N}_C \\ D_C - \hat{D}_C \end{bmatrix} W \right\|_\infty$$  (5)

where $W$ is a certain weighting. An analog of this procedure was re-
cently developed for the discretization problem. It handles problems
with unstable $C(s)$, i.e. where Assumption A2 was not fulfilled.

B. CLOSED-LOOP TRANSFER FUNCTION APPROACH

In deriving the index $||J_e||$ appearing in Fact 2.2, we used stability
robustness ideas and required, under Assumption A2, the open-loop
stability of $C(s)$. We shall here derive a different closed-loop measure,
but like $||J_e||$ the induced $L_2$ norm of a linear periodically time-varying
operator.

We retain the same use of the symbols $P(s), C(s), C_d(z), H(s), S$
and $F_a(s)$, and we retain Assumption set A1. We drop Assumptions A2
Let $G(s)$ denote interchangeably the transfer function $G(s) = PC(I + PC)^{-1}$ and the associated input/output operator of Figure 2. Let $G_h$ denote the closed-loop operator of the sampled-data system of Figure 3. The following criterion for performance deterioration will be used:

$$J_h = (G_h - G)W$$  \hspace{1cm} (6)$$

Here, $W(s)$ is a stable, strictly proper transfer function shaping the reference signal. The operator $J_h$ is depicted in Figure 6; it is a linear, periodically time-varying operator mapping $v$ into $e_h$.

![Fig. 6. Illustration of the operator $J_h$ whose induced norm minimization is sought.](image)

The problem of discretizing $C(s)$ becomes now the problem of selecting a $C_d(z)$ which will stabilize $G_h$ and will minimize the induced $L_2$ norm of $J_h$.

Notice that (6) can be rewritten formally as

$$J_h = PHC_dSF_a(I + PHC_dSF_a)^{-1} - GW$$
$$= PHC_d(I + SF_aPHC_d)^{-1}SF_aW - GW$$
$$= T_{11} + T_{12}C_d(I - T_{22}C_d)^{-1}T_{21}$$  \hspace{1cm} (7)$$

for certain operators $T_{ij}$ that are readily identifiable from the algebra and are known. Notice that were the $T_{ij}$ all discrete-time transfer functions, the problem of selecting $C_d$ to minimize $J_h$ would be a standard $H_\infty$
problem. What makes the problem unusual is the fact that the $T_{ij}$ are periodic time-varying operators. Similarly, the minimization of $||J_c||$, see (4) is like a standard $H_\infty$ problem (with $T_{22} = 0$), except that (operators defined by) discrete-time transfer functions are replaced by periodically time-varying operators.

III. A QUICK LOOK AT THE $H_\infty$ SAMPLED-DATA PROBLEM

In this section, we will indicate how the $H_\infty$ sampled-data problem acquires the same form as that of the controller discretization problem. [Let us remark that the phrase “$H_\infty$ sampled-data problem” is technically improper: $H_\infty$ functions are not the issue, once the operators become time-varying. The concern is with induced $L_2$-norm optimization, which only involves $H_\infty$ functions in the time-invariant situation].

Recall Figure 1. Suppose that the controller block is a cascade of antialiasing filter, sampler, discrete-time controller and hold. Then we can redraw the set up as shown in Figure 7. The operator defining the systems in dashed lines is linear and periodically time-varying. The input...
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and output spaces are $L_2 \otimes l_2$ (disregarding whether vector or scalar functions are involved). Define $T_{ij}$ by

$$
\begin{bmatrix}
  z \\
  u
\end{bmatrix} = \begin{bmatrix}
  T_{11} & T_{12} \\
  T_{21} & T_{22}
\end{bmatrix} \begin{bmatrix}
  w \\
  \psi
\end{bmatrix}
$$

We see that the closed-loop operator is

$$
G_h = T_{11} + T_{12}C_d(I - T_{22}C_d)^{-1}T_{21}
$$

and it is the induced $L_2$ norm of this operator which is to be minimized. The parallel with (7) is very clear. Note in fact just how close the parallel is, see Table 1.

**Table I. Comparison of Discretization and $H_\infty$ Sampled-Data Problem**

<table>
<thead>
<tr>
<th>Meaning in Discretization</th>
<th>Meaning in Sampled-Data Problem (7)</th>
<th>$H_\infty$ Problem (9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{11}$</td>
<td>$-GW$</td>
<td>$T_{11c}$</td>
</tr>
<tr>
<td>$T_{12}$</td>
<td>$PH$</td>
<td>$T_{12c}H$</td>
</tr>
<tr>
<td>$T_{21}$</td>
<td>$SF_aW$</td>
<td>$SF_aT_{21c}$</td>
</tr>
<tr>
<td>$T_{22}$</td>
<td>$-SF_aPH$</td>
<td>$SF_aT_{22c}H$</td>
</tr>
</tbody>
</table>

The above table puts in evidence the fact that the controller discretization problem is simply a special $H_\infty$ sampled-data problem, something that is almost evident when one realizes that Figure 6 can be redrawn as Figure 7, when $v$ and $e_h$ of Figure 6 are identified with $w$ and $z$ of Figure 7.

We remark that it is easy to demand that $C_d(z)$ necessarily contain a delay in the $H_2$ sampled-data problem (a common practical constraint). One simply replaces $T_{21} = SF_aT_{21c}$ by $z^{-1}SF_aT_{21c}$ and $T_{22} = SF_aT_{22c}H$.
by \( z^{-1}SF_nT_{22}K \). Let \( \mathcal{C}_{dn}^* \) be the optimum with \( T_{21} \) and \( T_{22} \) so modified; then the desired \( \mathcal{C}_d \) (with delay and unmodified \( T_{21} \) and \( T_{22} \)) is simply \( z^{-1}\mathcal{C}_{dn}^* \).

Such a delay is often introduced because it is too hard to meet speed-of-computation requirement for a \( \mathcal{C}_d(z) \) that is not strictly proper (needing calculations that are theoretically infinitely fast but practically achieved in a much shorter time than \( T \)).

If calculations can be achieved in time less than \( T \) (but not very much smaller than \( T \)), use of a delay of \( T \) is wasteful, and may impede the achieving of satisfactory performance. In Section V, a technique is described for handling a delay between 0 and \( T \) in the controller.

IV. OPERATOR APPROXIMATION

In this section, we shall describe a procedure whereby a linear periodically time-varying operator of the type covered in the previous sections can be approximated in a two-step procedure, first by a multi-rate time-invariant discrete-time operator and then by a single-rate time-invariant discrete-time operator. The approximation procedure brings with it several key features.

- the (induced \( L_2 \)) norm of the single-rate discrete-time operator is simply the \( H_\infty \) norm of the associated transfer function, the determination of which is standard

- the problem of choosing \( \mathcal{C}_d(z) \) to minimize the norm of the periodically time-varying operator becomes a standard \( H_\infty \) problem of choosing (the same) \( \mathcal{C}_d(z) \) to minimize the \( H_\infty \) norm of a transfer function

Even though an approximation is involved in this second step, it can be made arbitrarily accurate.

The word approximation is being somewhat misused here, since one cannot approximate one operator by another with different domain and range, and yet that is apparently what we are doing. Certainly though,
as will be seen, we are approximating norms, and we are approximating the interval behaviour of one system by the internal behaviour of another, even when input and output spaces are distinct.

A. MOTIVATION OF THE APPROXIMATION PROCEDURE

In the two versions of the controller discretization problem formulated, and in the sampled-data $H_\infty$ problem, we are interested in computing the induced norm of an operator mapping $L_2[0,\infty)$ into $L_2[0,\infty)$, which is obtained by combining different sorts of simpler operators, including discrete-time operators with period $T$. This operator is exemplified by that from $w$ to $z$ in Figure 7.

Now suppose that this operator $J$ is replaced by an operator mapping $l_2[0,\infty)$ to $l_2[0,\infty)$ and obtained by very fast sampling. More precisely:

- the $L_2[0,\infty)$ input $w(\cdot)$ is replaced by an $l_2[0,\infty)$ input which passes into a hold element of length $\tau$ (with $\tau$ very small and a submultiple of $T$), the output of the hold element entering the original system
- the $L_2[0,\infty)$ output $z$ is sampled by a sampler of period $\tau$, resulting in an $l_2[0,\infty)$ output.

![Diagram](image)

**Fig. 8.** Replacement of continuous time operator $J$ by discrete-time (multi-rate operator) $S_\tau J H_\tau$
See Figure 8 for the idea. Then, as established in Appendix B, one can conclude that

$$\lim_{T \to 0} \|S_r J H_r\| = \|J\|$$

Thus, from the point of view of gains, the system with discrete-time input and output behaves like the system with continuous-time input and output.

We will argue below in Section B that because $\tau$ is taken as a submultiple of $T$ the new system becomes a standard multirate system.

This means that the problem of norm evaluation (and minimization through controller selection) for the original system with continuous-time input and output becomes a problem of norm evaluation (and minimization) for a multirate discrete-time system, albeit with some approximation, since equality of the continuous-time system and multirate system norms only holds in the limit as $\tau$ goes to zero. However, for small enough $\tau$, a good approximation must result.

The next step in the approximation is to recognize that, using a reordering of inputs and outputs, a multi-rate system can be reorganized as a single-rate discrete-time system (with sampling interval equal to the slowest rate of the multi-rate system). The single-rate discrete-time system, having different input and output sequences to the multi-rate system, has also a different input-output operator. However, the operator is easy to find, has the same induced norm as that of the multi-rate system, and most importantly has a standard transfer function matrix description. The induced $L_2$-norm of the operator is therefore a standard $H_\infty$ norm, is equal to the induced $L_2$-norm of the operator for the multirate system, and approximates the induced $L_2$-norm of the original system, the quantity we are seeking.

Minimization of the $L_2$-norm of the single-rate system operator is turned into a standard $H_\infty$ problem. The controller solving this problem is the controller solving (approximately, but with arbitrarily small error) the original problem.
B. GENERATING A MULTI-RATE SYSTEM AND ASSOCIATED OPERATOR

Let us consider first the operator $J_c$ associated with the first approach to controller discretization of Section II A:

$$J_c = \Delta (I + PC)^{-1} P$$  \hspace{1cm} (2)

$$= H_T C_d(z) S_T F_a W - CW$$  \hspace{1cm} (11)

where

$$W = (I + PC)^{-1} P$$  \hspace{1cm} (12)

We have indicated the time-interval associated with the sample and hold by subscripting as shown. Let $\tau = T/N$ for some integer $N$ and suppose that the two samplers are synchronous at time $t = 0, T, 2T, \ldots$. Then we generate a multi-rate system with operator $J_{cr}$ by

$$J_{cr} = S_T H_T C_d(z) S_T F_a W H_\tau - S_T H_\tau S_T C W H_\tau$$  \hspace{1cm} (13)

By the result quoted earlier, we know that

$$\lim_{\tau \to 0} \|J_{cr}\|_2 = \|J\|_2$$

Actually, it proves convenient to make a very minor change. Recognize that $C(s)W(s)$ is strictly proper and stable. Hence

$$\lim_{\tau \to 0} \|(I - H_\tau S_T) C W\|_2 = 0$$

Therefore instead of working with $J_{cr}$ above, we can equally work with

$$J_{cr}' = S_T H_T C_d(z) S_T F_a W H_\tau - S_T H_\tau S_T C W H_\tau$$  \hspace{1cm} (14)

and we will have

$$\lim_{\tau \to 0} \|J_{cr}'\|_2 = \|J\|_2$$  \hspace{1cm} (15)
Fig. 9. The multirate operator $J_{\tau \nu}$; the operator $\Sigma_N$ selects one out of every $N$ samples, and $\Sigma_N^*$ takes its input at time $kT$ and reproduces this input at its output at times $kT, kT + T/N, kT + 2T/N, \ldots (k + 1)T - T/N$. Sampling intervals of $\tau$ and $T$ are indicated for each signal.

Consider now what (14) means. Figure 9, depicting $J_{\tau \nu}'$, will assist in understanding. Evidently, $S_\tau \hat{C}W \hat{H}_\tau$ is representable by a standard $z$-transform associated with sampling interval $\tau$, labelled $\hat{C}W(z_{\tau})$ in the figure, while $S_\tau H_\tau$ is simply a unit delay (of time $\tau$), labelled $z_{\tau}^{-1}$ in the figure. Also $S_\tau F_\nu WH_\tau$ can be represented by cascading an operator $S_\tau F_\nu WH_\tau$ with one selecting 1 out of every $N$ successive samples; call such an operator $\Sigma_N$. The operator $S_\tau F_\nu WH_\tau$ has a representation by a standard $z$-transform associated with sampling interval $\tau$, labelled $\hat{F}_\nu W(z_{\tau})$ in the figure. The operator $H_\tau$ copies the output of $C_d(z)$ at each time $kT$ for the next $T$ seconds as a constant (continuous-time) signal. The operator $S_\tau H_\tau$ evidently copies the output of $C(z)$ at time $kT$, replicating its value at $kT + \tau, kT + 2\tau, kT + 3\tau, \ldots (k + 1)T$. This is equivalent to replicating the value of the output of $C(z)$ at time $kT$ at times $kT, kT + \tau, \ldots, (k + 1)T - \tau$ and then delaying by time $\tau$. Call the replicating operator $\Sigma^*_{\tau \nu}$.

Of course, the $z_{\tau}$-transform representations referred to are straightforward to obtain in state-variable form from state-variable descriptions.
of the continuous-time transfer function matrices $C_W$ and $P_aW$.

Let us now consider the alternative operator for controller discretization of Section II.B, based on matching of closed-loop operators:

$$J_h = PH_T C_d(z) S_T P_a (I + PH_T C_d(z) S_T P_a)^{-1} W - GW$$  \hspace{1cm} (16)

The associated operator $J_h$ is depicted in Figure 10.

---

**Fig. 10.** The multirate operator $J_h$

The operator is redrawn in Figure 11. As before, the operator $\Sigma_N$ selects one out of every $N$ successive samples and the operator $\Sigma_N^*$ reproduces the output of $C(z)$ at times $kT, (k+1)T, \ldots, (N-1)T$. Notice that the subblock in dashed lines has a $z_T$-transform description that is easily found.

For the general $H_\infty$ problem, the operator depicted in Figure 12 results. Figure 12 has a redrawing like Figure 11.
Fig. 11. Redrawing of the operator $J_{hr}$ of Figure 10. The subsystem in dashed lines has a $z_r$-transfer function description.

Fig. 12. The multi-rate system associated with the sampled-data $H_\infty$ problem.
C. CHANGING THE RATE OF A SINGLE RATE SYSTEM

Before considering how a system with two sampling rates can be represented as a system with one sampling rate, let us observe how a single fast-rate system can be converted to a single slow-rate system. It will then be easier to understand the procedure applicable to multirate systems. Consider a system with underlying sample-time \( \tau \), defined by

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k \\
    y_k &= Cx_k + Du_k
\end{align*}
\]

We construct a system with sampling time \( T = N\tau \), by grouping \( N \) successive inputs of the fast system (17) into a single input of the slower system, and \( N \) successive outputs of the fast system into a single output of the slower system. Thus the slow system input sequence is:

\[
\begin{align*}
    \bar{u}_0 &= [u'_0 \; u'_1 \; \ldots \; u'_{N-1}]'
    \\
    \bar{u}_1 &= [u'_N \; u'_{N+1} \; \ldots \; u_{2N-1}]'
    \\
    & \vdots
\end{align*}
\]

and the output sequence is

\[
\begin{align*}
    \bar{y}_0 &= [y'_0 \; y'_1 \; \ldots \; y'_{N-1}]'
    \\
    \bar{y}_1 &= [y'_N \; y'_{N+1} \; \ldots \; y_{2N-1}]'
    \\
    & \vdots
\end{align*}
\]

The slow system's inputs and outputs are a rearranged version of those for the fast system. The beauty of this scheme is that the slow system is described by state-variable equations which are easily obtained from (17):

\[
\begin{align*}
    \bar{x}_{k+1} &= A^N \bar{x}_k + [A^{N-1}B \; A^{N-2}B \; \ldots \; B] \bar{u}_k \\
    \bar{y}_k &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} \bar{x}_k + \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & CA^{N-4}B & \cdots & D \end{bmatrix} \bar{u}_k
\end{align*}
\]
Notice also that with $T = N\tau$, because the $\bar{u}_i$ sequence is merely a rearranged version of the $u_k$ sequence, $||\bar{u}_k||_2 = ||u_k||_2$, and likewise $||\bar{y}_k||_2 = ||y_k||_2$. Hence the induced $l_2$-operator norms of (17) and (19) are the same; these norms are the same as the $H_\infty$ norms of the associated transfer function matrices. The above technique is an old one in signal processing. Various references can be found in [10].

D. CONVERTING MULTIRATE SYSTEMS TO SINGLE RATE SYSTEMS

Our focus is now on the multirate systems of Figures 9, 11 and 12. Consider that of Figure 9. We have explained how one can change the rate of the individual blocks $c^o(z)$ and $F^oW(z)$ in Figure 9, so that after the change, the underlying sampling time is $T = N\tau$ rather than $\tau$. Call the resulting $z$-transform transfer function matrices $\bar{c}^o(z)$ and $\bar{F}^oW(z)$. The state-variable matrices are found from those of $\bar{c}^o(z)$ and $\bar{F}^oW(z)$ as indicated in Subsection C.

Recall that $\Sigma_N$ copies every $N$th input sample to its output. In the single rate system, clearly $\Sigma_N$ is replaced by $[I \ 0 \ 0 \ldots \ 0]$, there being $N$ block entries. Likewise, $\Sigma_N^r$ is replaced by $[I \ I \ldots \ I]^T$. Overall, the arrangement of Figure 9 is replaced by that of Figure 13. [The delays $z\tau^{r-1}$ in the upper and lower arms of Figure 9 do not affect the norm of the operator at all; Figure 13 differs from Figure 9 by the suppression of the effect of these delays]

\[\begin{align*}
\bar{F}^oW(z) &\rightarrow [I \ 0 \ldots \ 0] &\rightarrow C_d(z) &\rightarrow [I \ I \ldots \ I]^T \\
\bar{c}^oW(z) &\rightarrow [I \ 0 \ldots \ 0] &\rightarrow \overline{C}d(z) &\rightarrow [I \ I \ldots \ I]^T \\
&\rightarrow \overline{C}W(z)
\end{align*}\]

Fig. 13. Single rate system with induced $l_2$ norm approximately that of $J_G$
The optimum choice of $C_d(z)$ is now the solution of the following (standard) $H_\infty$ problem:

$$\inf_{C_d(z)} \left\| \begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \end{bmatrix} C_d(z) \left[ I \ 0 \ \cdots \ 0 \right] \bar{W}(z) - \bar{W}(z) \right\|_\infty$$

As a second example, consider the set-up of Figure 11. The subblock in dashed lines has a $z_\tau$-transform description, and on change of the rate, an associated $z$-transform description, call it

$$\begin{bmatrix} \tilde{T}_{11}(z) & \tilde{T}_{12}(z) \\ \tilde{T}_{21}(z) & \tilde{T}_{22}(z) \end{bmatrix}$$

One must now recognize that the plant which $C_d(z)$ sees is a modification of this, due to the action of $\Sigma_N$ and $\Sigma_N^*$. The effective plant is:

$$\begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix} \begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11}(z) & \tilde{T}_{12}(z) \\ \tilde{T}_{21}(z) & \tilde{T}_{22}(z) \end{bmatrix} \begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \end{bmatrix}$$

and $C_d(z)$ solves

$$\inf_{C_d(z)} \left\| \tilde{T}_{11} + \tilde{T}_{12}C_d(I - \tilde{T}_{22}C_d)^{-1}\tilde{T}_{21} \right\|_\infty$$

The general sampled-data $H_\infty$ problem is handled just the same way.
V. ADDITIONAL POINTS

A. THE $H_\infty$ PROBLEM

In the previous section, we have explained how the controller discretization problem and the $H_\infty$ design problem lead to a conventional $H_\infty$ problem. Let us observe immediately that it is a 4-block $H_\infty$ problem. For solvability of an $H_\infty$ problem, certain standard conditions must be satisfied, relating to existence of a stabilizing controller, and nonoccurrence of unit circle zeros. Certainly, a stabilizing controller exists, by virtue of the assumptions we have made, especially Assumption A1(v), introduced in Section II and explained in Appendix A. The unit circle zero condition seems very hard to track through the various operations of discretization. Suffice it to say that in all examples attempted, no difficulty has arisen.

B. CONTROLLER ORDER

Consider the controller discretization problem. Typically, the discrete controller will have order like the sum of the plant order and the continuous controller order. Thus order reduction may very well be needed.

C. CHOICE OF INTEGER $N$

In principle, the larger $N$ is, the more accurate is the method. However, numerical problems can arise if $N$ is taken very large. Experience reflected in examples is that $\tau$ should be something like $\frac{1}{20}$ of the inverse of the closed-loop bandwidth. Usually, with $N = 4$ intersampling is under perfect control. Unusually, $N = 10$ might have to be taken. Larger values of $N$ only need to be used if some inequality condition has to be met such as in the stability margin approach, with a large $T$. If the value of $N$ is large enough to cause numerical problems, one can use a two step procedure. Suppose $\tau$ and $T = N\tau$ are known. Let $N = N_1N_2$, with $N_1$ and $N_2$ integers of comparable value. First find a discrete controller
with sampling time $N_1\tau$. Then one can apply a second discretization to approximate it by a controller with sampling time $N_2N_1\tau$. The details are not hard to work out.

D. ALLOWING FOR COMPUTATIONAL DELAY IN THE CONTROLLER

A controller cannot normally accept a measurement and instantaneously produce an output. Suppose that $mr$ for some integer $m$ is the computational delay. Figure 14 depicts how this can be handled. Figure 14b shows the adjustment to $\Sigma_N^*$ as it appears in a multi-rate situation, Figures 9 and 11 for example. Figure 14c depicts the adjustment to the single rate equivalent of $\Sigma_N^*$, as in Figure 13 for example; the single rate equivalent of $\Sigma_N$ is unadjusted.

![Figure 14](image)

Fig. 14. (a) Controller with no computational delay
(b) Controller with computational delay of $mr$, input and output sequences with sampling interval $\tau$
(c) Controller with computational delay of $mr$, input and output sequences with sampling interval $T$
E. ADJUSTMENT OF T

It will often be the case that a value of $T$ as large as possible will be sought. None of the ideas of this chapter can really speed up the process of examining different values of $T$.

It is pertinent to ask what is an appropriate value of $T$? This is an issue addressed often in the classical texts; often the guidelines given for the choice of $T$ are unnecessarily restrictive, especially if input-output matching is the main objective. Thus the classical texts can be regarded as suggesting an underbound. In addition, we note that in using the stability margin approach, an index with $||J_c|| < 0.1$ results in almost perfect discretization. Thus one can choose $T$ via

$$T = \arg \max_T ||J_c|| \leq 0.1$$

If the value of 0.1 is increased to 0.5, prepassive deterioration can be expected, but performance may well still be acceptable.

F. OTHER APPROACHES TO CONTROLLER DISCRETIZATION

Apart from the standard (textbook) approaches to controller discretization referred to in the introduction, there is a scattering of other techniques. One is directly aimed at allowing large $T$; if the sampling interval $T$ is increased, one might imagine that for certain problems, controller discretization could become almost impossible. It is therefore worth noting the approach of [14] to controller discretization where stability is guaranteed for all $T$ (overshoot, ripple and the like being another matter). In considering the examples, we shall also have occasion to recall two methods which attempt to mimic closed-loop behaviour, i.e. in some way reflect the presence of the plant in the discretization process, see [15,16]. Many of the ideas of this chapter appear in [17-19].

G. STATE VARIABLE REALIZATION OF A DISCRETE-TIME CONTROLLER

A discrete-time controller will normally be implemented in a com-
puter or equivalent, so that all numbers will have finite word-length representation. Because of the associated truncation error, performance depends on the choice of state-variable realization for the controller transfer function \[20\]. In principle, one could pose the problem of simultaneously discretizing and choosing a state-variable realization.

H. CONTROLLER REDUCTION PROBLEM

We noted above that the discrete-time controller order may be high, even if the continuous-time controller from which it came has modest degree. Order reduction can be contemplated. Existing order reduction methods however are all based on purely continuous-time or purely discrete-time loops. Accordingly, there is scope for using the ideas of this chapter for tackling the sampled-data controller reduction problem — perhaps simultaneously with the discretization problem or the optimum state-variable realization problem.

I. OTHER PROBLEMS

Broadly speaking, the methods of this chapter allow almost any sampled-data design problem to be recast as a design problem for a purely discrete-time but always multirate plant. Issues of combined \(H_2/H_\infty\), optimum \(L_1\) design, or design based on closed-loop convexity could all in principle be examined.

VI. EXAMPLES

In this chapter, discrete-time controller design methods of three different levels of complexity have been proposed. The controller approximation approach of section II A makes use of a general stability criterion. There are no discretization parameters to select and therefore the method is very easy to use. Further, the minimum value of the approximation criterion is a good indicator for performance. A value of \(\|J_c\|\) smaller than 0.1 indicates good matching with the continuous-time system. As shown in the following, the results of the method are good, but
there is no possibility of guaranteeing that some particular closed-loop property is maintained (apart from stability). The closed-loop transfer function approach does not suffer from this shortcoming, because the goal transfer function can be selected to reflect the desired closed-loop property. But a cost of the approach is that one has to select the weighting function \( W(s) \). This is not a simple task and only a clever weighting strategy leads to a low order controller with improved performance. The direct controller design touched in Section III offers all the possibilities for controller design but the complexity of the design is significantly increased again.

The discretization methods are compared with the results of [15] and [16]. The authors propose discretization methods that try to mimic continuous-time closed-loop behaviour. A brief description of these methods is as follows. The discretization technique of [15] uses complex curve fitting to minimize the error between the continuous-time and the sampled data system. Aspects of closed loop stability are completely ignored. Kennedy and Evans [16] propose a controller redesign based on pole-zero matching. The resulting controller consists of a feedforward and a feedback part. Closed-loop stability and step response matching at the sampling instants can be achieved, but intersampling behaviour and closed-loop properties such as disturbance rejection are unaddressed.

We will borrow the examples of [15] and [16]. The example of [16] was first investigated in Katz [21]. The plant and controller transfer functions are listed in Appendix C. For both examples it was reported that common discretization methods produce either nonstabilizing controllers or systems with very poor closed-loop performance. The discretization methods of section II will be compared with the methods of [15] and [16]. For that purpose a discrete-time controller without anti-aliasing filter and with the same sampling periods as in [15] and [16] will be determined. Additionally, the discretization is repeated to obtain discrete-time controllers with anti-aliasing filter or with computational delay. The \( H_\infty \) sampled-data problem of Section III will be solved for the example of Safonov et al [22].
There arises the problem of comparison of a continuous-time system with a sampled data system, consisting of a continuous-time plant, an anti-aliasing filter and a discrete-time controller possibly with computational delay. The easy interpretable impulse response will be used, because it is not absolutely clear how to calculate a comparable frequency response of the sampled data system. In order to investigate how robustness properties are changed with discretization, the Nyquist plots of the continuous-time open-loop and the purely discrete-time open-loop are compared. This is a sensible comparison, because stability of the discrete-time control system (with the observability assumption $A_1(v))$ is a necessary and sufficient condition for stability of the sampled data system.

A. RATTAN'S EXAMPLE

Let us first investigate the example of [15]. In Figure 15 the magnitude of the complementary sensitivity function $T(s) = PC(I + PC)^{-1}$ is plotted. The dashed line indicates half the sampling frequency $\omega_s/2$. The sampling time of [15] is $T = 0.157s$. The plot reveals that the sampling frequency is at the lower limit that results from the sampling theorem [2]. A rule of thumb [2] suggests that the sampling frequency be selected 10 times larger than the closed-loop bandwidth. With half the sampling frequency close to the system's bandwidth, discretization problems have to be expected.

Using the stability margin (SM) and the transfer function (TF) approach, the continuous-time controller was discretized. The controller data and the weights are listed in Appendix C. The weighting function $W(s)$ for the transfer function approach has a terrace shape and consists of a product of two transfer functions, a 3rd order Butterworth filter and a second order transfer function. The filter is used to cut off frequencies above $\omega_s/2$. A higher order, e.g. 2 or 3, is necessary to be able to force the system to follow also signals with frequencies in the upper part of the system's bandwidth. The second transfer function has a real pole close to the imaginary axis and a zero, which can be used to
place the flat part of the terrace. This shape of the weighting function has proved very versatile in achieving desirable closed loop properties. The controller order was reduced with unweighted balanced truncation as long as there were no changes of performance.

At the input, a reference signal $r$ in the form of an impulse of one sampling period duration was applied. The responses of the output $y$ and plant input $u$ are shown in Figures 16 and 17. The transfer function approach leads to the best agreement with the continuous-time system. In comparison to Rattan’s method the controller approximation approach results in better impulse response matching but with a small intersampling ripple. The corresponding plots of the plant inputs in Figure 17 show that there are no big control oscillations as they can be observed with some discrete-time controller design methods. The Nyquist plots in Figure 18 reveal no significant differences in robustness between the different discretization methods. In practice, the measurements are often corrupted with noise and an anti-aliasing filter is usually necessary.

Fig. 15. Rattan’s example: $T(j\omega)$
Fig. 16. Plant output impulse response

Fig. 17. Plant input impulse response
If half the sampling frequency is close to the system's bandwidth, the filter has a large impact on performance. This follows from the requirement that the anti-aliasing filter must attenuate frequencies above $\omega_s/2$. Figures 19 and 20 show the impulse response and the Nyquist plots of the control loop with filter. The Nyquist plot discloses the inevitable rotation of the curve as $\omega$ increases towards $\omega_s/2$ caused by the phase lag in the anti-aliasing filter. The implication is that if the sampling frequency is minimal, the rotation becomes effective at the cross-over frequency, largely destroying all gain and phase margin. With a 2nd order Butterworth filter with gain of -20 dB at $\omega_s/2$, the system with Rattan's controller becomes unstable.

If a discretization method does not allow the inclusion of a prefilter, the continuous-time controller design has to be performed with a filter. There results an iteration procedure, because the sampling frequency and therefore the filter's bandwidth is determined by the closed-loop bandwidth, which in turn can be considerably influenced by the filter.
Fig. 19. Output impulse response with anti-aliasing filter

Fig. 20. Nyquist plot with anti-aliasing filter
It is therefore very convenient if the effects of an anti-aliasing filter are directly taken into account in the discretization step.

For the transfer function approach (Section II.B), we have the choice of placing the filter either into the feedforward path as shown in Figure 6 or in the feedback path, i.e. filtering only $y_h$. The latter is more appropriate if we already have a frequency bounded reference signal $r$ (i.e. which does not need to be filtered again). In this case the system has a considerably reduced response time. This is also confirmed by the impulse responses plotted in Figure 19. The stability margin and the transfer function approach show similar impulse responses as the continuous-time system, but they are delayed by one sampling period. Almost perfect agreement is achieved with the transfer function approach with the prefilter only in the feedback path. A computational delay in the controller has a similar effect to that of additional phase lag inserted into the control loop, but in contrast to the prefilter, the gain at critical frequencies is not reduced. Unless one has to accept a very
Fig. 22. Katz' example: $T(j\omega)$

large delay, i.e. much greater than $T$, the additional phase-lag at $\omega_s/2$ is small compared to the lag of an anti-aliasing filter of second order. The plots in Figure 21 show the good performance of the discretization methods for a controller with a delay of less than one sampling period (discussed earlier in Section V.D).

B. KATZ EXAMPLE

Katz [21] shows that for his example only prewarped bilinear transformation leads to a stable discrete-time system. A closer look at the frequency response of the continuous-time system, shown in Figure 22, reveals that at $\omega_s/2$ the gain is 0.32. (The sampling frequency is 33 Hz.) Consequently, there are considerable aliasing effects to be expected. The example shows clearly that if $\omega_s/2$ is not much larger than the closed-loop bandwidth severe problems with common discretization methods are predictable. As for the previous example, the impulse responses and the Nyquist plots of different discretization approaches are compared.
Since the controls $u$ are, as before, similar to the continuous-time ones, they are omitted. The weighting strategy was the same as in the previous example. The transfer functions of the weightings are given in Appendix C. As a reference we consider the discretization of [16]. Figure 22 shows the impulse responses of the closed-loop systems without anti-aliasing filter and without controller delay. The closest match is achieved with the discretization of [16]. This is due to the additional degree of freedom obtained from the feedforward controller. The Nyquist plot in Figure 23 discloses large differences between the discretization methods. Since the distance of the Nyquist plot from the point (-1,0) is the inverse magnitude of the sensitivity function, the Nyquist plot for the design of [16] indicates bad disturbance attenuation. The impulse responses and the Nyquist plots of the two new proposed discretization methods show good results. For the example of Katz, it is interesting to investigate if an anti-aliasing filter can improve discretization or not, because at the chosen sampling rate, there is already considerable alias-
ing with the continuous-time design. For the controller approximation approach, there was no controller with anti-aliasing filter for which the stability condition $\| J_L \| < 1$ could be satisfied. Figure 25 shows the results of the closed-loop transfer function approach with a filter either in the feedback loop or as shown in Figure 6 in the forward path. The plots are compared to the continuous-time design and to the discretization without filter. It can be seen that the plot of the system without a filter coincides with the plot of the system with filter. In the optimization of $\| J_H \|$ of (6), the discretization with filter allowed a weighting $W(j\omega)$ of slightly larger norm than the one without filter to achieve the same minimum. Therefore, with an anti-aliasing filter a smaller transfer function approximation error is achieved. This is possibly because the performance degradation caused by aliasing effects is more severe than the degradation from the filter's phase lag. The system with forward path filter as in Figure 6 has an impulse response with a considerable time lag, which results from the additional filtering of the reference input.
Fig. 25. Output impulse response with anti-aliasing filter

The stability margins of the discretizations with filter are considerably smaller than for the continuous-time design, as can be seen in the Nyquist plots of Figure 26. The position of the filter has a considerable influence on the system's open loop gain. The filter effects, i.e. the additional phase lag combined with attenuation of high frequencies, become obvious. Both discretization methods, i.e. the stability margin and the transfer function approach, allow the designer to include a computational delay into this difficult discretization problem. The impulse responses for a delay of $0.2 \times T$ and $0.4 \times T$ are shown in Figure 27. The maximum values of the responses are smaller the larger the delay. This corresponds to an increase of rise time for the step response and to a loss of bandwidth in the frequency domain. This closed-loop behaviour is also described in [2]. If additionally an anti-aliasing filter is inserted in the feedback path, the system's performance remains unchanged.
Fig. 26. Nyquist plot with anti-aliasing filter

Fig. 27. Output impulse response for different computational delays
C. $H_{\infty}$ SAMPLED-DATA PROBLEM

A multivariable $H_{\infty}$ design problem of a 4th order plant with two inputs and two outputs was proposed by Safonov et al [22]. The design aims to optimize the sensitivity function with a constraint on the bandwidth of the complementary sensitivity function. The authors perform first a purely continuous-time controller design and then discretize the controller using the following transformation:

$$s = 6000 \left( \frac{z - 1}{1 + z/0.3} \right)$$

The sampling frequency is just fulfills the bandwidth constraint and is 3000 Hz. Hold-input discretization leads to instability and bilinear transformation to very bad closed loop properties.

For the direct discrete-time controller design, the system is represented as in Table 1 and approximated by a multirate system with $N = 4$. Since no anti-aliasing filter is used and $T_{21}$ is not strictly proper
a weighting is given in Appendix C. Figure 28 and 29 show the impulse response (impulse at the first input) of the two sampled-data systems in comparison with the continuous-time system. Safonov et al [22] use the upper bound on the complementary sensitivity function as a measure of robustness. The peak value of the impulse response is an indicator for this property. The first plot shows that the direct design has a similar maximum value as the continuous-time system. This is achieved at the cost of a marginally slower approach to zero. The impulse response of the second output is shown in Figure 27. The continuous-time system is decoupled. This property is not completely maintained in both of the sampled-data systems.

D. SUMMARY

The example of Katz [21] shows that even if the bandwidth condition to ensure considerable aliasing is not met, it is possible to discretize the continuous-time controller with the proposed methods, but
under realistic assumptions that a noise filter is necessary and a computational delay inevitable, the achievable closed-loop properties may not be acceptable. If the bandwidth condition is met, as in Rattan’s [15] example, there results good closed-loop matching with the original continuous-time system.

Using the proposed discrete-time controller design methods it is not necessary to have a sampling frequency of 20-100 times the system’s bandwidth (as proposed in [2]), if an anti-aliasing filter and a delay are included in the control loop.

VII. REFERENCES


A APPENDIX

SAMPLED-DATA SYSTEMS: STATE, STABILIZABILITY AND STABILITY

In this Appendix, we are going to review (in a self-contained way) a number of facts concerning sampled-data systems. A number of the results are variants on ones to be found in [10].

We shall work with a plant with rational proper transfer function matrix

\[
T(s) = \begin{bmatrix}
T_{11}(s) & T_{12}(s) \\
T_{21}(s) & T_{22}(s)
\end{bmatrix}
\]

which is controlled by a discrete-time compensator \(C(z)\), in conjunction with a strictly proper anti-aliasing filter \(F_a(s)\), sampler \(S\) and zero order hold \(H(s)\), as depicted in Figure A1.

![Diagram](image)

**Fig A1.** Basic set up considered in Appendices A and B
Fig A2. Equivalent discrete time plant obtained from combination of $T_{22}, F$, sampler and hold in Figure 1.

Suppose that the plant with anti-aliasing filter has description

$$\begin{align*}
\dot{x}_c &= A_c x_c + B_{cl} u_1 + B_{c2} u_2 \\
y_1 &= C_{cl} x_c + D_{cl} u_1 + D_{c2} u_2 \\
y_2 &= C_{c2} x_2
\end{align*}$$

Notice that the strict properness of the antialiasing filter means that no direct feedthrough terms appear in the expression for $y_2$. The controller equations are

$$\begin{align*}
\xi(k+1) &= A_d \xi(k) + B_d v(k) \\
\psi(k) &= C_d \xi(k) + D_d v(k)
\end{align*}$$

As defined in Section II, $B_T$ denotes the space of (possibly vector) functions which are continuous from the left, and piecewise continuous over intervals of length $T$. The sampler is assumed to have period $T$ and maps signals in $B_T$ into a discrete sequence, in accordance with

$$v(k) = y_2(kT-) \quad k \geq 0$$

The zero-order hold element produces from a discrete sequence signals which are continuous from the left and piecewise constant, (and so in
We shall argue below that for virtually all $u_1(\cdot)$ of interest, $y_2(\cdot)$ will be in $B_\infty$.

A. THE CONCEPT OF A HYBRID STATE

Recall that a state functions as a set of information which, together with knowledge of future inputs, allows the subsequent behaviour of the system to be determined. The following Lemma identifies a state for the above closed-loop system.

Lemma A.1 For the closed-loop sampled data system described above, at time $kT+$, the information $x_c(kT)$ and $\xi(k)$, together with $u_1(t)$ for $t > kT$ is sufficient to solve the system for $t > kT$. At time $s$ with $kT < s \leq (k+1)T$, the information $x_c(s), \xi(k)$ and $v(k)$ together with $u_1(t)$ for $t > s$ is sufficient to solve the system for $t > s$.

Proof. Obtain the following quantities in sequence.

$$y_2(kT) = C_2x_c(kT)$$
$$v(k) = y_2(kT)$$
$$\psi(k) = C_4 \xi(k) + D_4 v(k)$$
$$u_2(t) = \psi(k) \quad kT < t \leq (k+1)T$$

$x_c(t)$, using $x_c = A_c x_c + B_{c_1} u_1 + B_{c_2} u_2$, over $kT < t \leq (k+1)T$

$$\xi(k+1) = A_4 \xi(k) + B_4 v(k)$$

In addition, one has

$$y_1(t) = C_{el} x_c(t) + D_{el} u_1(t) + D_{el} u(t) \quad kT < t \leq (k+1)T$$

This process repeats each interval of length $T$.

The argument is trivial to vary in case the starting time is $s \neq kT$. 

$$B_T): \quad u_2(t) = \psi(k) \quad kT < t \leq (k+1)T$$
For future reference, let us define the hybrid state

\[ x_{\text{hybrid}}(t) = \begin{bmatrix} x_c(t) \\ \xi(k) \\ v(k) \end{bmatrix} \quad kT < s \leq (k+1)T \]

Notice that

\[ x_{\text{hybrid}}[(k+1)T] = \begin{bmatrix} x_c[(k+1)T] \\ \xi(k) \\ v(k) \end{bmatrix} \]

from which is obtainable the pair \( x_c[(k+1)T], \xi(k+1) \).

The above lemma shows that the definition is a valid one for a state.

**B. THE OUTPUT SPACE**

In the sequel, we will consider conditions under which bounded inputs produce bounded outputs. Here, we shall isolate a continuity property of the output.

Let \( L^p \) denote the extended \( L_p[0, \infty) \) space, i.e. the space of measurable functions \( f(t) \) on \( [0, \infty) \) for which \( \chi_E f \in L_p \) for any subset \( E \subset [0, \infty) \) of finite measure. The content of the following result is that, (neglecting the direct feedthrough component), the closed-loop hybrid system smooths \( L^p \) signals to make them \( B_T \):

**Lemma A.2** Consider the closed-loop sampled-data system described above, with \( D_{C1} = 0 \). Then \( u_1 \in L^p \) implies \( y_1 \in B_T \), and \( y_2 \in B_\infty \). In case \( D_{C1} \neq 0 \), \( u_1 \in L^p \) implies \( y_2 \in B_\infty \).

**Proof** Any finite-dimensional, strictly causal time-invariant system maps \( L^p \) to \( B_\infty \). Since \( L^p \subset L^p \) for all \( p \in [1, \infty) \), it map \( L^p \) into \( B_\infty \), see eg [12]. Now the plant can be regarded as being driven by two inputs, \( u_1 \in L^p \) and \( u_2 \in B_T \) (with this property of \( u_2 \) following from the fact that it is the output of a hold element). Hence the plant’s output \( y_1 \) is the sum of \( y^1_1 \) and \( y^2_1 \), defined by

\[
\begin{align*}
\dot{x}_{c1} &= A_c x_{C1} + B_{c1} u_1 \\
y^1_1 &= C_{c1} x_{c1}
\end{align*}
\]
and
\[
\begin{align*}
\dot{x}_2 &= A_c x_2 + B_c u_2 \\
y_1^2 &= C_c x_2 + D_c u_2
\end{align*}
\]
Obviously, \(y_1^2 \in B_\infty\) and \(y_2^2 \in B_T\), so the sum \(y_1 \in B_T\), as required. The properties of \(y_2\) follow trivially, using the fact that \(y_2 = C_2 x_2\).
Notice that this lemma validates an earlier assertion that for virtually all \(u_1(\cdot)\) of interest, \(y_2(\cdot)\) will be in \(B_\infty\).

C. STABILIZABILITY

Consider the plant of Figure A1, but with neglect of its input \(u_1\), and output \(y_1\). It is well known (and almost obvious) that a necessary and sufficient condition for stabilizability with a continuous-time, time-invariant controller acting on \(y_2\) and generating \(u_2\) is the stabilizability and detectability of the triple \((A_c, B_2, C_2)\). The problem is a little more subtle when a discrete-time controller is used, to produce a sampled data system. To consider this situation, the following Assumption A1(v) is introduced, as in Section II:

Assumption A1(v)

If \(\mu\) is an eigenvalue of \(A_c\) with positive real part, none of the points \(\mu + j2\pi k/T\) for integer \(k\) is an eigenvalue of \(A_c\);

further, none of the points \(j2\pi k/T\) is an eigenvalue of \(A_c\).

Obviously, if \(T\) is small enough, this condition is satisfied. Now suppose that we neglect the input \(u_1\) and output \(y_1\) of \(T(s)\) to consider a reduced \(T(s)\), \(T_{red}(s)\) say, with \(T_{red}(s) = T_{22}(s)\). [The antialiasing filter is lumped into \(T_{red}(s)\)]. Consider the discrete plant \(T_{red}(z)\) found by cascading \(T_{red}(s)\) with the hold and sampling element. We can define \(\hat{T}_{red}(z)\) via

\[
\begin{align*}
x_{red}(k + 1) &= \exp[A_c T]x_{red}(k + 1) + \left[ \int_0^T \exp(A_c s)B_c ds \right] u_{red}(k) \\
y_{red}(k) &= C_2 x_{red}(k)
\end{align*}
\] (6)
In [10], the following is established:

**Lemma A.3** Let \((A_c, B_c, C_c)\) be stabilizable and detectable. Then 
\[
\left[ \exp(A_cT), \int_0^T \exp(A_c\tau)B_c d\tau, C_c \right]
\] is stabilizable and detectable if Assumption A1(v) is satisfied.

Assumption A1(v) is virtually also necessary for stabilizability and detectability of (6). Hence it must be fulfilled if there is to exist a discrete time stabilizing \(C(z)\) for \(\bar{T}_{\text{red}}(z)\).

Let \(C(z)\) in fact stabilize \(\bar{T}_{\text{red}}(z)\). Suppose there is nonzero input and nonzero initial condition. Under Assumption A1(v), there are no unstabilizable or undetectable modes, and so the joint state of \(\bar{T}_{\text{red}}(z)\) and \(C(z)\) will decay exponentially fast. The state sequence of \(\bar{T}_{\text{red}}(z)\) is nothing but the sampled values \(x_c(kT)\) of the state of \(T_{\text{red}}(s)\), and in turn, if \(u_1 \equiv 0\), these are just the sampled values of the state of \(T(s)\).

Thus Assumption A1(v) guarantees that we can find a discrete-time controller for \(T(s)\) such that under zero input and arbitrary nonzero initial state, the resulting sequence of sampled values of the state of \(T(s)\) decay to zero.

This is almost the same thing, but not quite, as the property that the hybrid state \([x'_c(s), \xi'(k), v'(k)]'\) decays exponentially to zero under zero input, for arbitrary nonzero initial state condition. This property in fact is proved in [10], and the proof is easy. To sum up

**Lemma A.4** Suppose \((A_c, B_c, C_c)\) is stabilizable and detectable. Under Assumption A1(v), there exists a discrete-time controller for the sampled-data system producing zero-input exponential stability of the hybrid state.

### D. Input-Output Stability

Connections between input-output stability and exponential Lyapunov stability are well known for finite-dimensional systems which are completely continuous-time or completely discrete-time. Here, we shall
show that the connection carries over to sampled-data systems. The results are similar to results of [10].

The main result (Lemma A.6 below) demonstrates BIBO stability for $L_1$ and $L_\infty$ inputs. Our interest in most of this chapter is $L_2$ inputs. The connection is provided by the following result, see [11-12], which is a special case of a convexity theorem of M Riesz see eg. [13].

**Lemma A.5** Suppose $R$ is a linear transformation on $L_1$. If $R$ maps $L_1[0,\infty)$ to $L_1[0,\infty)$ and $L_\infty[0,\infty)$ to $L_\infty[0,\infty)$ and the restrictions are bounded operators with norms $M_1$ and $M_\infty$, then $R$ maps $L_p[0,\infty)$ to $L_p[0,\infty)$ for all $p \in [1,\infty]$, and the restriction is a bounded operator with norm bounded by $M_1^pM_\infty^{p-1}$.

Now the main result is

**Lemma A.6** Consider the interconnection of the plant (1) with discrete-time controller $C(z)$ including sampler, hold and antialiasing filters. Suppose under zero-input conditions, the hybrid state is exponentially stable. Then (with zero initial conditions)

(i) $L_\infty$ inputs $u_1(\cdot)$ are mapped boundedly into $L_\infty$ outputs of $y_1(\cdot)$

(ii) $L_1$ inputs $u_1(\cdot)$ are mapped boundedly into $L_1$ outputs of $y_1(\cdot)$

(iii) $L_p$ inputs $u_1(\cdot)$ are mapped boundedly into $L_p$ outputs of $y_1(\cdot)$ for all $p \in (1,\infty)$

**Proof** Suppose $u_1(\cdot) \in L_\infty[0,\infty)$. Decompose $u_1(\cdot)$ as

$$ u_1 = \sum_{k=0}^{\infty} u_k^k $$

where the support of $u_k^k$ is contained in $(kT,(k+1)T]$. Let $x_{\text{hybrid}}^k$ denote the additive component of $x_{\text{hybrid}}$ due to $u_k^k$ alone. The component $u_k^k$ will drive the hybrid state $x_{\text{hybrid}}^k(kT) = [x_{\text{eq}}^k(kT) = 0, \xi_k^k(k-1) = 0, v^k(k-1) = 0]'$ to $x_{\text{hybrid}}^k(kT + T) = [x_{\text{eq}}^k(kT + T), \xi_k^k(k) = $
0, \( v'(k) = 0 \) where 
\[
x^k_c(kT + T) = \int_{kT}^{(k+1)T} \exp(A_c s) B_{c1} u^k_1(s) ds.
\]
Thereafter, \( x^k_{\text{hybrid}}(t) \) decays exponentially by hypothesis. Obviously, \( x^k_{\text{hybrid}} \) is bounded on \([kT, (k + 1)T]\), in the form
\[
\| x^k_{\text{hybrid}}(t) \| = \| x^k_c(t) \| \leq M_1 \| u^k_1 \|_\infty
\]
Hence for all \( t \), and some \( \beta > 1, \alpha > 0 \),
\[
\| x^k_{\text{hybrid}}(t) \| \leq M_1 \| u^k_1 \|_\infty \{ \beta \exp(-\alpha [t - (k + 1)h]) \}
\]
It follows, as per [10], that
\[
\| x_{\text{hybrid}}(t) \| \leq M_2 \| u_1 \|_\infty
\]
(The argument is not difficult) The remainder of the proof of (i) is trivial.

Next, suppose \( u_1(s) \in L_1[0, \infty) \). Break up \( u_1 \) as before. For \( kT \leq t \leq (k + 1)T \), we will have
\[
x^k_c(t) = \int_k^t \exp A_c(t - s) B_{c1} u_1(s) ds
\]
and
\[
\| x^k_c(t) \| \leq M_2 \int_{kT}^{(k+1)T} \| u_1(s) \| ds
\]
In view of the exponential decay of \( \| x^k_{\text{hybrid}}(t) \| \) for \( t > (k + 1)T \), it is evident that for some \( M_3 \),
\[
\int_{kT}^\infty \| x^k_{\text{hybrid}}(s) \| ds \leq M_3 \int_{kT}^{(k+1)T} \| u_1(s) \| ds
\]
whence
\[
\int_0^\infty \| x_{\text{hybrid}}(s) \| ds \leq M_3 \int_0^\infty \| u_1(s) \| ds
\]
and part (ii) is then trivial.

Part (iii) is an immediate consequence of parts (i) and (ii) and Lemma A5.

E. SAMPLERS FOLLOWING A STRICTLY PROPER SYSTEM

This subsection treats a technical result which is first used in Section II. The argument is similar to that of the last subsection, and appears in [11].
Lemma A.7 Consider a finite-dimensional stable \( W(s) \) with \( W(\infty) = 0 \) followed by a sampler \( S \). Then \( SW(s) \) defines a BIBO operator from \( L_p[0, \infty) \) to \( l_p(Z_+) \) for all \( p \in [1, \infty) \).

Proof The case \( p = \infty \) is trivial. We only need to prove the case \( p = 1 \) in view of Lemma A5. Let \( u \in L_1[0, \infty) \) denote the input to \( W(s) \) and decompose \( u \) as

\[
u = \sum_{k=0}^{\infty} u^k\]

where the support of \( u^k \) is contained in \( kT, (k+1)T \]. If \( W(s) = C(sI - A)^{-1}B \) with \( \Re \lambda_i(A) < 0 \), the state response to \( u^k \) will be

\[
x^k(jT) = 0 \quad j \leq k
\]

\[
x^k[(k+1)T] = \int_0^T \exp(A(T-s)Bu^k(s+kT)ds
\]

so that

\[
\|x^k[(k+1)T]\| < M_1 \int_0^T \|u^k(s+kT)\| ds
\]

for some \( M_1 \). Also

\[
\|x^k[lT]\| \leq M_2 \alpha^{l-k+1}
\]

for some \( \alpha < 1 \), all \( l \geq k+1 \), and some \( M_2 \). It is easy to derive then that

\[
\sum_{k=0}^{\infty} \|x(kT)\| \leq M_3 \|u\|_1
\]

and the lemma claim follows for \( p = 1 \).

Notice that a sampler \( S \) alone does not map \( L_p[0, \infty) \) into \( l_p(Z_+) \) for \( p < \infty \), (though it does for \( p = \infty \)).
B APPENDIX

INTRODUCTION OF MULTIRATE SAMPLING

In this section, we shall study the effects of introducing additional sampling into a hybrid system. The additional sampling will involve a sampling interval $\tau$ which is a submultiple of the interval $T = N\tau$ for some $N$. Further, we shall assume synchronicity, in the sense that every $T$ seconds, both samplers operate simultaneously.

We continue to work with the basic set-up of Figure A1 defined in equations (2) and (3). It is assumed that Assumption A1(v) holds, and the discrete-time compensator is stabilizing.

We shall have occasion to work with discrete-time sequences associated with different sampling intervals, and for a reason to be made apparent in a moment, we shall adjust the norm definition. Let $a_0, a_1, a_2, \ldots$ be a sequence in $Z_+$ obtained with sampling interval $\tau$. Then

$$\|a\|_p = \tau^{\frac{1}{p}} \left[ \sum |a|^p \right]^{\frac{1}{p}} \quad (7)$$

To flag this departure from convention, we shall name the containing space $l_{pr}(Z_+)$. The advantage of the scaling factor $\tau^{\frac{1}{p}}$ is the following. Let $a_i \in l_{pr}(Z_+)$ be the input sequence to a hold element (with hold-interval $\tau$) and let $b$ be the element's output. Then $b \in L_p[0, \infty)$ and $\|b\|_p = \|a\|_p$. (The notation $\| \cdot \|_p$ is used for both continuous functions and sequences, with obvious differentiation). A consequence of this fact is that if $K$ is some operator into $l_{pr}(Z_+)$ from some space, then

$$\|H_{\tau} K\|_p = \|K\|_p \quad (8)$$

where the norms here are of course induced norms.

**Lemma B.1** Let $J : u_1(\cdot) \rightarrow y_1(\cdot)$ be the operator defining the sampled-data system of Figure A1 for which Assumption A1(v) holds, and the compensator is stabilizing. Suppose that the direct feedthrough term $D_{\alpha}$ is zero. Then $H_{\tau} S_{\tau} J$ is a bounded operator on $L_p[0, \infty)$ for all $1 \leq p \leq \infty$ and, with $\tau = T/N$ a submultiple of $T$, $S_{\tau} J$ is a bounded operator $L_p[0, \infty) \rightarrow l_{pr}(Z_+)$ for all $1 \leq p \leq \infty$ with the same norm.
**Proof** It is trivial that $H_T S_T$ is bounded on $B_T \cap L_\infty[0,\infty)$ (with norm 1) and $J$ is a bounded operator from $L_\infty[0,\infty)$ to $B_T \cap L_\infty[0,\infty)$. [Here functions in $B_T \cap L_p[0,\infty)$ are assigned their $L_p$ norm]. Hence $H_T S_T J$ is a bounded operator on $L_\infty[0,\infty)$. Similarly, $S_T J : L_\infty[0,\infty) \rightarrow l_{\text{cor}}(Z^\perp_\text{c})$ is bounded.

Next suppose $u \in L_1[0,\infty)$. Decompose $u_1(\cdot)$ (as in the earlier discussion of input-output stability) as

$$u_1 = \sum_{k=0}^{\infty} u_k^T$$

where the support of $u_k^T$ is contained in $(kT, (k+1)T]$. Let $x_{\text{hybrid}}^k$ denote the additive component of $x_{\text{hybrid}}$ due to $u_k$ alone. Recall that we showed in Appendix A that

$$\left\| x_{\text{hybrid}}^k(t) \right\| = \left\| x_{\text{hybrid}}^k(t) \right\| \leq M_1 \int_{kT}^{(k+1)T} \|u_1(s)\| \, ds$$

for

$$kT \leq t \leq (k+1)T$$

while $\|x_{\text{hybrid}}^k(t)\|$ decays exponentially fast in $t > (k+1)T$.

Accordingly,

$$\left\| y_1^k(kT) \right\| + \left\| y_1^k(kT + \tau) \right\| + \cdots + \left\| y_1^k(kT + (N - 1)\tau) \right\| \leq N M_2 \int_{kT}^{(k+1)T} \|u_1(s)\| \, ds$$

where $M_2$ is independent of $N$, and

$$\left\| y_1^k(kT + m\tau) \right\| \leq M_3 \int_{kT}^{(k+1)T} \|u_1(s)\| \, ds \cdot \beta \exp(-\alpha m\tau)$$

for all $m > N$ and some $\alpha > 0, \beta > 1$, with $M_3, \alpha$ and $\beta$ independent of $\tau$.

It follows that

$$\sum_{m=0}^{\infty} \left\| y_1^k(kT + m\tau) \right\| \leq \left[ M_4 \frac{\tau}{\tau} + \frac{M_5}{1 - e^{-\alpha\tau}} \right] \int_{kT}^{(k+1)T} \|u_1(s)\| \, ds$$
Recognize that the left side is precisely the $L_1[0,\infty)$ norm of the output of $H_\tau S_\tau J$ as well as the $l_1(\mathbb{Z}^+)$ norm of the output $S_\tau J$.

Hence boundedness for $L_1[0,\infty)$ inputs is proved.

Boundedness on $L_p[0,\infty)$ then follows for all $p$ as per Lemma A5.

In the next Corollary, we vary the set-up considered in Lemma B1 by including a hold element at the front of the system.

**Corollary B.1** Under the same hypothesis as Lemma B1, save that $D_{cl}$ is not necessarily zero, $H_\tau S_\tau JH_\tau$ and $S_\tau JH_\tau$ are bounded operators from $l_{pr}(\mathbb{Z}^+)$ to $L_p[0,\infty)$ and $l_{pr}(\mathbb{Z}^+)$ respectively, with the same induced norm.

**Proof** Suppose first $D_{cl} = 0$. It is trivial to see that $H_\tau S_\tau JH_\tau$ and $S_\tau JH_\tau$ will be bounded (and have the same norm), because $H_\tau S_\tau J$ and $S_\tau J$ have this property. Now suppose $J = J_1 + D_{cl}$. Although $H_\tau S_\tau D_{cl}$ is not bounded on $L_1[0,\infty)$, since $H_\tau S_\tau$ is not bounded on $L_1[0,\infty)$, it is trivial to see that $H_\tau S_\tau D_{cl} H_\tau$ is bounded from $l_{pr}(\mathbb{Z}^+)$ to $L_p[0,\infty)$ for all $p \in [1,\infty)$ and $S_\tau D_{cl} H_\tau$ is bounded from $l_{pr}(\mathbb{Z}^+)$ to $l_{pr}(\mathbb{Z}^+)$. The remainder of the corollary is then trivial to prove.

In the next Lemma, we examine the effect of letting $\tau$ tend to zero. We shall retain the assumption that $\tau = T/N, N$ integer with synchronous sampling. This assumption is probably not necessary. The following result is known [11] for the case of an open loop stable plant, with no controller.

**Lemma B.2** Let $J$ be the operator defining the sampled-data system of Figure A1, for which Assumption A1(v) holds and the compensator is
stabilizing. Suppose that the direct feedthrough term $D_{ci}$ is zero. With $\tau = T/N$, $N$ integer, there holds for all $p \in [1, \infty]$, \[
abla \lim_{\tau \to 0} \|J - H_\tau S_\tau J\|_p = 0 \quad (10)
\]

Proof Consider first the case of $p = \infty$. When $\|u_1\|_\infty$ is finite, $\|x_{hybrid}\|_\infty$ and $\|u_2\|_\infty$ are bounded correspondingly. Since \[
\dot{x}_c = A_c x_c + B_{c1} u_1 + B_{c2} u_2
\] it follows that for all $s \in [0, \tau]$ and all integer $m$ \[
\|x_c(s + m\tau) - x_c(m\tau)\| \leq M_1 \tau \|u_1\|_\infty
\] where $M_1$ is some constant independent of $\tau$. Since $u_2$ is piecewise constant on intervals of length $T = N\tau$, it follows that \[
y_1(s + m\tau) - y(m\tau+) \leq M_2 \tau \|u_1\|_\infty \quad \text{for all } s \in (0, \tau)
\] where $M_2$ is independent of $\tau$. It follows easily that \[
\|\langle J - H_\tau S_\tau J\rangle u_1\|_\infty \leq M_2 \tau \|u_1\|_\infty \quad (11)
\] so the claim of the Lemma is true for $p = \infty$.

Now take $p = 1$. A similar argument to the above shows that \[
\int_0^\tau \|x_c(s + m\tau) - x_c(m\tau)\| ds \leq M_3 \tau \int_0^\tau \|u_1(s + m\tau)\| ds + M_4 \tau \int_0^\tau \|u_2(s + m\tau)\| ds
\] for $M_3, M_4$ independent of $\tau$.

From this, one obtains \[
\int_0^\tau \|y_1(s + m\tau) - y_1(m\tau)\| ds \leq \tau [M_5 \int_0^\tau \|u_1(s + m\tau)\| ds + M_6 \int_0^\tau \|u_2(s + m\tau)\| ds]
\]
for $M_5, M_6$ independent of $\tau$. Adding such relations for all $s$ yields

$$
\| (J - H_T S_T J) u_1 \|_1 \leq \tau [M_5 \| u_1 \|_1 + M_6 \| u_2 \|_1] \\
\leq \tau M_7 \| u_1 \|_1
$$

(12)

since $\| u_2 \|_1 \leq M_6 \| u_1 \|_1$ by the stability of the loop. Hence the claim of the Lemma is true for $p = 1$. Since it is true also for $p = \infty$, it is true for all $p \in [0, \infty]$.

**Remark** The key in proving the above Lemma was to establish (11) and (12). Taken together, we can in fact write (by Lemma A5)

$$
\| (J - H_T S_T J) u_1 \|_p \leq \tau M_5^\frac{1}{p} M_7^\frac{p-1}{p} \| u_1 \|_p
$$

(13)

\forall p \in [1, \infty]$, where $M_2$ and $M_7$ are independent of $u_1(\cdot)$ and $\tau$.

The operator $J : L_p[0, \infty) \longrightarrow L_p[0, \infty)$ is of course bounded. In Corollary B1, we showed that $S_T J H_T : l_{pT}(Z_1) \longrightarrow l_{pT}(Z_1)$ was bounded. The theorem following connects their norms. It is the main result of this appendix.

**Theorem B.1** Let $J$ be the operator defining the sampled-data system of Figure A1, for which Assumption A1(a) holds, the compensator is stabilizing and $D_{\tau_1}$ may be nonzero. With $\tau = T/N, N$ integer, there holds for all $p \in [1, \infty],$

$$
\lim_{\tau_1 \to 0} \| J \|_p = \| S_T J H_T \|_p
$$

(14)

**Proof** Let $\epsilon > 0$ be an arbitrary small number. There exists a $u_1 \in L_p[0, \infty)$ with $\| u_1 \|_p = 1$ and

$$
\| J u_1 \|_p > \| J \|_p - \epsilon
$$

(15)

It is standard that the set of continuous functions is dense in $L_p$, so there exists a continuous $u_{1c}$ for which $\| u_1 - u_{1c} \|_p \leq \epsilon/2 \| J \|_p$. Since continuous functions are Riemann integrable, there exists a $\tau_1$, that is a submultiple of $T$ such that for all smaller submultiples $\tau$ of $T$, a $\bar{u}_{1c}(\cdot)$ can
be found, depending on $u_{1c}, \varepsilon$ and $\tau$, for which $\|u_{1c} - \bar{u}_{1c}\|_p \leq \varepsilon/2\|J\|_p$ and $\bar{u}_{1c}$ is piecewise constant on intervals of length $\tau$. Then
\[
\|u_1 - \bar{u}_{1c}\|_p \leq \varepsilon/\|J\|_p \tag{16}
\]

It follows that
\[
\varepsilon \geq \|J\|_p \|u_1 - \bar{u}_{1c}\|_p \\
\geq \|J(u_1 - \bar{u}_{1c})\|_p \\
\geq \|Ju_1\|_p - \|J\bar{u}_{1c}\|_p
\]

Let $\bar{u}_{1c} = H_r\bar{u}_1$, for some $\bar{u}_1 \in l_{pr}(Z_r)$ with the same $l_{pr}$ norm as $u_{1c}$ has $L_p$ norm. Then
\[
\varepsilon \geq \|Ju_1\|_p - \|JH_r\bar{u}_1\|_p \tag{17}
\]

Now write $J = J_0 + D_{c1}$, where $J_0$ is the strictly proper part of $J$. In the course of proving Lemma B2, we established, see (13),
\[
\|(J_0 - H_rS_rJ_0)u_1\|_p \leq \tau M_p \|u_1\|_p
\]
for arbitrary $u_1$ and some $M_p$ independent of $u_1$ and $\tau$. Also, $H_rD_{c1} = D_{c1}H_r$. Hence,
\[
\|JH_r\bar{u}_1\|_p = \|(J_0 - H_rS_rJ_0)H_r\bar{u}_1 + H_rS_rJ_0S_r\bar{u}_1 + D_{c1}H_r\bar{u}_1\|_p \\
\leq \tau M_p \|\bar{u}_{1c}\|_p + \|H_r(S_rJ_1H_r + D_{c1})\bar{u}_1\|_p \\
= \tau M_p \|\bar{u}_{1c}\|_p + \|(S_rJ_1H_r + D_{c1})\bar{u}_1\|_p \\
\leq \tau M_p \|\bar{u}_{1c}\|_p + \|(S_rJ_1H_r + D_{c1})\|_p \|\bar{u}_1\|_p
\]

Hence in (17)
\[
\varepsilon \geq \|Ju_1\|_p - \tau M_p \|\bar{u}_{1c}\|_p - \|S_rJ_1H_r + D_{c1}\|_p \|\bar{u}_1\|_p \\
\geq \|J\|_p - \varepsilon - \tau M_p \|\bar{u}_{1c}\|_p - \|S_rJ_1H_r + D_{c1}\|_p \|\bar{u}_1\|_p
\]

[Note that we have also used (15).] Thus
\[
\|S_rJ_1H_r + D_{c1}\|_p \geq \frac{\|J\|_p - 2\varepsilon - \tau M_p \|\bar{u}_{1c}\|_p}{\|\bar{u}_1\|_p}
\]
Now \( \|\tilde{u}_1\|_p = \|\tilde{u}_1\|_p \leq \|u_1\|_p + \|u_1 - \tilde{u}_1\|_p = 1 + \|u_1 - \tilde{u}_1\|_p \), so that by (16),
\[
\|S_{\tau}J_{\tau}\|_p + D_{cl} \geq \frac{\|J\|_p - 2\varepsilon}{1 + \varepsilon/\|J\|_p} - \tau M_p
\]
Recall this holds for all submultiples \( \tau \) of \( T \) which are less than \( \tau_1 \), where \( \tau_1 \) depends on \( \varepsilon \). Hence
\[
\lim_{\tau \downarrow 0} \|S_{\tau}J_{\tau}\|_p + D_{cl} \geq \frac{\|J\|_p - 2\varepsilon}{1 + \varepsilon/\|J\|_p}
\]
Since \( \varepsilon \) is arbitrary but positive,
\[
\lim_{\tau \downarrow 0} \|S_{\tau}J_{\tau}\|_p + D_{cl} \geq \|J\|_p
\] (18)
Next for any small submultiple \( \tau \) and \( \varepsilon > 0 \), there exists an \( l_p(Z_+) \) sequence \( \tilde{u} \), with norm 1 and associated \( L_p \) function \( \tilde{u}_1(\cdot) = H_{\tau}\tilde{u}_1 \) piecewise constant on intervals of length \( \tau \) and also of norm 1 such that
\[
\|S_{\tau}J_{\tau}\tilde{u}_1\|_p \geq \|S_{\tau}J_{\tau} + D_{cl}\|_p - \varepsilon
\]
Equivalently,
\[
\|H_{\tau}S_{\tau}J_{\tau}\|_p + H_{\tau}D_{cl}\tilde{u}_1\|_p \geq \|S_{\tau}J_{\tau} + D_{cl}\|_p - \varepsilon
\]
or
\[
\|(J_{\tau}H_{\tau} + D_{cl}H_{\tau})\tilde{u}_1 + (I - H_{\tau}S_{\tau})J_{\tau}H_{\tau}\tilde{u}_1\|_p \geq \|S_{\tau}J_{\tau} + D_{cl}\|_p - \varepsilon
\]
Hence
\[
\|(J_{\tau}H_{\tau} + D_{cl}H_{\tau})\tilde{u}_1\|_p \geq \|S_{\tau}J_{\tau}H_{\tau} + D_{cl}\|_p - 2\varepsilon
\]
where \( \tau \) is chosen small enough that \( \|(I - H_{\tau}S_{\tau})J_{\tau}H_{\tau}\tilde{u}_1\|_p < \varepsilon \), this being possible by Lemma B2. It follows that
\[
\|J_{\tau}H_{\tau}D_{cl}\|_p = \|J_{\tau}D_{cl}\|_p \|\tilde{u}_1\|_p \geq \|(J_{\tau}H_{\tau}D_{cl})\tilde{u}_1\|_p \geq \|S_{\tau}J_{\tau}H_{\tau} + D_{cl}\|_p - 2\varepsilon
\]
from which

\[ \|J\|_p = \|J_1 + D_{c1}\|_p \geq \lim_{\tau \to 0} \|S_\tau J_1 H_\tau + D_{c1}\|_p \]

Together with the previous reverse inequality (18), this means that

\[ \|J\|_p = \|J_1 + D_{c1}\|_p = \lim_{\tau \to 0} \|S_\tau J_1 H_\tau + D_{c1}\|_p \tag{19} \]

Now observe also that because \(S_\tau H_\tau\) functions as a unit delay of a sequence, there holds

\[ \|S_\tau J_1 H_\tau + D_{c1}\|_p = \|S_\tau H_\tau S_\tau J_1 H_\tau + S_\tau H_\tau D_{c1}\|_p \]
\[ = \|S_\tau J_1 H_\tau + S_\tau D_{c1} H_\tau + S_\tau [(H_\tau S_\tau - I) J_1] H_\tau\|_p \]

Hence letting \(\tau \to 0\) and using Lemma B2 again,

\[ \lim_{\tau \to 0} \|S_\tau J_1 H_\tau + D_{c1}\|_p = \lim_{\tau \to 0} \|S_\tau (J_1 + D_{c1}) H_\tau\| \]
\[ = \lim_{\tau \to 0} \|S_\tau J H_\tau\| \]

The result is now proven using (19).
C APPENDIX

CONTROLLER DATA

Abbreviations:

- **SM**: Stability margin approach
- **SMf**: Stability margin approach with anti-aliasing filter
- **SMdex**: Stability margin approach with computational delay $x$
- **TF**: Transfer function approach
- **TFF**: Transfer function approach, filter in feedforward path
- **TFb**: Transfer function approach, filter in feedback path
- **TFdex**: Transfer function approach with computational delay $x$

For the discretizations SMxx the approximation sampling time was $N = 20$ times smaller than the controller sampling time. For the TF discretizations, $N = 10$ was used.

Transfer functions of Rattan's example

**continuous-time**:

plant:

$$G(s) = \frac{10}{s(s + 1)}$$

controller:

$$C(s) = \frac{0.416s + 1}{0.139s + 1}$$

anti-aliasing filter, being a Butterworth-filter with $\omega_o = \pi/(5T)$:

$$F_a(s) = \frac{16.0162}{s^2 + 5.56s + 16.0162}$$

Rattan's controller ($T = 0.157s$):

$$C_d(z) = \frac{3.436z - 2.191}{z + 0.2390}$$
Weightings:

Poles of the Butterworth-filter: $p_k = \omega_0 e^{j2\pi(2k+1)}$, $k = 0, 1, ..., n - 1$

<table>
<thead>
<tr>
<th>TYPE</th>
<th>$n_{\text{Butterworth}}$</th>
<th>$\omega_0$</th>
<th>ZERO</th>
<th>POLE</th>
<th>GAIN</th>
</tr>
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<tbody>
<tr>
<td>TF</td>
<td>3</td>
<td>$\pi/(0.9T)$</td>
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<tr>
<td>TFFf</td>
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Controllers:

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<th>b0</th>
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<th>a2</th>
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<td>-0.265</td>
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Transfer functions of Katz/Kennedy’s example

Plant:

$$G(s) = \frac{863.3}{s^2}$$

Controller:

$$C(s) = \frac{2940s^2}{(s + 86436)^2}$$
anti-aliasing filter, being a Butterworth filter with $\omega_c = \pi/(5T)$:

$$F_a(s) = \frac{429.92}{s^2 + 29.32s + 429.92}$$

Kennedy's controller ($T = 0.030s$):

feedforward: feedback:

$$F(z) = \frac{f(z)}{r(z)} \quad C(z) = \frac{s(z)}{r(z)}$$

with

$$r(z) = z^2 + 1.1773z + 0.7418$$
$$s(z) = 2.6403z - 1.8714$$
$$t(z) = 1.3084z^2 - 0.5390z - 0.0007$$

Weightings:

<table>
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<tr>
<th>TYPE</th>
<th>$n_{\text{Butterworth}}$</th>
<th>$\omega_c$</th>
<th>ZERO</th>
<th>POLE</th>
<th>GAIN</th>
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<tbody>
<tr>
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<td>0.265</td>
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Controllers:

<table>
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<tr>
<th>TYPE</th>
<th>b3</th>
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</table>

Weightings for Safonov's example

$$W_w(s) = 7.4338 \times 10^{-10} \frac{(s + 2150)}{(s + 1300)(s + 4000)} \times \frac{1}{(s + 3920)(s + 3920)(s + 3920 - 4890)}$$