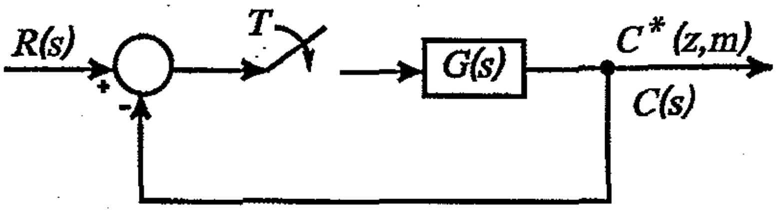
# FUNDAMENTALS OF DISCRETE-TIME SYSTEMS

## A Tribute to Professor Eliahu I. Jury

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# Optimal FWL Design of a Digital Controller*

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## ABSTRACT

The optimal Finite Word Length ($FWL$) state-space digital compensator design problem is investigated. Instead of the usual sensitivity measure, it is argued that it may be desirable to minimize a frequency weighted sensitivity measure over all similarity transformations. The set of optimal realizations minimizing this weighted sensitivity is completely characterized, and an algorithm is proposed to find the optimal solution set.

*Dedicated to Professor Eli Jury, on the occasion of his 70th birthday, and with deepest appreciation for his technical leadership and personal encouragement to younger workers, especially the first author, over decades of outstanding professional activity

## 1. INTRODUCTION

It has long been recognized that in realizing digital transfer functions, finite word length (FWL) effects can manifest themselves. It is well known that any linear system can be represented by state-space equations and that this state-space model is not unique. In the infinite precision case, all these realizations are equivalent since they yield one and the same transfer function. But different realizations have different numerical properties such as sensitivity and error propagation. This means that they are no longer equivalent in the finite precision case. The optimal $FWL$ state-space design task is to identify those realizations which minimize the degradation of the system performance due to the $FWL$ effects, [1-5].

In [3] a global sensitivity measure of a transfer function w.r.t. the parameters of the state space model was proposed by Tavsanoglu and Thiele, and a reasonable and easily computable upper bound for this measure was studied. It was shown in [5] that the realizations that minimize the upper bound also minimize the sensitivity measure itself and that, under a dynamic range constraint, this sensitivity measure and the roundoff noise gain are simultaneously optimized. The set of optimizing structures was characterized in [1]-[3] and [5].

In controller design, it may be that large errors in the controller realization at one frequency have little effect on the closed-loop gain, while small errors at other frequencies may have a large effect. In quantitative terms, let $R(z), P(z)$ and $T(z) = PR(1+PR)^{-1}$ denote the controller, plant and closed-loop transfer functions (the scalar case only will be considered). If $\alpha$ is a controller parameter, then

$$\frac{\partial T}{\partial \alpha} = (1+PR)^{-2} P \frac{\partial R}{\partial \alpha} \tag{1}$$

Evidently, the magnitude of $(1+PR)^{-2}P$ has the effect of frequency-weighting errors in the controller transfer function.

Evidently, a procedure is needed which can select an optimum FWL realization, having regard to the introduction of frequency weighting. This is the subject of the paper.

A frequency weighted measure has already been introduced by Thiele [5], but with a specific relationship between the weightings of the various terms of the measure. This relationship can virtually never be satisfied when the weightings are derived as in (1).

## 2. WEIGHTED SENSITIVITY MEASURE OF A REALIZATION

In this paper we consider the implementation of a discrete linear time-invariant single input, single output compensator having the following transfer function :

$$R(z) = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 + \sum_{i=1}^{n} a_i z^{-i}}. \tag{2}$$

This system can be implemented by a minimal state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + du(t) \end{aligned} \tag{3}$$

with $A \in \gtrless^{n\times n}, B \in \gtrless^{n}, C^T \in \gtrless^{n}$ and $d \in \gtrless$. The transfer function can be expressed in terms of the state matrices as

$$H(z) = C(zI - A)^{-1}B + d. \tag{4}$$

We now define a realization set $S_H$ of this system as follows:

$$S_H = \{(A, B, C, d) : (A, B, C, d) \text{ satisfies } (4)\}.$$

Clearly, if $(A, B, C, d)$ belongs to $S_H$, so does $(T^{-1}AT, T^{-1}B, CT, d)$ for any similarity transformation $T$. This means that $S_H$ is an infinite set.

There are of course several ways of defining an overall sensitivity measure. Here we present a measure proposed by Tavsanoglu and Thiele [3]. It is an absolute rather than a relative sensitivity measure and is therefore based on a fixed-point arithmetical implementation; alternative floating-point implementations are discussed in [6, 7].

*Definition 1.* : Let $M \in \gtrless^{n\times m}$ be a matrix and let $f(M) \in \lhd$ be a scalar complex function of $M$, differentiable w.r.t. all the elements of $M$. We then define the sensitivity function of $f$ w.r.t. $M$ as

$$S_M \triangleq \frac{\partial f}{\partial M} \text{ with } (S_M)_{ij} \triangleq \frac{\partial f}{\partial m_{ij}} \tag{5}$$

where $m_{ij}$ denotes the $(i,j)^{th}$ element of the matrix $M$. ∎

With these notations it is easy to show [3] that

$$\begin{aligned} S_A(z) &\triangleq \frac{\partial R(z)}{\partial A} = G(z)F^T(z) \\ S_B(z) &\triangleq \frac{\partial R(z)}{\partial B} = G(z) \\ S_C(z) &\triangleq \frac{\partial R(z)}{\partial C^T} = F(z) \end{aligned} \tag{6}$$

where

$$F(z) \triangleq (zI - A)^{-1}B = [f_1(z) \dots f_n(z)]^T$$
$$G^T(z) \triangleq C(zI - A)^{-1} = [g_1(z) \dots g_n(z)] \tag{7}$$

Note that the direct term $d$ and the sensitivity function w.r.t. it are coordinate independent, so they have nothing to do with the optimal realization problem.

*Definition 2.* : Let $f(z) \in \mathcal{C}^{n \times m}$ be any complex matrix valued function of the complex variable $z$. We then define the $L_p$-norm of $f(z)$ as

$$\| f \|_p \triangleq \left( \frac{1}{2\pi} \int_0^{2\pi} \| f(e^{j\omega}) \|_F^p \, d\omega \right)^{1/p} \tag{8}$$

where $\| f(e^{j\omega}) \|_F$ is the Frobenius norm of the matrix $f(e^{j\omega})$:

$$\| f(e^{j\omega}) \|_F \triangleq \left( \sum_{i=1}^{n} \sum_{k=1}^{m} |f_{ik}(e^{j\omega})|^2 \right)^{1/2} = \{tr[f^T(e^{-j\omega}) f(e^{j\omega})]\}^{1/2}. \tag{9}$$

∎

Tavsanoglu and Thiele [3] have proposed the following overall sensitivity measure of the transfer function $R(z)$ w.r.t. the parameters in the realization $A, B, C$:

$$M_a \triangleq \| \frac{\partial R}{\partial A} \|_1^2 + \| \frac{\partial R}{\partial B} \|_2^2 + \| \frac{\partial R}{\partial C^T} \|_2^2 . \tag{10}$$

The mixing of different measures ($L_1$ and $L_2$) in the overall sensitivity measure above is motivated by the analytic properties of the first term on the right of (10), which allow one to derive an analytic minimization procedure for $M_a$: see [3] and [5]. The optimization of a more logical $L_2$ measure is much harder and has only recently been solved in [8] and, independently, by Helmke and Moore [9].

Note that the measure in Definition 2 is in fact a frequency independent mean value of a matrix function in the whole frequency range. Therefore, the sensitivity measure $M_a$ defined in (10) considers the sensitivity behavior of the transfer function at one frequency point to be as important as at another frequency point. However, we want to introudce frequency weighting. This is done via the definition of a weighted sensitivity and hence a weighted sensitivity measure. We shall provide a procedure applying for arbitrary weights.

Let $W_A(z)$, $W_B(z)$ and $W_C(z)$ be three integrable scalar functions of the complex variable $z$. Then the weighted sensitivity functions corresponding to those given in

(6) are defined as

$$\begin{aligned}\frac{\delta R(z)}{\delta A} &\triangleq W_A(z)\frac{\partial R(z)}{\partial A}\\ \frac{\delta R(z)}{\delta B} &\triangleq W_B(z)\frac{\partial R(z)}{\partial B}\\ \frac{\delta R(z)}{\delta C^T} &\triangleq W_C(z)\frac{\partial R(z)}{\partial C^T}.\end{aligned} \tag{11}$$

Note that the notation is not meant to suggest that $\delta$ is a derivative operator.

$$W_A(z) = W_1(z)W_2(z) \tag{12}$$

be any factorization of $W_A(z)$. With Definition 2, the over-all weighted $L_1/L_2$ sensitivity measure is defined as

$$M_a^* \triangleq \| \frac{\delta R(z)}{\delta A} \|_1^2 + \| \frac{\delta R(z)}{\delta B} \|_2^2 + \| \frac{\delta R(z)}{\delta C^T} \|_2^2 . \tag{13}$$

Now using (6), (11) and (12), $M_a^*$ can be rewritten as

$$M_a^* = \| W_1(z)G(z)(W_2(z)F(z))^T \|_1^2 + \| W_B(z)G(z) \|_2^2 + \| W_C(z)F(z) \|_2^2 . \tag{14}$$

A similarity transformation $x = Tz$ transforms $(A, B, C, F(z), G(z))$ into $(T^{-1}AT,$-$T^{-1}B, CT, T^{-1}F(z), T^TG(z))$. The optimal *FWL* state-space design can then be formulated as follows:

$$\min_{(A,B,C)\in S_H} M_a^* \tag{15}$$

## 3. OPTIMAL FWL REALIZATIONS

The difficulty in solving (15) is due to the fact that the first term on the right of (13) is a complicated function of the realization $(A, B, C)$. To overcome this, note that by the Cauchy-Schwartz inequality

$$\| \frac{\delta H(z)}{\delta A} \|_1^2 = \| W_1(z)G(z)(W_2(z)F(z))^T \|_1^2 \le \| W_1(z)G(z) \|_2^2 \| W_2(z)F(z) \|_2^2 \tag{16}$$

where equality holds if and only if

$$\rho^2 G^H(z)G(z)|W_1(z)|^2 = F^H(z)F(z)|W_2(z)|^2 \; \forall z \in \{|z| = 1\}, \tag{17}$$

for some $\rho \neq 0 \in \gtreqless$. We will study the following upper bound of $M_a^*$ :

$$M_a^* \le \bar{M}_a^* \triangleq \| W_1(z)G(z) \|_2^2 \| W_2(z)F(z) \|_2^2 + \| \frac{\delta R(z)}{\delta B} \|_2^2 + \| \frac{\delta R(z)}{\delta C^T} \|_2^2 . \tag{18}$$

We consider methods for minimizing $\bar{M}_a^*$. It is easy to show with (8)-(9) that

$$\bar{M}_a^* = tr(K_{o1})tr(K_{c2}) + tr(K_{oB}) + tr(K_{cC}), \tag{19}$$

where $K_{o1}$, $K_{c2}$, $K_{oB}$ and $K_{cC}$ can be obtained by the following general expression:

$$K = \frac{1}{2\pi j}\oint_{|z|=1} X(z)X^H(z)z^{-1}dz \tag{20}$$

with $X(z) = G(z)W_1(z), F(z)W_2(z), G(z)W_B(z)$ and $F(z)W_C(z)$, respectively. We call these four matrices $K_{o1}, K_{c2}, K_{oB}, K_{cC}$ weighted Gramians. Several algorithms for computing a weighted Gramian are available in [5] and [10]. A similarity transformation $x = Tz$ transforms $(A, B, C, K_{cC}, K_{c2}, K_{oB}, K_{o1})$ into $(T^{-1}AT, T^{-1}B, CT, T^{-1}K_{cC}T^{-T}, T^{-1}K_{c2}T^{-T}, T^TK_{oB}T, T^TK_{o1}T)$. So, the optimal $FWL$ design problem of (15) is replaced by the following upper bound minimization:

$$\min_{T:\det T\neq 0}\{\bar{M}_a^* = tr(T^TK_{o1}T)tr(T^{-1}K_{c2}T^{-T}) + tr(T^TK_{oB}T) + tr(T^{-1}K_{cC}T^{-T})\}. \tag{21}$$

Now, with $P = TT^T$, it is easy to see that

$$\bar{M}_a^* = tr(K_{o1}P)tr(K_{c2}P^{-1}) + tr(K_{oB}P) + tr(K_{cC}P^{-1}) \triangleq \mathcal{M}(P) \tag{22}$$

$$\min_{T:\det T\neq 0}\bar{M}_a^* \iff \min_{P:P=TT^T,\det P\neq 0}\mathcal{M}(P). \tag{23}$$

We can now state (without proof here) the following result

**Main Result:** Suppose that $K_{oB}$ and $K_{cC}$ are nonsingular.[1] Then the minimum of $\mathcal{M}(P)$ defined in (23) exists and is achieved by a nonsingular $P$. Further, the optimum $P$ can be obtained through a gradient algorithm iteration

$$P(k+1) = P(k) - \mu\frac{\partial\mathcal{M}(P)}{\partial P}\bigg|_{P=P(k)} \tag{24}$$

$$\frac{\partial\mathcal{M}(P)}{\partial P} = -tr(K_{o1}P)P^{-1}K_{c2}P^{-1} - P^{-1}K_{cC}P^{-1} + tr(K_{c2}P^{-1})K_{o1} + K_{oB} \tag{25}$$

## Remarks

1. Thiele considered the case $W_1(z) = W_B(z)$ and $W_2(z) = W_0(z)$. In this case, the minimization is analytically achievable. If further, $W_1(z) = W_2(z)$, then $M_a^* = \bar{M}_a^*$. In compensator implementation, we have $W_A = W_B = W_C = (1+PR)^{-2}P$.
2. Knowledge of the optimum $P$ does not define an optimum coordinate basis change $T$ through $P = TT'$; rather $T$ is only unique up to right multiplication by an orthogonal matrix. The additional freedom in $T$ can be used to ensure the advantageous incorporation in the matrices $A, b, c$ of $\frac{1}{2}n(n-1)$ zero elements.

[1]This condition is generally satisfied. Space limitations prevent fuller analysis here

3. There are appropriate, easily computable, initial conditions for (24)

## 4. THE NEXT STEP - SAMPLED DATA CONTROL

A discrete time controller $C(z)$ will normally be used with a continuous-time plant $P(s)$, there being also present in the loop an antialiasing filter $F(s)$, sampler $S$ and hold element $H$. The analysis of the loop using a discrete-time transfer function approach alone suffers well known disadvantages, particularly the neglect of inter-sample ripple. A scheme is needed for selection of an optimal FWL realization of the controller $R(z)$ which allows consideration of intersample behaviour. We outline very briefly how this can be done. *Formally*, we have a closed-loop operator $T = PHCSF(1+PHCSF)^{-1}$ and

$$\begin{aligned}\frac{\partial T}{\partial \alpha} &= (1+PHCSF)^{-1}PH\,\frac{\partial C}{\partial \alpha}SF(1+PHCSF)^{-1} \\ &= W_1\,\frac{\partial C}{\partial \alpha}\,W_2\end{aligned}$$

The operators $W_1, W_2$ do <u>not</u> have transfer function representations, and this is a difficulty. The difficulty can be tackled by a two-step procedure: very fast sampling [at a multiple of the sampling frequency of $C(z)$] of the continuous-time parts of $W_1$ and $W_2$, followed by "blocking" or "lifting" to turn the resulting multirate discrete-time system into a single rate system. The use of very fast sampling followed by blocking has been used also for the problem of passing from a continuous-time to a discrete time controller, and for solving the sampled-data $H_\infty$ problem, [11-12].

The details of these ideas will be presented in a forthcoming publication, together with examples, [13]

## 5. CONCLUSIONS

The main ideas of the paper are these: Optimum FWL realization of a digital controller can be viewed as a frequency weighted sensitivity minimization problem; this problem has a unique solution, obtainable via a gradient algorithm iteration; a more sophisticated problem statement, reflecting inter-sample performance, is possible. A fuller version of many of the ideas of this paper is available as [14].

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A realisation of (A.1) is given by

$$A_1=\begin{bmatrix} 0 & 1 & & \\ 0 & & \ddots & \\ & & & 1 \\ -a_0 & -a_2 & \cdots & -a_{n-2} \end{bmatrix},\quad b_{L1}=\begin{bmatrix} 0\\ \vdots \\ 0 \\ 1\end{bmatrix},\quad c_{L1}^{T}=[a_1 \quad a_3 \quad \cdots \quad a_{n-1}]. \tag{A.2}$$

Define also

$$C^*=JCJ,\qquad D^*=JDJ. \tag{A.3}$$

The observability matrix is given by

$$S_1=\begin{bmatrix} c_{L1}^{T} \\ c_{L1}^{T}A_1 \\ \vdots \\ c_{L1}^{T}A_1^{m-1}\end{bmatrix}=S_1(A_1, c_{L1}^{T}). \tag{A.4}$$

Let $e_{L1}, e_{L2},\dots, e_{Lm}$ be the unit vectors; then it is easily verified that

$$S_1(A_1, e_{L1}^{T})=I,\quad S_1(A_1, e_{L2}^{T})=A_1,\quad \dots,\quad S_1(A_1, e_{Lm}^{T})=A_1^{m-1}.$$

Therefore,

$$S_1(A_1, c_{L}^{T})=a_1 I+a_3 A_1+\cdots+a_{n-1}A_1^{m-1}=g(A_1). \tag{A.5}$$

But

$$C^*e_{Lm}=\begin{bmatrix} a_1\\ a_3\\ \vdots \\ a_{n-3}\\ a_{n-1}\end{bmatrix}=c_{L1}$$

and by (17),

$$C^*A_1^k=(A_1^{T})^k C^*,\quad k=1,2,3,\dots.$$

Then it can be seen by inspection that

$$C^*[e_{Lm} \quad A_1 e_{Lm} \quad \cdots \quad A_1^{m-1}e_{Lm}]=S_1^{T}. \tag{A.6}$$

But $[e_{Lm} \quad A_1 e_{Lm} \quad \cdots \quad A_1^{m-1}e_{Lm}]$ is clearly a lower triangular matrix with second diagonal elements equal to one. Therefore, $\det C^*=(-1)^{m(m-1)/2}\det S_1^{T}$ and from (A.5),

$$\det C^*=(-1)^{m(m-1)/2}\det g(A_1). \tag{A.7}$$

From a well known result in matrix theory $\det g(A_1)=\prod g(\alpha_i)$ where $\alpha_i$ are the eigenvalues of $A_1$. Therefore,

$$\det C=(-1)^{m(m-1)/2}\prod g(\alpha_i) \tag{A.8}$$

where $\alpha_i$ are the roots of $h(\lambda)$.

For the proof of the second equality of Lemma 1, consider the transfer function

$$w_2(\lambda)=\frac{h(\lambda)}{\lambda^2 g(\lambda)/a_{n-1}}=\frac{a_0+a_2\lambda+\cdots+\lambda^m}{(a_1/a_{n-1})\lambda^2+(a_3/a_{n-1})\lambda^3+\cdots+\lambda^{m+1}}. \tag{A.9}$$

A realisation of (A.9) is given by

$$A_2=\begin{bmatrix}0 & 1\\ 0 & B\end{bmatrix}\quad\text{where}\quad B=\begin{bmatrix}0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1\\ 0 & -\dfrac{a_1}{a_{n-1}} & \cdots & -\dfrac{a_{n-3}}{a_{n-1}}\end{bmatrix}, \tag{A.10}$$

$$b_{L2}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},\qquad c_{L2}^{T}=[a_0\quad a_2\quad\cdots\quad 1].$$

The observability matrix $S_2$ is given by

$$S_2=\begin{bmatrix}c_{L2}^{T}\\ c_{L2}^{T}A_2\\ \vdots\\ c_{L2}^{T}A_2^{m}\end{bmatrix}=\begin{bmatrix}a_0 & a_2 & \cdots & 1\\ 0 & a_0 & \cdots & \cdots\\ \vdots & & \ddots & \\ 0 & 0 & 0 & a_{0+\cdots}\end{bmatrix}=S_2(A_2,c_{L2}^{T}). \tag{A.11}$$

$h(B)$(established below)

Now $S_2(A_2, e_{L1}^{T})=I$, $S_2(A_2, e_{L2}^{T})=A_2,\ldots,S_2(A_2, e_{Lm+1}^{T})=A_2^{m}$, so that

$$S_2(A_2,c_{L2}^{T})=a_0 I+a_2A_2+\cdots+A_2^{m}=h(A_2). \tag{A.12}$$

If we denote the $m\times m$ submatrix of $A_2$ obtained by deleting the first row and column by $B$ then the corresponding submatrix of $S_2$ is evidently $h(B)$.

We have

$$\det h(A)=a_0\det h(B) \tag{A.13}$$

Now it can be easily established by direct calculation that

$$D^*B=B^{T}D^* \tag{A.14}$$

and

$$D^*e_{Lm}=a_{n-1}\begin{bmatrix}a_0\\ a_2\\ \vdots\\ \vdots\\ a_{n-2}\end{bmatrix}-\begin{bmatrix}0\\ a_1\\ a_3\\ \vdots\\ a_{n-3}\end{bmatrix}=f_L \tag{A.15}$$

Also

$$c_{L2}^{T}A_2=\frac{1}{a_{n-1}}[0\quad f_L^{T}]. \tag{A.16}$$

Recalling the construction of $S_2$ above and its submatrix $h(B)$, we see that

$$[f_{\mathrm{L}} \quad Bf_{\mathrm{L}} \quad \cdots \quad B^{m-1}f_{\mathrm{L}}] = a_{n-1}[h(B)]^{\mathrm{T}} \tag{A.17}$$

and accordingly,

$$D^*[e_{\mathrm{L}m} \quad \cdots \quad B^{m-1}e_{\mathrm{L}m}] = \left[D^*e_{\mathrm{L}m} \quad B^{\mathrm{T}}D^*e_{\mathrm{L}m} \quad \cdots \quad (B^{\mathrm{T}})^{m-1}D^*e_{\mathrm{L}m}\right] = a_{n-1}[h(B)]^{\mathrm{T}}. \tag{A.18}$$

Now $[e_{\mathrm{L}m} \ \cdots \ B^{m-1}e_{\mathrm{L}m}]$ is a lower triangular matrix with unity elements in the second diagonal. Therefore,

$$\begin{aligned}\det D^* &= (-1)^{m(m-1)/2}a_{n-1} \det h(B) = (-1)^{m(m-1)/2}a_{n-1}h(0)\Pi h(\beta_j)\\ &= (-1)^{m(m-1)/2}a_0a_{n-1}\Pi h(\beta_j)\end{aligned} \tag{A.19}$$

where $\beta_j$ are the roots of $g(\lambda)$. Hence

$$\det C = \frac{1}{a_0} \det D^* = (-1)^{m(m-1)/2}a_{n-1}\Pi h(\beta_j). \qquad \square$$

**Proof of Lemma 2.** It is easily verified that

$$\frac{h(\mu)g(\lambda) - h(\lambda)g(\mu)}{\mu - \lambda} = [1 \quad \mu \quad \cdots \quad \mu^{m-1}]C^* \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{bmatrix}. \tag{A.20}$$

It is straightforward to show that if we substitute $\mu = \lambda + \varepsilon$ and let $\varepsilon$ tend to zero, then

$$\frac{h(\lambda+\varepsilon)g(\lambda) - h(\lambda)g(\lambda+\varepsilon)}{\varepsilon} = \frac{[h(\lambda+\varepsilon) - h(\lambda)]g(\lambda) - [g(\lambda+\varepsilon) - g(\lambda)]h(\lambda)}{\varepsilon}$$

so that

$$g(\lambda)\frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda} - h(\lambda)\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}.$$

Therefore,

$$\lambda_{\mathrm{L}}^{\mathrm{T}}C^*\lambda_{\mathrm{L}} = g(\lambda)\frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda} - h(\lambda)\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}. \qquad \square \tag{A.21}$$

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