Lecture Notes in Control and Information Sciences

Edited by A.V. Balakrishnan and M. Thoma

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David J. Clements
Brian D. O. Anderson

Singular Optimal Control:
The Linear-Quadratic Problem

Springer-Verlag
Berlin • Heidelberg • New York
Lecture Notes in Control and Information Sciences

Edited by A.V. Balakrishnan and M. Thoma

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XIII, 512 pages. 1978
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Singular Optimal Control:
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Springer-Verlag
Berlin Heidelberg New York 1978
Series Editors
A. V. Balakrishnan · M. Thoma

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Singular Optimal Control: The Linear-Quadratic Problem is a monograph aimed at advanced graduate students, researchers and users of singular optimal control methods. It assumes prior exposure to the standard linear-quadratic regulator problem, and a general maturity in linear systems theory.

A number of advances in singular, linear-quadratic control have taken place very recently. The book is intended to present an up-to-date account of many of these advances. At the same time, the book attempts to present a unified view of various approaches to singular optimal control, many of which are apparently unrelated.

Acknowledgements

The research reported in this book was supported by the Australian Research Grants Committee, and our thanks for this support are gratefully extended.

The manuscript typing, from first draft to final version, was undertaken by Mrs. Dianne Piefke. The willing and expert participation by Mrs. Piefke constituted a vital link in the publication chain, and to her the authors offer their sincerest thanks.
# TABLE OF CONTENTS

## I SINGULAR LINEAR-QUADRATIC OPTIMAL CONTROL - A BROAD BRUSH PERSPECTIVE

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1</td>
<td>Problem origins</td>
<td>1</td>
</tr>
<tr>
<td>I.2</td>
<td>Historical aspects of singular linear-quadratic control</td>
<td>2</td>
</tr>
<tr>
<td>I.3</td>
<td>Objective of this book</td>
<td>3</td>
</tr>
<tr>
<td>I.4</td>
<td>Chapter outline</td>
<td>4</td>
</tr>
</tbody>
</table>

## II ROBUST LINEAR-QUADRATIC MINIMIZATION

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.1</td>
<td>Introduction</td>
<td>8</td>
</tr>
<tr>
<td>II.2</td>
<td>Quadratic property of the optimal performance index</td>
<td>9</td>
</tr>
<tr>
<td>II.3</td>
<td>Initial condition results and the Riemann-Stieltjes inequality</td>
<td>14</td>
</tr>
<tr>
<td>II.4</td>
<td>Robustness in problems with end-point constraints</td>
<td>28</td>
</tr>
<tr>
<td>II.5</td>
<td>Extremal solutions of Riemann-Stieltjes inequalities</td>
<td>36</td>
</tr>
<tr>
<td>II.6</td>
<td>Summarizing remarks</td>
<td>38</td>
</tr>
</tbody>
</table>

## III LINEAR-QUADRATIC SINGULAR CONTROL: ALGORITHMS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>III.1</td>
<td>Introduction</td>
<td>41</td>
</tr>
<tr>
<td>III.2</td>
<td>Control space dimension reduction and a standard form</td>
<td>45</td>
</tr>
<tr>
<td>III.3</td>
<td>Vector version of Kelley transformation</td>
<td>47</td>
</tr>
<tr>
<td>III.4</td>
<td>Computation of optimal control and performance index</td>
<td>54</td>
</tr>
<tr>
<td>III.5</td>
<td>Solution via Riemann-Stieltjes inequality</td>
<td>57</td>
</tr>
<tr>
<td>III.6</td>
<td>Summarizing remarks</td>
<td>61</td>
</tr>
</tbody>
</table>

### APPENDIX III.A

Dolezal's theorem

### APPENDIX III.B

Symmetry condition
IV DISCRETE-TIMELINEAR-QUADRATICSINGULARCONTROL
ANDCONSTANTDIRECTIONS

IV.1 Introduction 67
IV.2 Linear-quadraticcontrolindiscretetime 68
IV.3 Constantdirections—basicproperties 73
IV.4 Controlspacedimensionreduction 78
IV.5 Statesspacedimensionreduction 80
IV.6 Totalreductionoftheproblem 84
IV.7 Time-varyingproblems,miscellaneous
pointsandsummarizingremarks 87
APPENDIX IV.A Definitions of coefficientmatrices 90

V OPENQUESTIONS 92
CHAPTER I

SINGULAR LINEAR-QUADRATIC OPTIMAL CONTROL
- A BROAD BRUSH PERSPECTIVE

1. PROBLEM ORIGINS

Our concern throughout this book is with singular, linear-quadratic optimal control problems. In this section, we explore the origins of such problems, and in later sections of the chapter, we sketch some historical aspects, and describe how this book surveys some major aspects of the present state of knowledge.

We first review the notion of a linear-quadratic problem (without regard to whether or not it is singular), then we review the notion of a singular control problem (without regard to whether or not it is linear-quadratic), and then we tie the two notions together.

Linear-quadratic optimal control problems, singular or nonsingular, usually arise in one of two distinct ways. First, there is prescribed a linear system

\[ \dot{x} = F(t)x + G(t)u \quad x(t_0) = x_0 \quad \text{known} \]  

and a performance index quadratic in \( u \) and \( x \); for the linear regulation problem, for example,

\[ V[x_0, u(.)] = \int_{t_0}^{T} [x^TQ(t)x + u^TR(t)u]dt + x^T(T)Sx(T) \]  

in which, usually, \( Q, R \) and \( S \) are symmetric with \( Q \) and \( S \) nonnegative definite and \( R \) positive definite. Of course, the problem is to find a control \( u(\cdot) \) minimizing the value of \( V[x_0, u(\cdot)] \). The pioneering work of Kalman, see e.g. [1, 2] has given rise to an extensive study of this type of problem, see e.g. [3, 4]. Any performance index usually reflects some physically based notion of performance or quality of control, and (1.2) as a result of the listed constraints on \( Q, R \) and \( S \) is often very much physically based. However, it is possible to relax some of the constraints on \( Q, R \) and \( S \) to allow a crossproduct term, \( 2x^TR(t)u \) in the integrand of (1.2). In this way, the most general form of linear-quadratic control problem can be encountered.

The second way in which linear-quadratic problems arise is via a perturbational type of analysis (a second variation theory) of a general optimal control problem, in which the underlying system may not be linear and the underlying performance index not quadratic. Given a certain initial state and the corresponding optimal control for a general optimal control problem, one can seek the adjustment to the optimal control necessary to preserve optimality when the initial state is changed by a small amount; an approximation to the control adjustment follows as the solution to a linear-quadratic problem. For an exposition of the perturbation procedures including details of the cal-
Calculations for obtaining the linear-quadratic problem from the general problem, see e.g. [5].

**Singular optimal control problems** arise in the following way. With $H$ the Hamiltonian, recall that, in any optimal control problem, singular or nonsingular, extremal arcs are defined by the requirement that $H$ takes an extreme value. If this requirement does not allow the expression of the control vector in terms of state and costate vectors, the problem is singular. This can happen if $H_u$ vanishes and $H_{uu}$ is singular.

Johnson and Gibson demonstrated the existence of optimal singular solutions to certain problems in [6]; other examples are provided in [5, 7-9] and the excellent survey [10]. In most singular problems, the Hamiltonian is linear in $u(\cdot)$; this will be the case if the system equation and loss function involve $u(\cdot)$ linearly.

The notion of a singular linear-quadratic problem is obtained by a straightforward coalescing of the singular and linear-quadratic notions. Singularity of a linear-quadratic problem is equivalent to singularity of the matrix $R$ in the cost function $x^T Q x + 2 u^T H x + u^T R u$. Singular linear-quadratic problems can arise directly, or as a result of applying a second variation theory to a general optimal control problem.

In considering any optimal control problem, a number of questions tend to arise. Is the problem solvable, or can the performance index be made as negative as desired by some choice of control? If the problem has a solution, how may one compute the optimal value of the performance index and an optimal control?

In the case of nonsingular linear-quadratic problems, as most readers will know, there are tidy solutions to these problems, see e.g. [5, 11-14]. Most of these solutions involve a matrix Riccati differential equation in which $R^{-1}$ appears. As the survey [10] indicates, tidy solutions for singular problems in the main are much more recent or not yet available. We devote the next section to discussing some of the singular linear-quadratic results that have been determined.

2. **HISTORICAL ASPECTS OF SINGULAR LINEAR-SQUADRIC CONTROL**

Up till this point, at least four, largely disjoint, methods of attack on singular linear-quadratic problems have existed. In this section, we indicate what these are.

One thrust can be identified in the work of Goh [15, 16], Kelley [9, 17, 18] and Robbins [19]; this thrust may not have yet petered out, since further results are still forthcoming, [20]. The prime, but not exclusive, concern in this work is with the computation problem, and the recurring theme in the papers is to aim to replace a singular linear-quadratic control problem by a nonsingular one, such that solutions of the nonsingular problem somehow determine solutions of the singular problem.

The second thrust is exemplified by the work of Jacobson [21, 22], as amplified by Anderson [25] and rounded nicely by Molinari [24]. Here the emphasis is on finding necessary and (sometimes differing) sufficient conditions for the solvability of sing-
ular linear-quadratic problems. To be sure, conditions of one sort and another have been known for a long time (e.g. \( R \geq 0 \)); the point about the conditions in [21-24] is that they encompass all earlier known conditions, and when necessary and sufficient condition sets differ, the difference is quite clearly very minor. Actually, very recent work [25, 26] has been concerned with eliminating the differences.

The third method of attack is via regularization, i.e. one perturbs a singular problem to make it nonsingular, the perturbation being such that the solution of the nonsingular problem is, in some sense, close to that of the singular problem. This idea has been exploited especially by Jacobson, see e.g. [27, 28], with the latter of these references making contact with the problem of generating necessary and sufficient conditions for problem solvability. The precise regularization procedure used is a simple one - a term \( \epsilon u'u \) for small positive \( \epsilon \) is added to the performance index integrand. The effect is to perturb the optimal performance index slightly; however, the perturbation in the optimal control can be exceedingly hard to pin down, see e.g. [29].

We shall pay very little attention to the regularization idea in this book, not because of any inherent demerit of the idea, but rather because once a nonsingular problem has been obtained, the problem ceases to have much challenge about it. Further theoretical conditions concerning singular problems obtainable by studying a sequence of regularized problems with \( \epsilon \to 0 \) are in the main obtainable more simply by other procedures. [Of course, in a specific problem, the study of such a sequence to deduce the optimal control may be very attractive computationally].

The fourth thrust is rather a related direction of research than a method of attack on singular linear-quadratic problems. It turns out that there are certain problems in passive network synthesis and covariance factorization that are allied to the singular linear-quadratic control problem. More precisely, certain matrix differential and integral inequalities recently perceived to be relevant in studying control problems have been used for the network and covariance problems [30, 31].

By way of general comment, we note that experience shows that vector control problems are often much more difficult than problems with a scalar control. Throughout the book, we consider vector controls.

3. OBJECTIVE OF THIS BOOK

We have set ourselves the task of presenting solutions to the existence and computation questions associated with singular linear-quadratic problems. In the process however, we aim to show how the four directions of activity described in the last section may be made to coalesce, allowing the presentation of a unified theory.

Having done this, we attempt to translate a number of the ideas to discrete-time problems. More is said about this in the next section.
4. CHAPTER OUTLINE

We warn the reader that only brief comments on the background of the problems studied in this book are made in this section; we defer detailed comments to the introductory material in each chapter.

The central theme of Chapter II is robustness in linear-quadratic minimization problems. To understand why this should be so, we shall digress from a description of the actual chapter contents.

In studying the second variation problem, it is frequently the case that the initial state in the associated linear-quadratic problem is always zero, and one desires necessary and sufficient conditions for the performance index to be nonnegative for all controls $u(\cdot)$. With a controllability assumption, a necessity condition can be stated, and without the controllability assumption, a sufficiency condition can be found, [21-24]. The necessity and sufficiency conditions are very similar but not identical; the sufficiency condition is basically a generalization to the singular case of a condition that a Riccati equation appearing in the nonsingular problem have no escape times on $[t_0, t_f]$.

The question then arises as to how this aesthetically disquieting situation of differing necessary and sufficient conditions for the nonnegativity requirement can be remedied. The solution to the problem is quite simple — we slightly change the problem statement. Instead of seeking necessary and sufficient conditions for nonnegativity we seek necessary and sufficient conditions for the optimal performance index to be finite for all initial states. Equivalently, we demand that the minimization problem (or, more generally, the optimization problem) have a solution not just for zero initial state, but for all initial states close to zero (and hence for all initial states without restriction on size, by the linear-quadratic nature of the problem). We suggest that any realistic model of a physical control system that has a solution for zero initial condition should also have a solution for any initial condition in a neighborhood of zero. Otherwise for an arbitrarily small change in initial condition of the system the optimal cost in controlling the system would change from a finite value to $\infty$. Clearly, such a situation is unrealistic.

The first use of the robustness idea in Chapter II is therefore, in the relatively easy derivation of identical necessary and sufficient conditions for the linear-quadratic problem to have a finite optimal performance index for all initial states. Once having introduced the idea of robustness however, we are led to examine the extent of its applicability to other types of robustness, e.g. with respect to initial time, final time, end-point weighting, and initial and end-point constraints.

Actually, the notion of robustness with respect to initial time has been used elsewhere, as in [14] where it is termed "extendibility". It turns out that a minimization problem is robust with respect to the initial time if and only if it is robust with respect to the initial state.

The latter part of Chapter II is concerned with constrained end-point problems.
Here we study the existence question again, once more closing gaps between earlier necessity and sufficiency conditions by imposing a robustness assumption. It turns out that this time there are three possible robustness assumptions, not two, which are nontrivially equivalent: robustness with respect to final state, final time and terminal state weighting matrix in the performance index.

Throughout Chapter II, conditions are almost entirely expressed in terms of matrix integral inequalities, of the type that have evolved from the second general line of attack on singular problems described earlier [21-24]. The opportunity is therefore taken to derive several new properties of these inequalities. Some have appeared in [25, 26], while others are reported here for the first time.

Whereas Chapter II is devoted to the existence question, Chapter III is concerned with computation questions — more precisely with describing algorithms for checking nonnegativity of the performance index with zero initial state and arbitrary control, for evaluating the optimal performance index with arbitrary initial state (and in the process checking its existence), and/or computing optimal controls. The algorithms are shown to flow from the first, second and fourth approaches to singular problems set out earlier, i.e. all approaches save that of regularization. This provides therefore a significant unification of many ideas previously somewhat disjoint.

There is, unfortunately, a caveat. To execute the algorithms, certain smoothness and constancy of rank assumptions must be fulfilled, and this cannot always be guaranteed.

While we obviously leave precise specification of the algorithms to Chapter III, we mention several global aspects of them here. The algorithms proceed by replacing a singular linear-quadratic problem by another linear-quadratic problem with either lower control space dimension and/or lower state space dimension. The replacing problem may be nonsingular or singular. A series of such replacements ultimately leads to a problem which either has zero control space dimension, zero state space dimension, or is nonsingular. In all three cases, the problem is easily dealt with, and its solution can be somehow translated back to provide a solution to the originally posed singular problem.

In Chapter IV, singular discrete-time problems are defined and discussed. From one point of view, one might claim there is little to be gained from a close examination of discrete-time problems, since a Riccati difference equation defining the optimal cost and a formula for the optimal control are always valid. Nevertheless, many interesting parallels with the singular continuous-time results can be obtained, including a very general theory of degenerate and constant directions, first introduced and explored in [32-34].

Chapter V offers a very brief statement of some remaining directions for research.

REFERENCES


CHAPTER II

ROBUST LINEAR-QUADRATIC MINIMIZATION

1. INTRODUCTION

Conditions for the solvability of linear-quadratic continuous-time minimization problems have been studied in a number of papers, e.g. [1-7]. The more recent work has shown the value of characterizing problem solvability in terms of nonnegativity conditions involving certain Riemann-Stieltjes integrals. In this work, one somewhat bothersome problem remains – there are generally gaps between necessary conditions and sufficient conditions. In this chapter, we survey this material and show how to eliminate these gaps.

In the broadest terms, what we find is that the gap vanishes if one is studying problems which are in some way robust, that is, tolerant of small changes to some or all of the relevant parameters. To the extent that nonrobust problems can be regarded as qualitatively ill-posed (though in strict terms, or quantitative terms, they may be well-posed), we are saying that there is no gap in the case of a well-posed problem.

In what ways do we vary parameters in the robustness study? The answer depends somewhat on the problem in hand. For free end-point problems, we can consider a variation in the initial time, and in the case of those conditions applying to a zero initial state situation, we can consider a variation away from zero of that initial state. For constrained end-point problems, we can consider a variation in the final time, we can consider the use of a penalty function approach, and we can consider a relaxation of a constraint requiring that the final state be at a point to one requiring the final state to be in a (small) sphere around that point. For both classes of problems, we can also consider variations in the parameters of the matrices defining the system and performance index. (We do not however explore this last issue deeply; one immediate and major difficulty stems from deciding which quantities normally zero should remain at zero in a general parameter change, and which should not). Altogether then, in this chapter we study a range of robustness problems. In general, many of our conclusions are along the lines that if there is one kind of robustness, another kind is automatically implied.

An outline of this chapter is as follows. In Section 2, we study an arbitrary linear-quadratic problem with arbitrary constraints on the initial and terminal states. We show that if the constraints are achievable, the optimal performance index is a quadratic form in the independent variables which define the initial and terminal states. Various versions of this result are used through the later sections. In Section 3, we study problems with arbitrary but fixed initial state and free terminal state. Our main results equate two types of robustness (with respect to initial state and initial time) with an inequality condition involving Riemann-Stieltjes integrals. In Section 4, we turn to constrained end-point problems; the use of penalty functions provides yet another form of robustness, and most of the work is concerned with showing the
equivalence of this new type of robustness with robustness properties involving the terminal time or state, and with a Riemann-Stieltjes integral inequality condition. Finally in Section 5 we study the extremal solutions of the Riemann-Stieltjes inequality of Sections 3 and 4 and the relation between these extremal solutions and optimization problems. Section 6 offers some summarizing remarks.

2. QUADRATIC PROPERTY OF THE OPTIMAL PERFORMANCE INDEX

We study the system

\[
\dot{x} = F(t)x + G(t)u \quad t_0 \leq t \leq t_f
\]  

(2.1)

with various possible constraints on the initial and terminal states. Thus we postulate that for two fixed (possibly evanescent) constant matrices \( D_0, E_0 \), we have

\[
D_0 x(t_0) = \eta_0 \quad E_0 x(t_0) = 0,
\]

(2.2)

and that the controls are constrained such that

\[
D_f x(t_f) = \eta_f \quad E_f x(t_f) = 0.
\]

(2.3)

In (2.2) and (2.3), \( D_0, E_0, D_f \) and \( E_f \) are fixed constant matrices such that (without loss of generality) \( C_0 = [D_0^T \ E_0^T]^T \) and \( C_f = [D_f^T \ E_f^T]^T \) have full row rank. We allow the possibility of one or more of \( D_0, E_0, \) etc. evanescent. The vectors \( \eta_0 \) and \( \eta_f \) are arbitrary, but fixed in any given problem. The commonest situations are those where \( D_0 = I, E_0, D_f \) and \( E_f \) evanescent (the usual free terminal state problem) and \( D_0 = I, E_0 \) and \( D_f \) evanescent, and \( E_f = I \) (the usual zero terminal state problem).

Associated with (2.1) through (2.3) is the performance index

\[
V[x_0, t_0, u(\cdot)] = \int_{t_0}^{t_f} \left[ \dot{x}(t)^T Q(t)x(t) + 2u(t)^T R(t)u(t) + u(t)^T S(t)u(t) \right] dt + x^T(t_f) S x(t_f)
\]

(2.4)

where \( x_0 = x(t_0) \). The matrices \( Q(\cdot), R(\cdot), S(\cdot) \) all have dimensions consistent with (2.1) and (2.4) and are piecewise continuous. (This has the obvious meaning; the matrices have piecewise continuous entries). The matrix \( S \) is constant. Without loss of generality, we assume \( Q(\cdot), R(\cdot) \) and \( S(\cdot) \) are symmetric. The controls \( u(\cdot) \) are assumed to be piecewise continuous on \( [t_0, t_f] \). To begin with, we shall study classes of problems in which one or both of \( x_0 \) and \( x_f \) vary; later, variation of \( t_0 \) and \( t_f \) will be considered. In general, we shall be interested in the question of when (2.4) possesses an infimum which is finite.

However, in this section, we shall be concerned with establishing the functional form of the infimum. For each \( x_0 \) and \( u(\cdot) \), (2.4) takes a definite value. Now imagine that \( \eta_0, \eta_f \) are fixed but arbitrary, and that \( x_0 \) and \( u(\cdot) \) exist such
that (2.1) through (2.3) are satisfied. Then essentially (2.4) is being evaluated when \( x_0 \) and \( u(\cdot) \) are constrained linearly, by (2.1) through (2.3). In general, each \( \eta_\theta, \eta_\xi \) pair will permit an infinity of \( x_0, u(\cdot) \) satisfying (2.1) through (2.3). We define \( V[\eta_\theta, \eta_\xi] \) to be the infimum of the values of \( V[x_0, x_\xi, u(\cdot)] \) obtainable under the constraints (2.1) through (2.3). In case \( D_\xi = I \), and \( E_\theta, E_\xi \) evanesce, then \( V[\eta_\theta, \eta_\xi] \) simply becomes \( V[x_0] \), the usual free-end-point optimal performance index.

The main idea of this section is that the optimal performance index, if it is finite for all \( \eta_\theta \) and \( \eta_\xi \), is quadratic in these quantities. Special cases of the result will be used in later sections. We also postpone to later sections study of issues of existence and computation.

Before proving the quadratic property however, we consider a preliminary question: for a given \( \eta_\theta, \eta_\xi \) pair, is it always possible to find \( x(t_\xi) \) and \( u(\cdot) \) satisfying (2.1) through (2.3)? Various special answers are easily established. Thus if \( D_\theta \) and \( E_\theta \) evanesce, i.e. \( x(t_\xi) \) is free, then there always exists an \( x(t_\xi) \) and \( u(\cdot) \) such that \( D_\xi x(t_\xi) = \eta_\xi \), \( E_\xi x(t_\xi) = 0 \) for any \( \eta_\xi \). (Because \( [D_\xi E_\xi]' \) has full row rank, there is no possibility of the constraints on \( x(t_\xi) \) being inconsistent). On the other hand, if \( D_\xi = I \) and \( E_\xi \) evanesce and \( E_\theta = I \), so that controls are required to take an arbitrary \( x(t_\theta) = \eta_\theta \) to \( x(t_\xi) = 0 \), it is clear that a controllability condition of some description is required. The precise condition is pinned down in the following way.

**Lemma II.3.1.** Let \( V[\eta_\theta, \eta_\xi] \) denote \( \inf V[x(t_\theta), x_\xi, u(\cdot)] \) [defined in (2.4)], subject to (2.1) through (2.3) holding. Then \( V[\eta_\theta, \eta_\xi] < \infty \) for all \( \eta_\theta, \eta_\xi \). [equivalently: (2.1) through (2.3) are attainable for arbitrary \( \eta_\theta, \eta_\xi \) if and only if

\[
\begin{align*}
\text{Range} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} : C_\xi \Theta(t_\xi, t_\theta) C_\xi^T (C_\xi C_\xi^T)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} 
\end{align*}
\]

(2.5)

where \( X \) is a matrix whose columns constitute a basis for \( N(C_\theta) \), the nullspace of \( C_\theta \), and \( W \) is the controllability matrix

\[ W = \int_{t_\theta}^{t_\xi} \Theta(t_\xi, r) G(r) G^T(r) \Phi(t_\xi, r) dr. \]

**Proof:** Necessity: observe that for any \( u(\cdot) \), \( \int_{t_\theta}^{t_\xi} \Theta(t_\xi, r) G(r) u(r) dr \) has the form \( Wx \) for some \( x \) [8, see p. 75], and that any \( x(\xi) \) for which \( C_\xi x(t_\xi) = [\eta_\xi 0]' \) may be written as \( C_\xi (C_\xi C_\xi)^{-1} [\eta_\xi 0]' + K \beta \) for some \( \beta \). Now premultiply by \( C_\xi \) the basic equation

\[ x(t_\xi) = \Theta(t_\xi, t_\theta) x(t_\theta) + \int_{t_\theta}^{t_\xi} \Theta(t_\xi, r) G(r) u(r) dr. \]

This leads to
\[
\begin{bmatrix}
\eta_f \\
0
\end{bmatrix} = C_e^\theta(t_f, t_0)C_e^T(C_eC_e^T)^{-1}
\begin{bmatrix}
\eta_0 \\
0
\end{bmatrix} = C_e^\theta(t_f, t_0)\kappa + C_e^\lambda\kappa
\]  
(2.6)

for some \(\alpha\) and \(\beta\). Now in order that (2.3) be attainable given (2.1) and (2.2) there must be \(\alpha\) and \(\beta\) satisfying (2.6). In order that such \(\alpha\) and \(\beta\) exist for all \(\eta_0\) and \(\eta_f\), (2.5) must hold.

Sufficiency: for prescribed \(\eta_f\), \(\eta_0\), choose \(\alpha\), \(\beta\) satisfying (2.6), which is possible in view of (2.5). Take \(u(t) = G_e(t)\theta(t_f, t_0)\alpha\) and \(x(t_0) = G_e(C_0C_0^T)^{-1}[\eta_0, 0]^T\). Then (2.2) and (2.3) both hold. This proves the lemma.

Remarks 2.1: 1. In case \(x(t_0)\) is fixed but arbitrary i.e. \(D_0 = I\), while \(\eta_0\) and \(\eta_f\) remain, the condition of the lemma is equivalent to the positive definiteness of \(E_eW_e\), which is used in [4]. Whatever values \(D_0\), etc., take, positive definiteness of \(C_e\) certainly causes satisfaction of (2.5).

2. The question of whether or not \(V^\theta[\eta_0, \eta_f] > -\infty\) is entirely different from whether or not \(V^\theta[\eta_0, \eta_f] < -\infty\), and will be discussed in later sections. (It is a good deal harder to answer).

To establish the quadratic nature of \(V^\theta[\eta_0, \eta_f]\), we shall use the following characterization of quadratic functions. A similar characterization has been used elsewhere in studying linear-quadratic problems [4, 9].

Lemma II.2.2. Let \(K(\cdot)\) be a scalar function of an \(n\)-vector. Then \(K(\cdot)\) is quadratic, i.e. \(K(x) = x^T P x\) for some symmetric \(P\) if and only if for all scalar \(\lambda\) and \(n\)-vector \(x_1, x_2\)

\[
K(\lambda x_1) = \lambda^2 K(x_1)
\]

\[
K(x_1 + x_2) + K(x_1 - x_2) = 2K(x_1) + 2K(x_2)
\]

\[
K(x_1 + \lambda x_2) - K(x_1 - \lambda x_2) = \lambda K(x_1 + x_2) - \lambda K(x_1 - x_2).
\]

Proof: The "only if" part of the result is easily checked. For the "if" part, we proceed as follows. Below, it is shown that (2.7) imply

\[
K(x_1 + x_2) - K(x_1 - x_2) + K(x_1 + x_3) - K(x_1 - x_3)
\]

\[
= K(x_1 + x_2 + x_3) - K(x_1 - x_2 - x_3).
\]

(2.8)

Assuming this for the moment, set

\[
L(x, y) = K(x+y) - K(x-y).
\]

Then (2.8) shows that \(L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)\), the first equation in (2.7) yields that \(L(x, y) = L(y, x)\) since \(K(x-y) = K(y-x)\), and the third equation of (2.7) yields that \(L(x, y) = \lambda L(x, y)\). Therefore \(L\) is bilinear in \(x\) and \(y\). Consequently,
L(x, x) is quadratic and noting that \( L(x, x) = K(2x) - K(0) = 4K(x) \) [using the first equation of (2.7)], we also have \( K(x) \) quadratic.

We now verify (2.8). We have, from the second equation of (2.7),

\[
K(x_1 + x_2) + K(x_1 + x_3) = l_2 [K(x_2 - x_3) + K(x_1 + x_2 + x_3)]
\]

\[
K(x_1 - x_2) + K(x_1 - x_3) = l_2 [K(x_1 - x_2 - x_3) + K(x_2 - x_3)]
\]

whence

\[
K(x_1 + x_2) - K(x_1 - x_2) + K(x_1 + x_3) - K(x_1 - x_3) + K(x_1 - x_2 - x_3) - K(x_1 x_3 + x_3)
\]

\[
= l_2 [K(2x_1 + x_2 + x_3) - 2K(x_1 + x_2 + x_3)] - l_2 [K(2x_1 - x_2 - x_3) - 2K(x_1 - x_2 - x_3)].
\]

The right hand side of this equation is identical to

\[
l_2 [K(x_1 + x_3 + 2x_1) - K(x_1 + x_2 - 2x_1)]
\]

and so, using the third equation of (2.7) with \( \lambda = 2 \), it becomes zero. Thus, (2.8) follows.

The quadratic nature of \( V^* [\eta_0, \eta_f] \) can now be established.

**Theorem 11.2.1.** Suppose \( V^* [\eta_0, \eta_f] \) exists for all \( \eta_0 \) and \( \eta_f \). Then \( V^* [\eta_0, \eta_f] \) has the representation

\[
V^* [\eta_0, \eta_f] = [\eta_0, \eta_f] \begin{bmatrix} P_{\eta_0} & P_{ef} \\ P_{ef} & P_{ef} \end{bmatrix} [\eta_0, \eta_f]
\]

for some matrices \( P_{\eta_0}, P_{ef}, P_{ef} \).

**Proof:** As a consequence of the above lemma, the desired result follows if \( V^* [\eta_0, \eta_f] \) satisfies the three equalities

\[
V^* [\eta_0, \eta_f] = \lambda^2 V^* [\eta_0, \eta_f]
\]

\[
V^* [\eta_0 + \eta_0, \eta_{ef} + \eta_{ef}] + V^* [\eta_0 - \eta_0, \eta_{ef} - \eta_{ef}]
\]

\[
= 2V^* [\eta_0, \eta_{ef}] + 2V^* [\eta_0, \eta_{ef}]
\]

and

\[
V^* [\eta_0 + \lambda \eta_0, \eta_{ef} + \lambda \eta_{ef}] - V^* [\eta_0 - \lambda \eta_0, \eta_{ef} - \lambda \eta_{ef}]
\]

\[
= \lambda V^* [\eta_0, \eta_{ef}] - \lambda V^* [\eta_0, \eta_{ef}].
\]

(2.10)
The method of proof is very similar for each case and proceeds by contradiction; it relies on the quadratic nature of $\mathcal{V}(x(t_0), t_0, u(\cdot))$ in (2.4), which implies versions of (2.10) through (2.12) for $\mathcal{V}$ as opposed to $\mathcal{V}^\star$. We shall prove merely (2.12), this being slightly harder than (2.10) and (2.11).

The third equality is trivial for $\lambda = 1$ and zero, and if true for $\lambda > 0$ is easily extended to $\lambda < 0$. So assume $\lambda > 0$, $\lambda = 1$.

Suppose that $u_2(\cdot)$, $i = 1, 2$ are any controls ensuring satisfaction of the constraints (2.1) through (2.3) for $\eta_0$, $\eta_f$ replaced by $\eta_{1f}$, $\eta_{f1}$ for $i = 1, 2$ respectively. By linearity of (2.1) through (2.3) $u_1 + u_2$ will ensure satisfaction with $\eta_0$, $\eta_f$ replaced by $\eta_{01} + \eta_{02}$, $\eta_{1f} + \eta_{2f}$. From the quadratic nature of the performance index (2.4), it is easily checked that

$$
\mathcal{V}[x_{11} - \lambda x_{21}, t_0, u_1 + \lambda u_2] + \lambda \mathcal{V}[x_{01} - x_{02}, t_1, u_1 - u_2]
\leq \lambda \mathcal{V}[x_{01} - x_{02}, t_0, u_1 + u_2] + \mathcal{V}[x_{01} - \lambda x_{21}, t_0, u_1 - \lambda u_2].
$$

(2.13)

For arbitrary $\varepsilon > 0$, choose $u_3(\cdot)$ and $u_4(\cdot)$ so that

$$
\mathcal{V}[x_{01} + \lambda x_{02}, t_0, u_1] \leq \mathcal{V}^\star[\eta_{01} + \eta_{02}, \eta_{1f}^- \eta_{f2}^-] + \varepsilon \\
\mathcal{V}[x_{01} - \lambda x_{21}, t_0, u_1] \leq \mathcal{V}^\star[\eta_{01} - \lambda \eta_{02}, \eta_{1f}^- \eta_{f2}^-] + \varepsilon.
$$

(2.14)

Also, $u_3$ is to cause satisfaction of (2.1) through (2.3) with $\eta_0$, $\eta_f$ replaced by $\eta_{01} + \eta_{02}$, $\eta_{1f} + \eta_{2f}$ while $u_4$ is to cause satisfaction with replacement by $\eta_{01} - \lambda \eta_{02}$, $\eta_{1f} - \lambda \eta_{f2}$. Define $u_1(\cdot)$, $u_2(\cdot)$ by

$$
u_1 + \lambda u_2 = u_3 \\
u_1 - \lambda u_2 = u_4.
$$

Then (by linearity) $u_1 + \lambda u_2$ and $u_1 - u_2$ cause satisfaction of (2.1) through (2.3) for pairs $\eta_{01} + \lambda \eta_{02}$, $\eta_{1f} + \lambda \eta_{f2}$ and $\eta_{01} - \lambda \eta_{02}$, $\eta_{1f} - \lambda \eta_{f2}$ respectively. Equations (2.13) and (2.14) now yield

$$
\mathcal{V}^\star[\eta_{01} + \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}] + \lambda \mathcal{V}^\star[\eta_{01} - \lambda \eta_{02}, \eta_{1f} - \lambda \eta_{f2}]
\leq \mathcal{V}[x_{01} + \lambda x_{02}, t_0, u_1 + \lambda u_2] + \lambda \mathcal{V}[x_{01} - x_{02}, t_0, u_1 - u_2]
\leq \lambda \mathcal{V}^\star[\eta_{01} + \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}] + \mathcal{V}[\eta_{01} - \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}] + (1 + \lambda)\varepsilon.
$$

A similar argument yields an inequality going the other way and since $\varepsilon$ is arbitrary, one obtains

$$
\mathcal{V}^\star[\eta_{01} + \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}] + \lambda \mathcal{V}^\star[\eta_{01} - \lambda \eta_{02}, \eta_{1f} - \lambda \eta_{f2}]
= \lambda \mathcal{V}^\star[\eta_{01} + \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}] + \mathcal{V}[\eta_{01} - \lambda \eta_{02}, \eta_{1f} + \lambda \eta_{f2}].
$$
as required.

Remarks 2.2: In case $D_0 = I$ and $E_0$, $D_e$ and $E_e$ evanesce (the usual optimal control problem), a version of the above result is proved in [4] with an additional and substantial controllability constraint, over and above that of Lemma II.2.1 (which in this case evanesces). The first removal of this controllability constraint appears to be in [6].

3. INITIAL CONDITION RESULTS AND THE RIEMANN-STIELTJES INEQUALITY

In this section, we study the system

$$\dot{x} = F(t)x + G(t)u \quad t_0 \leq t \leq t_f$$

(3.1)

with $x(t_0) = x_0$ prescribed, $x(\cdot)$ of dimension $n$ and $u(\cdot)$ of dimension $m$. Associated with (3.1) is the performance index

$$V[x_0, t_0, u(\cdot)] = \int_{t_0}^{t_f} [x^TQ(t)x + 2u^TR(t)x + u^TR(t)u]dt + x^T(t_f)Sx(t_f).$$

(3.2)

The matrices $F(\cdot), G(\cdot), H(\cdot), Q(\cdot)$ and $R(\cdot)$ all have dimensions consistent with (3.1) and (3.2) and are piecewise continuous. (This has the obvious meaning; the matrices have piecewise continuous entries). In the next chapter, stronger smoothness conditions will be imposed. The matrix $S$ is constant. Without loss of generality, we assume $Q(\cdot), R(\cdot)$ and $S$ are symmetric. The controls $u(\cdot)$ are assumed to be piecewise continuous on $[t_0, t_f]$. We specialise the ideas of the last section, by assuming $x_f$ is free; with $x_0$ arbitrary fixed, this is a standard optimal control problem. It is immediate that $V$, and thus $V^*$, cannot be $\infty$. Here, we shall be interested in the question of when (2.2) possesses an infimum, denoted $V^*[x_0, t_0]$, which is not $\infty$.

The problem of determining when (3.2) has a finite infimum subject to the constraint (3.1) is called nonsingular, partially singular or totally singular according as $R(t) > 0$ on $[t_0, t_f], R(t)$ is singular at one point on $[t_0, t_f]$ but not identically zero, or $R(t) \equiv 0$ on $[t_0, t_f]$. Of course, a well-known necessary condition for a finite infimum in $R(\cdot) \geq 0$ on $[t_0, t_f]$, this being the classical Legendre-Clebsch necessary condition [10].

The nonsingular problem is much easier than the singular problem, which will take most of our attention. However, the earlier results of this section apply equally to nonsingular and singular problems.

We remark also that in this book we study various necessity and sufficiency conditions for $V^*[x_0, t_0]$ to be finite in reverse order to their historical development. In particular, in this chapter we are interested in general existence conditions for finiteness of $V^*[x_0, t_0]$ whereas in Chapter III, we turn our attention to a more
classical approach which, on the one hand, is restrictive in that not all problems can be covered, while on the other hand, lends itself to the derivation of computational algorithms for computing optimal controls and performance indices.

As noted above, we shall assume throughout this section that \( x(t_f) \) is free, taking up the possibility of constraining the value of \( x(t_f) \) in later sections. However, the reader should be aware that all the results of this section carry over to the constrained end-point problem with minor changes, and these minor changes do not involve the behaviour of quantities in the vicinity of \( t_0 \). Since the results of this section are almost all concerned with behaviour in the vicinity of \( t_0 \), these minor changes are also conceptually insignificant.

We now define further notation. With \( F(\cdot) \) an \( n \times n \) symmetric matrix with entries of bounded variation, set

\[
\operatorname{dM}(P) = \begin{bmatrix}
  \int t_2^t \Phi(t, t) G(t) G(t) \Phi(t, t) \, dt \\
  \int t_2^t (G(t) H) \, dt \\
  \int t_2^t dt
\end{bmatrix}
\]

The subsequent material will make heavy use of the following inequality:

\[
\int t_2^t [\dot{v}(t) \quad \dot{u}(t)] \, \operatorname{dM}(P) [\begin{bmatrix}
  v(t) \\
  u(t)
\end{bmatrix}] \geq 0
\]

for any piecewise continuous \( m \)-vector \( \dot{u}(\cdot) \) and continuous \( n \)-vector \( \dot{v}(\cdot) \) defined on \([t_1, t_2]\). The integral is defined in the usual Riemann-Stieltjes sense. We shall say that \( \operatorname{dM}(P) \geq 0 \) or \( \operatorname{dM}(P) \) is nonnegative within any interval \( I \), closed or open at either end, if the inequality holds on \([t_1, t_2]\) for all \([t_1, t_2] \subset I \).

With these preliminaries, we can indicate the principal known connections between the existence of finite infima for the minimization problem, and the existence of matrices \( P \) satisfying \( \operatorname{dM}(P) \geq 0 \) on certain intervals and satisfying a terminal condition.

A convenient reference for the first two theorems is [4]. However, Theorem II.3.1 does extend the main result of [4] in a manner we explain subsequently.

**Theorem II.3.1:** Suppose that \( V[0, t_f, u(\cdot)] \geq 0 \) for all \( u(\cdot) \) and that

\[
\int t_2^t \Phi(t, t) G(t) G(t) \Phi(t, t) \, dt > 0 \quad \forall \ t_1 \in (t_0, t_f)
\]

so that all states are reachable from \( x(t_0) = 0 \) at any time \( t_1 > t_0 \). Here, \( \Phi(\cdot, \cdot) \) is the transition matrix associated with \( F(\cdot) \). Then

\[
\Rightarrow V[0, x_1, t_f] \Rightarrow \forall \ x(t_1) = x_1, \quad \forall \ t_1 \in (t_0, t_f)
\]

and there exists a symmetric \( F(\cdot) \) of bounded variation defined on \((t_0, t_f]\) such that \( F(t_f) \geq 0 \) and \( \operatorname{dM}(P) \geq 0 \) within \((t_0, t_f]\).

The proof of this theorem will proceed by a series of lemmas. Broadly speaking,
the strategy is as follows. We first demonstrate (3.6); then we appeal to the material of the last section to conclude a quadratic form for $V^*[x_1, t_1]$ as $x^TP^*(t_1)x_1$ for some $P^*(t_1)$; we show that $P^*(\cdot)$ is of bounded variation and satisfies a restricted form of the requirement $dM(P^*) \geq 0$, and then finally we remove the restriction.

**Lemma II.3.1:** With the hypotheses of Theorem II.3.1, $V^*[x_1, t_1]$ is finite for all $x(t_1) = x_1$ and $t_1 \in (t_0, t_f]$.

**Proof:** The inequality $\inf > V^*[x_1, t_1]$ for each $x_1 = x(t_1)$ and each $t_1 \in (t_0, t_f]$ is trivial. For the other inequality $V^*[x_1, t_1] > -\infty$, we have by the reachability condition (3.3) that there exists a piecewise continuous control $u(t)$ on $[t_0, t_1]$ taking $x = 0$ at time $t_2$ to $x = x_1$ at time $t_1$. Then letting $u(t)$ be any piecewise continuous control on $[t_1, t_f]$, we have by assumption that

$$\int_{t_0}^{t_1} q(x, u) dt + \int_{t_1}^{t_f} q(x, u) dt + x'(t_1)P(t) x(t_1) \geq 0,$$

where $q(x, u) = x'Qx + 2x'Ru + u'Ru$. Clearly this gives a lower bound on $V[x_1, t_1, u(\cdot)]$ and hence $V^*[x_1, t_1]$.

As a result of the above lemma and the main result of the previous section, we know that we can write

$$V^*[x_1, t_1] = x'^2 P^*(t_1) x_1$$

(3.7)

for some $P^*(t_1)$, all $x_1$ and all $t_1 \in (t_0, t_f]$. Next, we demonstrate an inequality for $P^*(\cdot)$ and the fact that it has bounded variation.

**Lemma II.3.2:** With the hypotheses of Theorem II.3.1, let $P^*(\cdot)$ be defined by (3.7). Then with $q(x, u) = x'Qx + 2x'Ru + u'Ru$,

$$\int_{t_1}^{t_2} q(x, u) dt + x'(t)P(t) x(t) |_{t=t_1}^{t=t_2} \geq 0$$

(3.8)

for all $[t_1, t_2] \subseteq (t_0, t_f]$ and all $x(t_1) = x_1$, and $P^*(\cdot)$ is of bounded variation on $[t_0, t_f]$.

**Proof:** In view of (3.7), (3.8) is nothing but the principle of optimality:

$$V^*[x_1, t_1] \leq \int_{t_1}^{t_2} q(x, u) dt + V^*[x_2, t_2].$$

Next, let $W(t)$ be the solution on $(t_0, t_f]$ of

$$\dot{W} + WP + P'W + Q = 0 \quad W(t_f) = S,$$

(3.9)

let $x(t_1)$ be arbitrary, and let $u(\cdot)$ be zero on $[t_1, t_2]$. It follows that

$$\int_{t_1}^{t_2} q(x, u) dt = -x'(t)W(t) x(t) |_{t=t_1}^{t=t_2}.$$
Noting that \( x(t_2) = \Phi(t_2, t_1)x(t_1) \), we conclude from (3.8) that
\[
x''(t_1) [\Phi'(t_2, t_1) P^*(t_2) - W(t_2)] \Phi(t_2, t_1)x(t_1)
\geq x''(t_1) [P^*(t_1) - W(t_1)]x(t_1),
\]
since this holds for all \( x(t_1) \), it follows easily that \( \Phi'(t, t_2) P^*(t) - W(t) \Phi(t, t_2) \)
is monotone non-decreasing on \( [t_2, t_1] \). The same matrix is accordingly of bounded variation; it is trivial then that \( P^*(\cdot) \) has this property.

To complete the proof of Theorem II.3.1, we need to show that \( P^*(t_2) \leq S \) and \( \text{dM}(P^*) \geq 0 \) within \( (t_0, t_1] \). That \( P^*(t_2) \leq S \) follows trivially. In the next lemma, we prove a result close to \( \text{dM}(P^*) \geq 0 \); the result is restricted in that the vector \( v(\cdot) \) of the \( \text{dM}(P^*) \geq 0 \) definition must be related to \( u(\cdot) \).

**Lemma II.3.3:** Let \( P(t) \) be any \( n \times n \) matrix symmetric and of bounded variation on \( [t_1, t_2] \), let \( u(\cdot) \) be any piecewise continuous control on \( [t_1, t_2] \) and let \( x(t) \) be the state at time \( t \) of (3.1). Then

\[
\int_{t_1}^{t_2} [x'(t) \ u'(t) \text{dM}(P^*) \ P(t) x(t)] \geq 0 \text{ (3.10)}
\]

and with the hypothesis of Theorem II.3.1 and the definition (3.7),
\[
\int_{t_1}^{t_2} [x'(t) \ u'(t) \text{dM}(P^*)] \geq 0 \text{ (3.11)}
\]

**Proof:** The proof of (3.10) relies on a standard "integration by parts" result and the symmetry of \( P(t) \). The continuity of \( x(\cdot) \), needed for this integration by parts, follows from the piecewise continuity of \( F(\cdot), G(\cdot) \) and \( u(\cdot) \). Equation (3.11) is an immediate consequence of (3.8) and (3.10).

The main result of (4) is as stated in the theorem, save that the condition \( \text{dM}(P^*) \geq 0 \) is replaced by the requirement that (3.11) hold for all \( x(t_1) \) and \( u(\cdot) \), with \( x(\cdot) \) defined by (3.1). The last lemma in the chain proving Theorem II.3.1 shows that this restriction is not necessary.

**Lemma II.3.4:** With notation as defined above, let \( P^*(\cdot) \) be such that (3.11) holds for all \( x(t_1) \) and \( u(\cdot) \). Then \( \text{dM}(P^*) \geq 0 \).

**Proof:** Suppose that \( \text{dM}(P^*) \geq 0 \) fails. Then for some \( \varepsilon_1 > 0 \), \( [t_0, t_1] \subset (t_2, t_3] \) and \( u(\cdot) \) and \( v(\cdot) \) with appropriate continuity properties, one has
\[
\int_{t_0}^{t_1} [v'(t) \ u'(t)] \text{dM}(P^*) u(t) < -\varepsilon_1.
\]
Our strategy to deduce a contradiction will be to show that there exists a partition
of \([t_a, t_b]\) into intervals \([t_0, t_2], [t_2, t_3], \ldots, [t_{N-1}, t_b]\) such that
\[
\int_{t_0}^{t_b} [v''(t) \ u'(t)] d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} v(t) \\ u(t) \end{bmatrix}
\]
can be approximated by
\[
\sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} [x_i'(t) \ u(t)] d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} x_i(t) \\ u(t) \end{bmatrix}
\]
where the \(x_i(\cdot)\) are all state trajectories of \(\dot{x} = Fx + Gu\). Since the approximating quantity must be nonnegative, a contradiction obtains.

The details of the argument follow.

Because \(u(\cdot)\) is piecewise continuous, \(\|\Phi(t, \tau)G(t)u(t)\|\) is bounded on \([t_a, t_b] \times [t_a, t_b]\), and accordingly, given arbitrary \(\varepsilon_2 > 0\), there exists a \(\delta_1\) such that
\[
\int_{t-\delta_1}^{t} \Phi(t, \tau)G(t)u(t) d\tau < \frac{\varepsilon_2}{2}
\]
for all \([t-\delta_1, t] \subset [t_a, t_b]\). Also, because \(v(\cdot)\) is continuous on \([t_a, t_b]\), there exists a \(\delta_2\) such that
\[
\sup_{\tau \in [t-\delta_2, t]} \|v(t) - \Phi(t, \tau)v(\tau)\| < \frac{\varepsilon_2}{2}
\]
for all \([t-\delta_2, t] \subset [t_a, t_b]\). Let \(\delta = \min(\delta_1, \delta_2)\). Choose a finite set of points \(A_0 = \{t_1 = t_a, t_2, \ldots, t_N = t_b\}\) with \(t_i < t_{i+1}\) such that \(t_{i+1} - t_i < \delta\) and such that \(F\) does not have a jump at \(t_i\). (The latter requirement can be fulfilled since \(\mathcal{P}^*(\cdot)\) as a function of bounded variation is differentiable almost everywhere). Now define
\[
x_i(t_i) = v(t_i)
\]
\[
x_i(t) = \Phi(t, t_i)x_i(t_i) + \int_{t_i}^{t} \Phi(t, \tau)G(t)u(t) d\tau
\]
for \(t_i < t < t_{i+1}\). Observe that on \([t_i, t_{i+1})\), \(x_i(\cdot)\) is a state trajectory corresponding to \(u(\cdot)\), and also \(\|x_i(\cdot) - v(\cdot)\| < \varepsilon_2\) by virtue of the definition of \(\delta\).

Now we have
\[
\int_{t_i}^{t_{i+1}} [v''(t) \ u'(t)] d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} v(t) \\ u(t) \end{bmatrix} = \int_{t_i}^{t_{i+1}} [x'_i(t) \ u'(t)] d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} x'_i(t) \\ u(t) \end{bmatrix} + 2 \int_{t_i}^{t_{i+1}} [v''(t) - x''_i(t)] 0 d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} x_i(t) \\ u(t) \end{bmatrix} + \int_{t_i}^{t_{i+1}} [v''(t) - x''_i(t)] 0 d\mathcal{M}(\mathcal{P}^*) \begin{bmatrix} x_i(t) \\ u(t) \end{bmatrix}.
\]
The magnitude of the second and third terms can be overbounded by a quantity involving the total variation of \( P^* \) on \([\tau_1, \tau_{1+1}]\) and \( \varepsilon_2 \); moreover, the bound approaches zero as \( \varepsilon_2 \) approaches zero. Collecting the \([\tau_1, \tau_{1+1}]\) intervals, we conclude that

\[
\int_{\tau_1}^{\tau_{1+1}} [v(t) - u(t)] dM(P^*) \left[ \frac{v(t)}{u(t)} \right] - \frac{1}{2} \int_{\tau_1}^{\tau_{1+1}} [x_1(t)] \left[ \frac{x_1(t)}{u(t)} \right]
\]

\[
\leq K \varepsilon_2
\]

for some \( K \) reflecting, inter alia, the total variation of \( P^* \) on \([t_0, t_f]\). Choosing \( \varepsilon_2 \) such that \( K \varepsilon_2 < \varepsilon_1 \), we obtain a contradiction to the facts that

\[
\int_{t_0}^{t_f} [v(t) - u(t)] dM(P^*) \left[ \frac{v(t)}{u(t)} \right] < -\varepsilon_1
\]

and

\[
\int_{\tau_1}^{\tau_{1+1}} [x_1(t)] \left[ \frac{x_1(t)}{u(t)} \right] \geq 0
\]

for all \([\tau_1, \tau_{1+1}] = (t_0, t_f]\).

The chain of reasoning proving Theorem 11.3.1 is now completed.

Remarks 3.1: 1. The condition \( V[0, t_0, u(\cdot)] \geq 0 \) is equivalent to \( V^*[0, t_0] = 0 \).

(If \( V[0, t_0, u(\cdot)] < 0 \) for some particular \( u(\cdot) \), scaling that \( u(\cdot) \) scales \( V \), and so \( V \) can be made arbitrary negative, i.e. \( V^*[0, t_0] = -\infty \). Hence \( V^*[0, t_0] = -\infty \) implies \( V[0, t_0, u(\cdot)] \geq 0 \) for all \( u(\cdot) \). Next, \( V[0, t_0, u(\cdot)] \geq 0 \) for all \( u(\cdot) \) and the observation \( V[0, t_0, u(\cdot)] = 0 \) shows that \( V^*[0, t_0] = 0 \).

2. For future reference, we summarize the above result in loose but intuitively helpful language:

Nonnegativity + controllability \( \implies V^* \) finite within 
\[ (t_0, t_f]. \] (3.12)

Nonnegativity + controllability \( \implies \exists F \) such that \( F(t_0) \leq S \) and 
\( dM(F) = 0 \) within 
\[ (t_0, t_f]. \] (3.13)

3. The proof of the second part of the theorem proceeds by exhibiting a particular \( F(\cdot) \), viz. \( F = F^* \) where \( V^*[x_1, t_1] = x_1 F^*(t_1) x_1 \), satisfying the listed constraints. The reader should be aware however that there are normally other matrices \( F \), different from \( F^* \), satisfying the constraints. This point will be explored later in the chapter.
4. It is possible to have a situation in which the nonnegativity and controllability conditions hold, and for which $V^*[x_0, t_f] = \infty$ for all nonzero $x_0$. This shows the futility of attempting to improve (3.12) to the extent of obtaining finiteness of $V^*[x_1, t_1]$ for all $x_1$ and for all $t_1 \in [t_0, t_f]$. The following example is to be found in [4]. The dynamics are $\dot{x} = u, t_0 = 0, t_f = \pi/2$ and $V[x_0, 0, u(\cdot)] = \int_{t_0}^{t_f} (-x^2 + u^2)dt$.

We first construct a sequence of piecewise continuous controls such that $V[x_0, 0, u_n(\cdot)] \to \infty$ for any nonzero $x_0$. Let $\epsilon_n$ be a monotone decreasing sequence with $\epsilon_n \to 1$ and $\epsilon_n \to 0$ as $n \to \infty$. Define $u_n(t) = 0$ on $[t_0, \epsilon_n)$, $x_0 (\cos \epsilon_n) (\cos t)$ on $[\epsilon_n, T]$. It is easy to verify that $V[x_0, 0, u_n(\cdot)] = -(\epsilon_n + \cot \epsilon_n)x_0^2$, which diverges to $\infty$ as $n \to \infty$.

We next show the nonnegativity of $V[0, 0, u(\cdot)]$. With $u(\cdot)$ piecewise continuous, $V^*(\cdot)$ is continuous on $[0, \pi/2]$. Define $\bar{V}(\cdot) = [0, \pi]$ by reflecting $V^*(\cdot)$ in $t = \pi/2$, i.e., $\bar{V}(t) = V(t)$ for $0 \leq t \leq \pi/2$, $x_0 (\pi - t)$ for $\pi/2 \leq \pi \leq \pi$. There is then a Fourier series expansion of $\bar{V}^*(\cdot)$ on $[0, \pi]$ with $\bar{V}(t) = \sum_{k=1}^{\infty} a_k \sin kt$. One then computes $V[0, 0, u(\cdot)] = \sum_{k=1}^{\infty} a^2_k (k^2 - 1)$, from which the nonnegativity is evident. Notice incidentally that all controls of the form $a \sin t$ lead to $V[0, 0, u(\cdot)] = 0$.

5. By taking $v(t) \equiv 0$ in the definition of $dM(P) \geq 0$, we obtain independent verification of the fact that $R(t) \geq 0$ is necessary for a finite $V^*$.

As noted earlier, nonnegativity of $\bar{V}$ has long been recognized as necessary for the existence of a finite optimal performance index.

6. The matrix $P^*$ need not be continuous. Consider for example dynamics $\dot{x} = u$, with $V[x_0, 0, u(\cdot)] = \int_{t}^{t_f} [k(t)x(t)u(t) dt + \frac{1}{2} x^2(t)]$, where $k(t) = 0$ on $[0, \frac{\pi}{4}]$, $k(t) = 1$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$. It is readily verified that $V[x(t), t, u(\cdot)] = \frac{1}{2} x^2(t)$ for $t \leq \frac{\pi}{4}$ and $V[x(t), t, u(\cdot)] = \frac{1}{2} x^2(t) + \frac{1}{2} x^2(\frac{\pi}{2})$ for $t \geq \frac{\pi}{4}$. This implies that $V[x(t), t] = 0$ for $t \leq \frac{\pi}{4}$ and $V[x(t), t] = \frac{1}{2} x^2(t)$ for $t \geq \frac{\pi}{4}$.

We can however establish that $P^*(\cdot)$ and indeed any $P(\cdot)$ satisfying the Riemann-Stieltjes inequality can have jumps in only one direction:

Lemma II.3.5: Let $P(\cdot)$ be a matrix, symmetric and of bounded variation on $(t_0, t_f)$ and satisfying the conditions $P(t) \neq S$ and $dM(P) \geq 0$ within $(t_0, t_f)$. Then all jumps in $P(\cdot)$ are nonnegative i.e.,

$$\lim_{t^+} P(t) \leq P(t)$$
$$\lim_{t^-} P(t) \geq P(t)$$

for $t \in (t_0, t_f)$.

Proof: Let $t \in (t_0, t_f)$ be a point of discontinuity of $P(\cdot)$. Let $v(\cdot)$ be an arbitrary constant and $u(\cdot)$ zero in $[t-\delta, t) \subset (t_0, t_f)$. Using the fact that $dM(P) \geq 0$ on $(t_0, t_f)$ and therefore $[t-\delta, t)$, and letting $\delta \to 0$ yields $V(\cdot)P(t) - \lim_{t^-} P(t)v(t) \geq 0$. Since $v(t)$ is arbitrary, the first result follows.
The second is proved the same way.

We shall later use without explicit comment trivial variations obtained by closing the interval at \( t_0 \) or opening it at \( t_f \).

The proof of a result similar to Theorem 3.1 appears in [3] as an extension of the totally singular case studied in [7]. The approach in [7] is to regularize the singular problem, replacing it with a nonsingular one obtained by adding the positive quantity \( \varepsilon \int_{t_0}^{t_f} u u' \, dt \) to the cost in the singular problem and allowing \( \varepsilon \) to approach zero. The nonsingular problem is of course much easier to solve, but one naturally has to prove things concerning the limit as \( \varepsilon \to 0 \). (The same idea is also used in [3]). A cleaner derivation, bypassing the need to obtain conditions for the totally singular case prior to the partially singular case, is to be found in [11, and 12] (modulo minor changes such as time reversal); an important feature of the proof is the use of the Helly convergence theorem for sequences of functions of bounded variation.

Now we turn to the second main result of this section. As a partial converse to Theorem 3.1, we have the following:

**Theorem 11.3.2:** Suppose that there exists a symmetric \( P(\cdot) \) of bounded variation defined on \([t_0, t_f]\), with \( P(t_f) \leq S \) and with \( dM(F) \geq 0 \) within \([t_0, t_f]\).

Then \( V[0, t_0, u(\cdot)] \geq 0 \) for every \( u(\cdot) \).

**Proof:** By the result of Lemma 3.4 we can write

\[
\int_{t_0}^{t_f} [x'(t) u'(t)] dM(F) = x'(t_f)P(t_f) x(t_f) \bigg|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} q(x, u) \, dt
\]

for each \([t_1, t_f] \subseteq [t_0, t_f]\). By assumption \( dM(F) \geq 0 \) and so for the interval \([t_0, t_f]\), we obtain

\[
x'(t_0) P(t_0) x(t_0) \leq x'(t_f) P(t_f) x(t_f) + \int_{t_0}^{t_f} q(x, u) \, dt
\]

for each \( x(t_0) \) and each \( u(\cdot) \). In particular, for \( x(t_0) = 0 \), and noting \( P(t_f) \leq S \), we obtain \( V[0, t_0, u(\cdot)] \geq 0 \) for each \( u(\cdot) \).

**Remarks 3.2:**

1. Summarizing the result in loose language, we have

\[
P(t_f) \leq S
\]

and \( dM(F) \geq 0 \) \( \implies \) nonnegativity condition \( (3.14) \)

within \([t_0, t_f]\).

2. Statements (3.12) through (3.14) highlight the extent to which Theorems 3.1 and 3.2 fail to be complete converses. There are basically two aspects of this failure, one residing in the need for controllability in Theorem 3.1 and its absence in Theorem 3.2, the other resulting from the fact that a left-closed interval condition is required to guarantee nonnegativity, which only implies a left-open
interval condition.

3. In fact the hypotheses of Theorem 3.2 imply the slightly stronger result that $V^*[x_0, t]$ is finite for all $x_0$. This follows easily on working with the inequality contained in the proof of the theorem.

In order to get tidier results (with tidiness measured by the occurrence of conditions which are both necessary and sufficient, not one or the other), we suggest a change of viewpoint, based on the observations of Section 1 concerning robust and non-robust problems. Thus we should be interested in not merely conditions for $V^*[0, t]$ to be finite, but in conditions for $V^*[0, t]$ to be finite for $t'$ in a neighborhood of $t_0$ (robustness as far as initial time is concerned), and in conditions for $V^*[x_0, t]$ to be finite for $x_0$ in a neighborhood of 0. (Because of the linear-quadratic nature of the problem, this means that $V^*[x_0, t]$ is finite for all $x_0$).

One step in this direction is provided by Theorem 3.3 below, which connects finiteness of $V^*[x_0, t]$ for all $x_0$ with the existence of a matrix $P$ satisfying certain conditions.

**Theorem 3.3**: $V^*[x_0, t] > -a$ for all $x_0$ if and only if there exists on $[t_0, t_f]$ a symmetric $P(\cdot)$ of bounded variation such that $P(t_f) \geq 0$ and $\text{dm}(P) \geq 0$ within $[t_0, t_f]$.

**Proof**: Since $V[0, t, u(\cdot)] \geq 0$ for all $u(\cdot)$ if and only if $V^*[0, t_0] = 0$, we know from Theorem 3.1 that $V^*[x_1, t_1] \geq \omega$ for all $x_1$ and $t_1 \in [t_0, t_f]$. Since also $V^*[x_0, t_0] \geq \omega$ for all $x_0$, we have $V^*[x_1, t_1] = x_1P'(t_1)x_1$ for some symmetric $P'(t_1)$, all $x_1$ and all $t_1 \in [t_0, t_f]$. Trivial variation on the lemma used in proving Theorem 3.1 yields the necessity claim of the theorem. Sufficiency is a simple consequence of the proof of Theorem 3.2, as noted in Remark 2.3.3.

The first necessity and sufficiency result on the existence of $V^*[x_0, t_0]$ appears to be that of [6]. It is the same as that of Theorem 3.3, save that the condition $\text{dm}(P) \geq 0$ is replaced by the restricted condition $\text{(3.11)}$.

**Remarks 3.3**: 1. It is an immediate consequence of this result that if $V^*[x_0, t_0] > -a$ for all $x_0$, then $V^*[x_1, t_1] \geq \omega$ for all $x_1$ and all $t_1 \in [t_0, t_f]$. (Simply use the fact that if $\text{dm}(P) \geq 0$ on all closed intervals contained in $[t_0, t_f]$, then $\text{dm}(P) \geq 0$ on all closed intervals contained in $[t_1, t_f]$).

2. We describe this result as extended by Remark 1, as

$$V^* \text{ finite on } [t_0, t_f] \iff P(t_f) \leq S$$

and

$$\text{dm}(P) \geq 0 \text{ within } [t_0, t_f].$$

$$\iff V^* \text{ finite within } [t_0, t_f]. \tag{3.15}$$

3. A comparison of Theorems 3.1 and 3.3 shows that the interval within which $\text{dm}(P) \geq 0$ is open or closed at $t_0$ according as we restrict the values of $x(t_0)$ of interest, or $x(t_0)$ is free. It will be seen subsequently that the interval
is open or closed at $t_f$ according as $x(t_f)$ is or is not restricted.

4. The example of Remarks 3.1 shows, in conjunction with Theorem 3.3, that it is possible to have a symmetric $P$ of bounded variation, satisfying $P(t_f) \leq S$ and with $dM(P) \geq 0$ within $[t_0, t_f]$ but not within $[t_0, t'_f]$.

We now state two corollaries to Theorem 3.3 which show that the necessary and sufficient conditions of that theorem are indeed just generalizations of better known, but less general, conditions for the existence of a solution to the control problem. For the first of these corollaries, which is virtually self-evident, we assume that the $P$ matrix is differentiable on some $[t_1, t_2] \subseteq [t_0, t_f]$ and obtain a linear matrix differential inequality. For the second we assume that the problem is nonsingular on the interval $[t_0, t_f]$; it is then possible to show that the necessary and sufficient conditions of Theorem 3.3 are equivalent to the well-known condition that $P^*(t)$ satisfy the Riccati differential equation on $[t_0, t_f]$. The interested reader can consult [4], [3] and [10] for these corollaries and other closely related results.

**Corollary II.3.1:** Let $P(t)$ be a matrix, symmetric and of bounded variation on $[t_0, t_f]$ with $dM(P) \geq 0$ within $[t_0, t_f]$. Further suppose that $P(t)$ is differentiable in a neighborhood of $t$. Then

$$
\begin{bmatrix}
P + PP + PP + Q & PG + H \\
(PG + H)' & R
\end{bmatrix} \geq 0
$$

in a neighborhood of $t$. Conversely, satisfaction of this inequality in a neighborhood of $t$ implies that $dM(P) \geq 0$ within this neighborhood.

**Corollary II.3.2:** Assume that $R(t) > 0$ on $[t_0, t_f]$. If $V^*[x_0, t_0]$ is finite for each $x_0$, then the matrix $P^*(t)$ defined by $V^*[x(t), t] = x^*(t)P^*(t)x(t)$ on $[t_0, t_f]$ is continuously differentiable and satisfies

$$
P^* + PP + PP + Q - (P^*G + H)R^{-1}(P^*G + H)' = 0, \quad P^*(t_f) = S
$$

on $[t_0, t_f]$. Conversely, if the solution of this Riccati equation has no escape time on $[t_0, t_f]$, $V^*[x_0, t_0]$ is finite for each $x_0$ and is given by $x_0P^*(t_0)x_0$.

A variant of Corollary II.3.1 has found extensive use in problems of time-varying network synthesis and covariance factorization, see [11, 12]. This completes our discussion of the idea of "robustness with respect to initial state". We turn now to a consideration of "robustness with respect to initial time", with the goal of connecting the notions of initial state and time robustness. Remark 3.3.1 considered a change of initial time from $t_0$ to some $t \in (t_0, t'_0]$. We need now to consider the possibility of taking an initial time $t_{-1} < t_0$. The next theorem provides the main result.

**Theorem II.3.4:** Suppose that $V^*[x_0, t_0] > -\infty$ for all $x_0$. Then there exists
\[ t_1 < t_0 \] and definitions on \([t_1, t_0)\) of \(F(\cdot), G(\cdot), H(\cdot), Q(\cdot)\) and \(R(\cdot)\) such that these quantities are continuous on \([t_1, t_0]\), such that

\[
\int_{t_1}^{t_0} \phi(t_1, t_0)G(t_1)G(t_1)\theta(t_1, t_0)dt > 0 \quad \forall t_1 \in [t_1, t_0],
\]

(3.16)

such that \(\nu(0, t_1, u(\cdot)) \geq 0\) for all \(u(\cdot)\), and in fact \(\nu^*\left[x_{-1}, t_{-1}\right] > \infty\) for all \(x(t_{-1}) = x_{-1}\).

**Proof**: We consider the proof of the theorem first for a special situation; then we show that the general situation can always be reduced to the special situation.

Let \( P(\cdot) \) be the matrix whose existence on \([t_0, t_2]\) is guaranteed by Theorem 3.3, and suppose for the moment that \( P(\cdot) \) exists in a neighborhood \([t_0, t_0 + \varepsilon]\) of \(t_0\). Take \( t_1 < t_0 \) and otherwise arbitrary, and take \( F(\cdot), G(\cdot) \) on \([t_1, 1/2(t_0 + t_2)]\) to be any constant, completely controllable pair; define \( F(\cdot), G(\cdot) \) on \([t_1, t_0)\) so as to ensure smooth joins, and continuity on \([t_1, t_0]\).

Let \( F(t) = F(t_0) \) on \([t_1, t_0)\); this ensures that \( F(\cdot) \) is continuous on \([t_1, t_0]\). Choose \( Q(\cdot) \) on \([t_1, t_0)\) such that \( F + FF + FP + Q \) is constant on \([t_1, t_0]\); this ensures \( Q(\cdot) \) is continuous on \([t_1, t_0]\). Choose \( H(\cdot) \) on \([t_1, t_0)\) so that \( F + H \) is constant on \([t_1, t_0]\); again, \( H(\cdot) \) is continuous on \([t_1, t_0]\).

Finally, choose \( R(\cdot) \) on \([t_1, t_0)\) to be constant and equal to \( R(t_0) \), ensuring thereby continuity on \([t_1, t_0]\).

These choices guarantee that

\[
\begin{bmatrix}
\dot{H} = \\
\dot{P} + PP + FP + Q & PG + H \\
(PG + H)^{-1} & R
\end{bmatrix}
\]

is constant on \([t_1, t_0]\). The fact that \( \dot{H}(P) \geq 0 \) on \([t_0, t_1]\) for all \( t_1 \in [t_0, t_0 + \varepsilon] \) and that \( \dot{P} \) exists in a neighborhood of \( t_0 \) ensures that \( \dot{H}(P) \geq 0 \) -- see Corollary 3.1. Consequently \( \dot{H} \geq 0 \) on \([t_0, t_1]\) and then \( \dot{H}(P) \geq 0 \) within \([t_1, t_0]\). By Theorem 3.3, \( \nu(t, t_1, x(\cdot)) \geq \infty \) for all \( t < t_0 \), while (3.16) holds because of the choice of \( P(\cdot), G(\cdot) \) on \([t_1, t_0 + \varepsilon]\).

Now suppose that \( \dot{P} \) does not exist in a neighborhood of \([t_0, t_0 + \varepsilon]\) of \( t_0 \). Consider the following equation for \( t < t_0 \):

\[
\dot{P} + FP + FP + Q = (PG + H)[R_0 + (t_0 - t)^{\frac{1}{2}}]^{-1}(PG + H) = 0
\]

(3.17)

where \( R_0 = R(t_0) \), and \( P(\cdot), G(\cdot), H(\cdot) \) and \( Q(\cdot) \) are arbitrary continuous extensions of these quantities into \( t < t_0 \), chosen to ensure continuity at \( t_0 \). Equation (3.17) is initialized by the known quantity \( \dot{H} = P(t_0) \).

In case \( R_0 \) is nonsingular, \( P(t) \) is guaranteed to exist in some interval \([t_2, t_3)\), with \( \dot{P} \) guaranteed to exist in \([t_2, t_3)\). However, in case \( R_0 \) is singular, \( \dot{P} \) is unbounded as \( t \to t_0 \), and so an existence question arises, which we now resolve. Set \( \tau = (t_3 - t)^{\frac{1}{2}} \). Then
and with \( \tau \) the new independent variable, (3.17) becomes

\[
- \frac{dp}{dt} + 2T(PP^T + Q) - (P + H)2T(R_0 + \tau I)^{-1}(P + H)' = 0. \tag{3.18}
\]

This equation is defined in the interval \( \tau \geq 0 \); strictly, we should have used different symbols for \( P(\tau) \), etc., to reflect their change of independent variable. The equation has \( P(\tau)|_{\tau=0} = P_0 \). Now \( (R_0 + \tau I)^{-1} \) is obviously continuous for \( \tau > 0 \), and it is not hard to check that it is continuous at \( \tau = 0 \). Therefore, \( P(\tau) \) exists in some interval \([0, t_2]\) with \( \frac{dp}{dt} \) continuous there. It follows that (3.17) has a solution in some interval \([t_{-2}, t_0]\) with \( \frac{dp}{dt} \) existing on \([t_{-2}, t_0]\) and, in fact, in a neighborhood of \( t_{-2} \).

Now (3.17) implies that

\[
\begin{bmatrix}
\tau + PP^T + Q & P + H \\
(P + H)' & R
\end{bmatrix} 
\geq 0
\]

on \([t_{-2}, t_0]\), where \( R = R_0 + (t_2 - t_2)^{1/2} \) and is nonsingular. Since \( \lim_{t \to t_0} P(t) = P_0 \), it is clear that \( \text{det}(P) \geq 0 \) within \([t_{-2}, t_0]\). Now we can use the first part of the proof to further extend on \([t_{-1}, t_{-2}]\), since \( P(t) \) exists in a neighborhood of \( t_{-2} \). In this way, the controllability assumption is fulfilled, and the theorem is proved.

Remarks 3.4: 1. This theorem is the first in the book to introduce an extendability criterion. The first use of the extendability idea of which we are aware is in [1], where nonsingular problems only were discussed.

2. In case \( R(t) \) is nonsingular throughout \([t_0, t_2]\), the above theorem is much easier to prove, for \( V^* [x_1, t_1] = x_1P^*(t_1)x_1 \) with \( P(t) \) the solution of a Riccati equation, \( t_1 \in [t_0, t_2] \). Then \( P(t) \) automatically exists for all \( t_2 \in [t_0, t_2] \).

3. We summarise the result as:

\[ V^* \text{ finite on } [t_0, t_2] \implies \text{ nonnegativity and controllability for the extended interval } [t_{-1}, t_2] \tag{3.19} \]

and

\[ V^* \text{ finite on } [t_0, t_2] \implies V^* \text{ finite on extended interval } [t_{-1}, t_2]. \tag{3.20} \]

4. An examination of the proof of Theorem 3.4 will show that \( t_{-1} \) may be taken arbitrarily close to \( t_0 \). This fact essentially makes (3.19) and (3.12) con-
verse statements; the converse to (3.13) is obtained by replacing $V^*$ finite on $[t_1, t_2]$ in (3.19) by the equivalent statement involving $F$ contained in (3.15).

5. The result contained in the preceding theorem might lead one to make the following conjecture, which we can readily show is false: suppose that $V^*[x_0, t_0] > \infty$ for all $x_0$, and that $F^*(\cdot), G^*(\cdot), H^*(\cdot), Q^*(\cdot), R^*(\cdot)$ are defined on $[t_{-1}, t_1]$ such that these quantities are continuous on $[t_{-1}, t_1]$; then there exists $t_{-2} \in (t_{-1}, t_1)$ such that $V^*[x_{-2}, t_{-2}] > \infty$ (effectively, it is being claimed that the set of $t_0$ for which $V^*[x_0, t_0] > \infty$ is open). By way of counterexample, consider $x = u, V[x(t), t, u(\cdot)] = \int_{t_0}^{t} [x u + p(t) u^2] dt$ where $p(t) = 0, t \in [t_0, t_1]$ and $p(t) < 0$ for $t < [t_{-1}, t_1]$, with $p(\cdot)$ continuous on $[t_{-1}, t_1]$. Certainly then, it is impossible that $V^*[x_{-2}, t_{-2}] > \infty$ for $t_0 \in [t_{-1}, t_1]$, since $p$ is negative on $(t_{-2}, t_0)$, $V[x(t_0), t_0, u(\cdot)] = \frac{1}{2} x^2(t_0) - \frac{1}{2} x^2(t_1)$ for all $u(\cdot)$, so that $V^*[x_0, t_0] = -\frac{1}{2} x_0^2$.

A second example, in which negative $p(t)$ is not used is provided by $x = g(t) u, V[x(t), t, u(\cdot)] = \int_{t_0}^{t} [x u + g(t) u^2] dt$ where $g(t) = 0$ for $t < 0$ and $g(t) = 1$ for $t > 0$.

It is unclear whether such examples can be constructed in case $R(t) \geq 0$ and $F, G, H, Q, R$ are all continuous.

Under the restriction $R(t) > 0$ on $[t_0, t_1]$ and a continuity requirement on $R(\cdot)$, the above conjecture is definitely true, for $V^*[x(t), t]$ is defined via the solution to a Riccati equation which, if it exists at $t_0$, exists in a neighborhood around $t_0$, including points to the left of $t_0$.

In this section we have so far separately considered robustness with respect to the initial state (Theorem 3.3) and robustness with respect to the initial time (Theorem 3.4). It is now convenient to summarize these theorems together with Theorem 3.1 as

**Theorem 3.5:** With notation as previously, the following conditions are equivalent:

(a) $V^*[x_0, t_0]$ is finite for all $x_0$.

(b) $V^*[x(t), t]$ is finite for all $x(t)$ and for all $t \in [t_0, t_1]$.

(c) There exist extensions of the interval of definition of $F^*(\cdot)$, etc., such that $V[0, t_{-1}, u(\cdot)] \geq 0$, for some $t_{-1} < t_0$, and with controllability on $[t_{-1}, t]$ for all $t \in (t_{-1}, t_1]$.

(d) There exists on $[t_0, t_1]$ a symmetric $F(\cdot)$ of bounded variation with $F(t_1) \leq S$ and $dM(F) \leq 0$ within $[t_0, t_1]$.

We emphasize the fact that the conditions involving the Riemann-Stieltjes integral are simultaneously necessary and sufficient.

We conclude this section with remarks of minor significance on another type of perturbation. To this point, we have considered the effect of perturbations of the initial state away from zero, and perturbations of the initial time $t_0$. Another type of perturbation that can be considered is a perturbation of the underlying matrices.
If \( R(t) \) is nonsingular on \([t_0, t_f]\), existence of a solution to a certain Riccati equation on \([t_0, t_f]\) is necessary and sufficient for \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \), and this existence condition is robust with respect to variations in \( F, G, H, Q \) and \( R \) which are suitably small. If the Riccati equation has a solution on \((t_0, t_f)\) with escape time at \( x_0 \), so that \( V^*[x_0, t_f] = -\infty \) for \( x_0 = 0 \) but is \(-\infty\) for some \( x_0 \neq 0 \), matters are not quite the same; variation in \( F, G, H, Q \) and \( R \) can cause \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \), or \( V^*[x_0, t_f] = -\infty \) for some \( x_0 \) and \( t_1 > t_0 \) with \( t_1 \) close to \( t_0 \). A third possibility arises if \( R(t_1) \) is singular for some \( t_1 \in [t_0, t_f) \). Then perturbations can make \( R(t_1) \) indefinite, and certainly then \( V^*[x_1, t_1] \) may be \(-\infty\); this will be the case even if \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \) prior to perturbation.

Evidently, two crucial issues affecting tolerance of perturbations are whether \( R(\cdot) \) is nonsingular on \([t_0, t_f]\), or singular somewhere in the interval, and whether \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \), or \( V(0, t, u(\cdot)) \geq 0 \) for all \( u(\cdot) \), with \( V(0, t, u(\cdot)) \) \(-\infty\) failing for some \( x_0 \). In this latter case, we can establish a result of minor consequence which applies both to nonsingular and singular \( R(\cdot) \) cases; it states that a perturbation in \( R(\cdot) \) can always be found to ensure \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \).

**Theorem III.3.6:** Suppose that \( V(0, t_0, u(\cdot)) \geq 0 \) for all \( u(\cdot) \) and that the controllability condition (3.5) holds. Suppose that \( V^*[x_0, t_2] > -\infty \) fails for some \( x_0 \). Let \( t_1 \in (t_0, t_f) \) be arbitrary (in particular, \( t_1 \) may be arbitrarily close to \( t_0 \)), and let \( \varepsilon > 0 \) be arbitrary. Let \( \rho(\cdot) \) be continuous on \([t_0, t_f]\) with \( \rho(t_0) = 1 \), \( \rho(t) > 0 \) on \([t_0, t_1) \), \( \rho(t) = 0 \) on \([t_1, t_f]\). With \( R(\cdot) \) replaced by \( R(t) = R(t) + \varepsilon \rho(t)I \) on \([t_0, t_f]\), \( V^*[x_0, t_f] > -\infty \) for all \( x_0 \).

**Proof:** Let \( t_2 \in (t_0, t_1) \). With \( x(t_0) = 0 \), we have

\[
0 \leq \int_{t_0}^{t_1} [x'Q(t)x + 2u'H(t)x + u'R(t)u]dt
+ \int_{t_1}^{t_2} x'Q(t)x + 2u'H(t)x + u'R(t)u]dt
+ x'(t_2)Sx(t_2)
\]

and so

\[
0 \leq \int_{t_0}^{t_2} [x'Q(t)x + 2u'H(t)x + u'R(t)u]dt
+ x'(t_2)P^*(t_2)x(t_2)
\]

where \( x'(t_2)P^*(t_2)x(t_2) = V^*[x(t_2), t_f] \).

A straightforward argument, set out in the main result of [1], show that the problem of minimizing
\begin{align*}
\int_{t_0}^{t_2} \left[ x^T Q(t)x + 2u^T H(t)x + u^T R(t)u + \epsilon p(t) u^T u \right] dt \\
+ x^T (t_2) P^*(t_2) x(t_2)
\end{align*}

with arbitrary \( x(t_2) = x_0 \), has for all \( x_0 \) a solution which is not \( \infty \), since there exists \( \eta > 0 \) such that with \( x(t_0) = 0 \), one has for all \( u(\cdot) \)

\begin{align*}
\int_{t_0}^{t_2} \left[ x^T Q(t)x + 2u^T H(t)x + u^T R(t)u + (\epsilon p(t) - \eta) u^T u \right] dt \\
+ x^T (t_2) P^*(t_2) x(t_2) \geq 0.
\end{align*}

Hence for all \( x_0 \),

\begin{align*}
\psi^*[x_0, t_2] &= \min_{u(\cdot)} \int_{t_0}^{t_2} \left[ x^T Q(t)x + 2u^T H(t)x + u^T R(t)u \\
&\quad + \epsilon p(t) u^T u \right] dt + \psi^*[x(t_2), t_2] \\
&\geq \min_{u(\cdot)} \int_{t_0}^{t_2} \left[ x^T Q(t)x + 2u^T H(t)x + u^T R(t)u \\
&\quad + \epsilon p(t) u^T u \right] dt + \psi^*[x(t_2), t_2]
\end{align*}

\( > \infty \).

(The first equality follows from the principal of optimality and the first inequality by monotonicity of \( \psi^* \) with \( \epsilon \)).

4. ROBUSTNESS IN PROBLEMS WITH END-POINT CONSTRAINTS

Throughout this section, we study the system (3.1) with performance index (3.2). As earlier, \( x(t_e) \) is fixed but arbitrary; now \( x(t_e) \) is no longer free but constrained by

\[ E_x(t_e) = 0 \quad (4.1) \]

where \( E_x \) is a matrix with full row rank, sometimes specialised to the identity. Of course, we are interested in minimizing (3.2), or (3.2) with \( t_e \) replaced by variable \( t \).

We shall begin by reviewing known results drawn from [2-4]. These results suffer from a degree of asymmetry - necessary conditions are not quite sufficient conditions. Then we shall observe that by introducing robustness requirements, this asymmetry can be removed. A new form of robustness enters the picture, additional rather than alternative to those encountered earlier.

The following result is drawn from [2-4]. Its proof can be obtained similarly to the proofs of Theorems 2.1 through 2.3.
Theorem 11.4.1: Assume that
\[ E_f^T \int_t^{t_f} \Phi(t, \tau) G(t) \Phi(t, \tau) \, d\tau \geq 0 \]  
(4.2)

for all \( t \in [t_0, t_f] \). Let \( Z \) be a matrix with columns constituting a basis for the nullspace of \( E_f \). A necessary condition for \( V(x(t), t, u(\cdot)) \) to have a finite infimum for all \( x(t) \) and \( t \in [t_0, t_f] \) is that there exists on \( [t_0, t_f] \) a symmetric \( P(\cdot) \) of bounded variation such that \( dM(P) \geq 0 \) within \([t_0, t_f] \) and

\[ \lim_{t \to t_f} Z' \Phi(t, t_f) P(t) \Phi(t, t_f) S Z = 0. \]  
(4.3)

A sufficient condition is that a symmetric \( P(\cdot) \) of bounded variation exist on \([t_0, t_f] \) with \( dM(P) \geq 0 \) within \([t_0, t_f] \) and

\[ Z'[P(t_f) S] Z \leq 0. \]  
(4.4)

Remarks 5.1: 1. Strictly, [2-4] are concerned with conditions which ensure \( V(x(t), t, u(\cdot)) \) has a finite infimum for all \( t \in (t_0, t_f) \) and \( V[0, t_0, u(\cdot)] \geq 0 \) for all \( u(\cdot) \). The methods of Section 2 however allow the derivation of Theorem 4.1 in the same way that Theorem 3.3 is derived from Theorems 3.1 and 3.2.

2. One \( P(\cdot) \) satisfying the necessary conditions is defined by \( x'(t) P(t) x(t) = \inf V(x(t), t, u(\cdot)) \) with \( E_f x(t_f) = 0 \).

3. The controllability condition (4.2) is the appropriate specialization of a general condition given in Section 2 which ensures that the optimal performance index exists, i.e. the state constraint is attainable.

There are at least three distinct ways in which robustness might be sought. First, one can study the effect of allowing variations in \( t_f \), much as we varied the initial time \( t_0 \) in Section 3. Second, we can study the effect of replacing (4.1) by a condition like \( \| E_x x(t_f) \| \leq \varepsilon \) for suitably small \( \varepsilon \). Third, making use of the idea of incorporating penalty functions [13] in the performance criteria, we can study the effect of eliminating (4.1) while adding the quantity \( \| E_x x(t_f) \|^2 E_x x(t_f) \) to the performance criterion (2.2), with \( N \) a large number. Robustness in this latter case corresponds to their existing an optimum performance index for all sufficiently large \( N \).

As noted in the last section, at least for \( t_0 \) replacing \( t_f \), there are problems lacking the first two kinds of robustness. [Strictly, in the last section, the second kind of robustness was viewed as replacing \( x(t_0) = 0 \) by arbitrary \( x(t_0) \), rather than \( \| E_x x(t_0) \| \leq \varepsilon \). Let us now observe that there are also problems lacking the third kind of robustness. Then we shall go on to discuss the equivalence of the types of robustness. Some of the results are drawn from [14].

Consider \( x = (t-1)u, V[x_0, 0, u(\cdot)] = \int_0^1 x(t) \delta(t) \, dt \), with side constraint \( x(1) = 0 \). We show first that, with this constraint, \( V \geq 0 \) for all \( x(0) \) and all piecewise
continuous $u(\cdot)$. Since the constraint is evidently attainable, it follows that $\inf V$ exists. Observe that

$$V = \lim_{T \to 0} \int_{t}^{T} \{1 - \int_{t}^{T} (t-1)u(t)dt\} u(t)dt$$

[using $x(1) = 0$]

$$= \lim_{T \to 0} \int_{t}^{T} \left\{ \int_{t}^{T} (t-1)u(t)dt \right\} (t-1)^{-1} \frac{d}{dt} \left\{ \int_{t}^{T} (t-1)u(t)dt \right\} dt$$

$$= \lim_{T \to 0} \left\{ \int_{t}^{T} (t-1)^{-1} \left[ \int_{t}^{T} (t-1)u(t)dt \right] dt + \int_{0}^{T} \left[ \int_{t}^{T} (t-1)u(t)dt \right] dt \right\}$$

$$+ \lim_{T \to 0} \int_{t}^{T} (t-1)^{-2} \left[ \int_{t}^{T} (t-1)u(t)dt \right] dt.$$

The second term on the right side is $-V$, whence

$$V = \frac{1}{2} \int_{0}^{T} (t-1)u(t)dt + \frac{1}{2} \lim_{T \to 0} (t-1)^{-1} \int_{0}^{T} (t-1)u(t)dt$$

$$+ \frac{1}{2} \lim_{T \to 0} \int_{0}^{T} (t-1)^{-2} \int_{t}^{T} (t-1)u(t)dt dt.$$

Using L'Hôpital's rule, the first limit is seen to be zero, the second to be nonnegative. Therefore $V \geq 0$ as claimed.

Now consider the minimization of $V[x_{0}, 0, u(\cdot); N]$ with the constraint $x(1) = 0$ removed. We shall show there is no finite $N$ for which $\inf V > -\infty$. If there were, by Theorem 3.3 there would exist $F(\cdot)$ of bounded variation on $[0, 1]$ with $\Delta M(F) = 0$, $F(1) = N$. Such a $F(\cdot)$, if it has jumps, must only have positive jumps as explained in Section 3. Now arguments as in, for example [7], show that for any problem for which $R(t) = 0 \forall t \in [t_{0}, t_{1}]$ and for which $G(\cdot)$ and $H(\cdot)$ are continuous, one has $PC + B = 0 \forall t \in (t_{0}, t_{1})$ for any $P$ such that $\Delta M(P) = 0$ within $[t_{0}, t_{1}]$. (The result is easy to establish.) This means that here, $F(\cdot)(t-1) + \frac{1}{2} = 0$, or $F(\cdot) = \frac{1}{2}(1-t)^{-1}$. Satisfaction of the endpoint constraint is accordingly impossible, since $F(\cdot)$ would have an infinitely negative jump there. In this way, we have a contradiction to the claim that $\inf V > -\infty$ for some $N$.

We now turn to the main task of this section, which is to illustrate the equivalence of the three kinds of robustness - robustness with respect to terminal time, terminal state constraint, and terminal weighting matrix in the performance index. The first theorem below shows the equivalence of the last two forms of robustness.

We shall make notational remarks. We recall that

$$V^{+}[x_{0}, t_{0}] = \inf_{u(\cdot)} V[x_{0}, t_{0}, u(\cdot)]$$

and we define also

$$V^{+}[x_{0}, t_{0}, \eta] = \inf_{u(\cdot)} V[x_{0}, t_{0}, u(\cdot)] \text{ subject to } E_{x_{0}} x(t_{0}) = \eta$$
Throughout the following $t_0$ and $t_f$ are considered fixed. Later in the section, we shall consider $t_f$ variable; permitting $t_0$ variable adds nothing.

**Theorem 11.4.2:** The following conditions are equivalent.

(a) The controllability condition (4.2) holds with $t$ replaced by $t_0$, and $V(x_0, t_0; N)$ exists for some $N$ and all $x_0$.

(b) $V(x_0, t_0; N)$ exists for all $N \geq N_0$ and is bounded above uniformly in $\bar{x} \times N$.

(c) $\bar{V}(x_0, t_0)$ exists for all $x_0$ and all $\epsilon > 0$.

(d) $v(x_0, t_0; N)$ exists for all $x_0$ and all $N$. 

Moreover, should any one condition hold, we have

$$\lim_{N \to \infty} V(x_0, t_0; N) = \lim_{\epsilon \to 0} V^*(x_0, t_0) = V^*(x_0, t_0; N = 0). \quad (4.5)$$

**Proof:** (a) $\Rightarrow$ (b). By the controllability condition, there exists a $u(\cdot)$ taking $x(t_0) = x_0$ to $x(t_f)$ with $E_{x}(t_f) = 0$. Then $V(x_0, t_0, u(\cdot)) \geq V^*(x_0, t_0; N)$.

(b) $\Rightarrow$ (c). We show first that $V^*(x_0, t_0) < \infty$.

Suppose this is not the case. The only way this can happen is that if, for some $\epsilon$, $||E_{x}(t_f)|| < \epsilon$ is not attainable. In order to show a contradiction assume this is the case, and let $M$ be such that $V^*(x_0, t_0; N) < M - \delta$ for some arbitrary $\delta > 0$ and all $N$. Assumption (b) of the Theorem statement guarantees existence of $M$ and $\delta$. Define $U^*_N$ as the set of piecewise continuous $u(\cdot)$ for which $V(x_0, t_0, u(\cdot); N) < M$. Then for $u(\cdot) \in U^*_N$ we have

$$V(x_0, t_0, u(\cdot)) < M - N ||E_{x}(t_f)||^2$$

$$< M - N \epsilon^2. \quad (4.6)$$

(We can never have $||E_{x}(t_f)|| < \epsilon$). Set

$$Q_N^* = \inf_{u(\cdot) \in U^*_N} V(x_0, t_0, u(\cdot)).$$

Since for $N_1 < N_2$, $U^*_N_1 \subset U^*_N_2$, we see that $Q_N^*$ is monotone increasing. On the other hand from (4.6) we have
\[ Q^*_N < M - N \varepsilon^2 \]

from which it is clear that \( \lim_{N \to \infty} Q^*_N = -\infty \). Therefore \( Q^*_N = -\infty \) for all \( N \), and given arbitrary \( K > 0 \), there exists \( \bar{u}(\cdot) \in U_{N+1} \) such that \( \nu(x_0, t_0, \bar{u}(\cdot)) < -K \). Then we have

\[
\nu^*(x_0, t_0; N) \leq \nu(x_0, t_0, \bar{u}(\cdot)) + \bar{N}||F_x(x_0)||^2
\]

\[
< -K + \bar{N}||F_x(x_0)||^2
\]

and

\[
\nu(x_0, t_0, \bar{u}(\cdot); \bar{N}+1) = \nu(x_0, t_0, \bar{u}(\cdot); \bar{N}) + \bar{N}||F_x(x_0)||^2
\]

\[
> \nu^*(x_0, t_0; \bar{N}) + \frac{1}{N} \nu^*(x_0, t_0; \bar{N}) + K.
\]

Since \( K \) is arbitrary, this violates the constraint that \( \nu(x_0, t_5, \bar{u}(\cdot)); \bar{N}+1) < M \), and the contradiction is established.

To show that \( V^*_C[x_0, t_5] > -\infty \) is much easier. We have, for all \( u(\cdot) \) such that

\[
||F_x(x_0)|| \leq \varepsilon,
\]

\[
\nu^*[x_0, t_0; N] \leq \nu(x_0, t_0, u(\cdot)) + M \varepsilon^2
\]

or

\[
\nu(x_0, t_0, u(\cdot)) \geq \nu^*[x_0, t_0; N] - M \varepsilon^2.
\]

The lower bound on \( V^*_C[x_0, t_5] \) is immediate.

(c) \( \Rightarrow \) (d). First observe that the finiteness of \( V^*_C[x_0, t_5] \) implies the controllability condition (4.2) holds with \( t \) replaced by \( t_5 \). For if it did not, there would exist an initial state \( x_0 \) and some value of \( \varepsilon \) such that for no control \( u(\cdot) \) could one ensure that \( ||F_x(t_0)|| \leq \varepsilon \). Now the fact that the controllability condition holds implies that \( V^*[x_0, \eta^*_f] < \varepsilon \), since all \( \eta^*_f \) are reachable. For fixed \( \eta^*_f \), choose \( \varepsilon \) such that \( ||\eta^*_f|| \leq \varepsilon \). Then it is clear that \( V^*[x_0, \eta^*_f] \geq V^*_C[x_0, t_5] > -\infty \).

(d) \( \Rightarrow \) (a). Controllability is trivial. Next, by Theorem 2.1, we have the representation

\[
V^*[x_0, \eta^*_f] = [x_0 \quad \eta^*_f] \begin{bmatrix} P_{x\eta} & P_{x\eta} \\ P_{\eta x} & P_{\eta x} \end{bmatrix} [x_0 \quad \eta^*_f]
\]

for some \( P_{x\eta}, P_{x\eta} = P_{x\eta}^* \) and \( P_{\eta x} = P_{\eta x}^* \) so that

\[

\nu^*[x_0, t_0; N] = \inf_{\eta^*_f} \left( \inf \{ \nu(x_0, t_0, u(\cdot)) + ||\eta^*_f||^2 \} \right)
\]
where the class of $u(\cdot)$ are those leading to $E_x(t_f) = \eta_f$. The inner infimum is precisely
\[ V^*[x_0, \eta_f] + N||\eta_f||^2, \]
so that
\[ V^*[x_0, t_0; N] = \inf_{\eta_f} \left\{ x_0 \left( \begin{array}{c} P_{x0} & P_{x1} \\ P_{x0} & P_{x1} + N \eta_f \end{array} \right) \left( \begin{array}{c} x_0 \\ \eta_f \end{array} \right) \right\} = x_0 [P_{x0} - P_{x1} (P_{x1} + N \eta_f)^{-1} P_{x0}] x_0. \] (4.8)
for suitably large $N$, one clearly has $V^*[x_0, t_0; N] > -\infty$ for all $x_0$. Obviously $V^*[x_0, t_0; N] < \infty$. So conditions (a) through (d) have been shown to be equivalent.

It remains to verify (4.5). From (4.7) and (4.8), it is clear that
\[ \lim_{N \to \infty} V^*[x_0, t_0; N] = V^*[x_0, \eta_f = 0]. \]
Further
\[ \lim_{N \to \infty} V^*_c[x_0, t_1] = \min_{\eta_f} [2x_0^T \eta_f + \eta_f P_{x1} \eta_f] + x_0^T P_{x0} x_0. \]

It is clear that
\[ \lim_{\varepsilon \to 0} \min_{\eta_f} [2x_0^T \eta_f + \eta_f P_{x1} \eta_f] = 0 \]
since $x_0$, $P_{x1}$ and $P_{x2}$ are fixed during the minimization and limiting operations. Thus
\[ \lim_{\varepsilon \to 0} V^*_c[x_0, t_1] = x_0 P_{x0} x_0 = V^*[x_0, \eta_f = 0] \]
as required.

Remarks 4.2: 1. Condition (c) can be replaced by
\[ (c') V^*_c[x_0, t_1] \exists \text{ for all } x_0 \text{ and all } \varepsilon > 0 \]
as some minor calculations will show.

2. Perhaps a little more surprisingly, condition (c) can also be replaced by
\[ (c'') V^*_c[x_0, t_0] \exists \text{ for all } x_0 \text{ and some } \varepsilon > 0. \]
The reason for this is that, by a simple scaling argument, one has $V^*_c[kx_0, t_0] = k^2 V^*_c[x_0, t_0]$ for all $k$.

In Section 3, we showed the equivalence of various conditions involving some kind of robustness with a single condition involving a Riemann-Stieltjes integral. At this point, we cannot quite do this; as we argue below, the sort of robustness studied is not quite adequate.

One can informally think of the problem of minimizing (2.2) subject to $E_x(t_f) = 0$ as one of minimizing the sum of the integral term in (2.2) and a terminal weighting.
The conditions (a) and (b) of Theorem 4.2 are seen to involve the replacement of this weighting term by \( x'(t_f) [S + NE_f E_f'] x(t_f) \), with \( N \) suitably large. In this way, some perturbation of the weighting matrix is being allowed but, if \( E_f \) has fewer rows than \( x(\cdot) \), it is clear that part of the weighting matrix is not perturbed. It is then reasonable to postulate a form of robustness in which \( S \) is replaced by \( \tilde{S} \) for some symmetric \( \tilde{S} \) with \( ||\tilde{S} - S|| < \eta \) for some small positive \( \eta \). Call this \( S \)-perturbation. The introduction of this type of robustness allows a connection of the ideas of Theorems 4.1 and 4.2.

**Theorem II.4.3:** The following conditions are equivalent.

(a) The controllability condition (4.2) holds with \( t \) replaced by \( t_0 \), and for some \( \eta > 0 \) and with \( S \) replaced by any symmetric \( \tilde{S} \) with \( ||\tilde{S} - S|| < \eta \), \( v^t [x(t), t_0; N] \) exists for some \( N \) and all \( x_0 \).

(b) For some \( \eta > 0 \) and any \( \tilde{S} \) with \( ||\tilde{S} - S|| < \eta \), there exists on \( [t_0, t_f] \) a symmetric \( P(\cdot) \) of bounded variation such that \( dM(P) \geq 0 \) within \( [t_0, t_f] \) and, with \( Z \) as in Theorem 4.1,

\[
Z'[P(t_f) - \tilde{S}] Z \leq 0. \tag{4.9}
\]

(c) An obvious modification of any of conditions (b) - (d) of Theorem 4.2 holds.

**Proof:** (a) \( \iff \) (c) is trivial. We show first that (a) \( \Rightarrow \) (b). By Theorem 3.3, there exists a symmetric \( P(\cdot) \) of bounded variation on \( [t_0, t_f] \), with \( dM(P) \geq 0 \) within \( [t_0, t_f] \) and with

\[
P(t_f) - \tilde{S} - NE_f E_f' \leq 0.
\]

Equation (4.9) is immediate. To show (b) \( \Rightarrow \) (a), observe that with \( \eta < \eta - ||\tilde{S} - S|| \), one has \( ||\tilde{S} - \eta I - S|| < \eta \) and so, using condition (b) with \( \tilde{S} \) replaced by \( \tilde{S} - \eta I \), there exists \( P(\cdot) \) with the various properties stated except that (4.1) is replaced by

\[
Z'[P(t_f) - \tilde{S} + \eta I]Z \leq 0
\]

whence

\[
Z'[P(t_f) - \tilde{S}]Z < 0.
\]

Some matrix algebra will then show that there exists \( N \) for which

\[
\begin{bmatrix}
Z & [P(t_f) - \tilde{S} - NE_f E_f'] \end{bmatrix} [Z \ E_f'] < 0
\]

or
Theorem 3.3, then implies condition (a).

Remarks 4.3: 1. The crucial step in the above proof is to replace (4.9) by a strict inequality. This is not possible without the S-perturbation idea.

2. In case $E_{\infty}$ is square, there is no need for the S-perturbation idea at all. Conditions (4.4) and (4.9) evanescence.

Now we examine the third type of robustness—that involving perturbation of the terminal state $t_{f}$. To simplify the results, we shall consider constraints of the type $x(t_{f}) = 0$.

Theorem 11.4.4: Suppose that the controllability condition (4.2) holds. Then the following conditions are equivalent.

(a) $V^{x}[x_{0}, t_{0}; N]$ exists for some $N$ and all $x_{0}$.

(b) There exists $t_{1} > tf$ with definitions of $F(\cdot), G(\cdot), \text{etc.}$ on $(t_{f}, t_{1})$ such that these quantities are continuous on $[t_{0}, t_{1}]$, such that

$$\int_{t}^{t_{1}} \phi(t_{1}, \tau) G(t) G'(t) \phi'(t_{1}, \tau) d\tau > 0$$

for all $t \in [t_{0}, t_{1})$ and such that the constrained end-point problem has a solution for all $x(t)$ and $t \in [t_{0}, t_{1})$.

Proof: (a) $\Rightarrow$ (b). By Theorem 3.3, there exists a symmetric $P(\cdot)$ of bounded variation on $[t_{0}, t_{f}]$ satisfying $\delta M(P) \geq 0$ within $[t_{0}, t_{f}]$ and $P(t_{f}) \leq S + NI$. Apply a minor modification of the extension procedure of the proof of Theorem 3.4 to define $F(\cdot), G(\cdot), \text{etc.}$ with the properties stated and $P(\cdot)$ such that $\delta M(P) \geq 0$ within $[t_{0}, t_{1}]$. By Theorem 4.1, the constrained end-point problem defined over $[t, t_{1}]$ has a solution.

(b) $\Rightarrow$ (a). Suppose that

$$x^{*}(t)P(t)x(t) = \inf_{u(\cdot)} \int_{t_{0}}^{t} [x^{*}Q(t)x + 2u^{*}H(t)x + u^{*}R(t)u] dt$$

subject to $x(t_{1}) = 0$. Then for $t \leq t_{f}$ and $N$ such that $NI \propto P(t_{f})$ we obtain a lower bound for $V^{x}[x_{0}, t_{0}; N]$:

$$x^{*}(t_{f})P(t_{f})x(t_{f}) = \inf_{u(\cdot)} \int_{t_{0}}^{t_{f}} [x^{*}Q(t)x + 2u^{*}H(t)x + u^{*}R(t)u] dt$$

$$+ x^{*}(t_{f})P(t_{f})x(t_{f})$$

$$\leq \inf_{u(\cdot)} \int_{t_{0}}^{t_{f}} [x^{*}Q(t)x + 2u^{*}H(t)x + u^{*}R(t)u] dt$$

$$+ N||x(t_{f})||^{2}$$

$$= V^{x}[x_{0}, t_{0}; N].$$
This proves the result.

Remark 4.4: Consideration of constraints of the type \( E^T x(t_f) = 0 \) for nonsquare \( E \) is more messy. It does not seem possible to obtain tidy results for which the extended interval constraint becomes \( E^T x(t_f) = 0 \); rather, one should consider \( E^T \phi(t_f) x(t) \) for \( \phi(t_f, t_1) x(t) \neq 0 \) and now \( \phi(\cdot, \cdot) \) depends on the particular \( \phi(\cdot) \) chosen.

5. EXTREMAL SOLUTIONS OF RIEMANN-STIELTJES INEQUALITIES

In this section, we study the inequality \( dm(P) \geq 0 \) within \([t_0, t_f]\) and determine maximum and minimum solutions of the inequality. To begin with, we have the following result:

Theorem 5.1.1: Suppose in relation to the unconstrained minimisation problem defined by (3.1) and (3.2) \([x(t_0), \infty) \) fixed but arbitrary and \( x(t_f) \) free that, for some terminal weighting matrix \( S_f \), \( V[x(t), t] = x^*(t) P(t) x(t) \) exists for all \( x(t) \) and \( t \in [t_0, t_f] \). Then for any \( P(\cdot) \) such that \( dm(P) \geq 0 \) and \( P(t) = S_f \), we have

\[
P(t; \ P(t) \leq S_f) \leq P^*(t; \ P^*(t) = S_f)
\]

\( \forall t \in [t_0, t_f] \). (5.1)

(The notation should be self explanatory; though perhaps needlessly complicated at this point, it will be helpful later).

Proof: From the proof of Theorem 3.2 we know that for \( t \in [t_0, t_f] \),

\[
\begin{align*}
\int_t^{t_f} \left[ x^*(t) P(t) x(t) - x^*(t) P(t) x(t) \right] - x^*(t) P(t) x(t) \right. \\
+ \left. \int_t^{t_f} \left[ x^*(t) P(t) x(t) - x^*(t) P(t) x(t) \right] \right] dt.
\end{align*}
\]

Therefore

\[
V[x(t), t, u(\cdot)] = x^*(t) P(t) x(t) + x^*(t) \left[ S_f - P(t) \right] x(t)
\]

Using the conditions on \( P(\cdot) \), it follows that

\[
V[x(t), t, u(\cdot)] \geq x^*(t) P(t) x(t).
\]

Equation (5.1) is immediate.
As we know, $P^*(\cdot)$ satisfies $dM(P^*) \geq 0$ and $P^*(t_f) \leq S_f$; thus $P^*(\cdot)$ is the 
maximal $P(\cdot)$ with this property, the ordering being defined by (5.1).

We obtain a minimal $P_\infty(\cdot)$ in the following way.

**Theorem 11.5.2:** Define the performance index

$$V_\infty[x(t), t, u(\cdot)] = \int_{t_0}^{t_f} \left[ x^\prime \Phi x + 2x^\prime Bu + u^\prime Ru \right] dt - x^\prime(t_0)S_0x(t_0) \quad (5.2)$$

in which $x(t)$ is arbitrary but fixed, $x(t_0)$ is free, and $u(\cdot)$ is free.

Set

$$V_\infty[x(t), t] = \inf_{u(\cdot)} V_\infty[x(t), t, u(\cdot)]. \quad (5.3)$$

If $V_\infty[x(t), t]$ exists for all $x(t)$ and $t \in [t_0, t_f]$, then $V_\infty[x(t), t]$ 
$= -x^\prime(t)P_\infty(t)x(t)$ for some $P_\infty(t)$ defined on $[t_0, t_f]$. Moreover, there exists 
$P(\cdot)$ such that $dM(P) \geq 0$ within $[t_0, t_f]$ and $P(t_f) \geq S_0$, and $P_\infty(\cdot)$ is 
the minimal such $P(\cdot)$:

$$P(t; P(t_0) \geq S_0) \geq P_\infty(t; P_\infty(t_0) = S_0) \quad (5.4)$$

Conversely, if there exists $P(\cdot)$ such that $dM(P) \geq 0$ with $[t_0, t_f]$ and 
$P(t_f) \geq S$, then $V_\infty[x(t), t]$ exists for all $x(t)$ and $t \in [t_0, t_f]$.

The proof of this theorem can be obtained by time reversal of Theorem 5.1, and 
Theorem 3.3, which relates the existence of $V^*$ to the existence of $P(\cdot)$ satisfying 
$dM(P) \geq 0$ within $[t_0, t_f]$ and $P(t_f) \leq S_f$.

Theorem 5.2 now provides information about those terminal weighting matrices 
$S_f$ for which $V^*[x(t), t]$ exists for all $x(t)$ and $t \in [t_0, t_f]$. In fact, the 
theorem was introduced not for its intrinsic content, but in order to provide this 
information.

**Theorem 11.5.3:** With notation as above, let $V_\infty[x(t), t]$ exist for all $x(t)$, 
all $t \in [t_0, t_f]$, and for some $S_0$. Then it exists for all $S_f \geq S_0$. Moreover, 
$V^*[x(t), t]$ exists for all $S_f$ with 

$$S_f \geq P_\infty(t_f; P_\infty(t_0) = S_0) \quad (5.5)$$

for some $S_0 \leq S_0$. If (5.5) fails for all $S_f \geq S_0$ and some $S_f$, then 
$V^*[x(t), t]$ does not exist for all $x(t)$ and $t \in [t_0, t_f]$.

**Proof:** From (5.2), we see that $V[x(t), t]$ increases as $S_0$ decreases. Therefore 
$V[x(t), t] = -x^\prime(t)P_\infty(t)x(t)$ increases as $S_0$ decreases. Hence if $V_\infty[x(t), t]$ 
exists for some $S_0$, it exists for all $S_f \leq S_0$. Moreover, we see that $P_\infty(t; P_\infty(t_0) = S_0)$ 
for each fixed $t$ is monotone with $S_f$. This proves the first claim.
To prove the second claim, suppose that (5.5) holds for some $\hat{S}$. Then $P(t) = P_0(t; P_0(t_0) = \hat{S})$ is such that $dM(P) \neq 0$ and $P(t_f) = S_f$. By Theorem 3.3, $V^*[x(t), t]$ exists for all $x(t)$ and $t \in [t_0, t_f]$.

Conversely, suppose that (5.5) fails for some $S_f$ and for all $\hat{S}$, but that $V^*[x(t), t]$ exists for all $x(t)$ and $t \in [t_0, t_f]$. We shall obtain a contradiction. Let $\hat{S} = P_0(t_0)$. Then since $dM(P) \neq 0$ within $[t_0, t_f]$ and $P_0(t_0) = \hat{S}$, we have by Theorem 5.2 that $V_0[x(t), t]$ exists for all $t$ and $x(t) \in [t_0, t_f]$, with $S_f$ replaced by $\hat{S}$ in (5.2). Moreover, from (5.4),

$$P_0(t) \leq P_0(t; P_0(t_0) = \hat{S})$$

for all $t \in [t_0, t_f]$. Taking $t = t_f$ yields that $S_f \leq S_0$ and $\hat{S} \leq S_f$. Now for $S_f$ suitably negative, $\hat{S} \leq \hat{S}$ and $\hat{S} \leq S_0$. Then $P_0(t_f; P_0(t_0) = \hat{S}) \leq P_0(t_f; P_0(t_0) = \hat{S}) \leq S_f$, which is a contradiction. This proves the theorem.

Remark 5.1: 1. In view of the monotonicity of $P_0(t; P_0(t_0) = S_f)$ with $S_f$, we see that any $S_f$ for which $S_f \geq P_0(t_f; P_0(t_0) = -nI)$ for some arbitrarily large $n$ is such that the corresponding $V^*[x(t), t]$ exists on $[t_0, t_f]$. Likewise, any $S_0$ for which $S_0 \leq P_0(t_f; P_0(t_0) = -nI)$ for arbitrarily large $n$ is such that the corresponding $V_0[x(t), t]$ exists on $[t_0, t_f]$. 2. It is possible to obtain results for constrained minimization problems in which part or all of the state vector is constrained at $t_0, t_f$. The most interesting of these is one that follows easily from the three theorems above:

$$P_0(t; x(t_0) = 0) \leq P(t) \leq P_0(t; x(t_f) = 0)$$

(5.6)

In this inequality, it is assumed that the requisite controllability conditions are satisfied and that the first and third quantities are well-defined on $[t_0, t_f]$ and $[t_0, t_f]$ respectively. The quantity $P(t)$ is any solution to $dM(P) \neq 0$ on $[t_0, t_f]$. The inequality is reminiscent of some known for time-invariant problems, see [15]. The bulk of the results of this section first appeared in [16].

6. SUMMARIZING REMARKS

The main thrust of the chapter has been to show that there exist conditions involving the nonnegativity of certain Riemann-Stieltjes integrals which are both necessary and sufficient for certain linear-quadratic optimization problems to have a solution. These problems are not identical, though they are closely related, to those conventionally examined in the literature; rather, they have an inherent quality of robustness, which makes them qualitatively well-posed.

One set of results relate to robustness around the initial time or state, and a second set to robustness around a final time or constrained state. In the latter context, we have shown that penalty function ideas can be employed, and, moreover, robust...
problems are the only class of problems to which they can be applied.

On the grounds then of mathematical tidiness and the rational appear of qualitatively well-posed problems, we suggest a change of viewpoint as to which linear-quadratic minimization problems should be thought of as standard.

We also have pointed out the applicability of Riemann-Stieltjes type conditions to further linear-quadratic control problems, including those requiring transfer from an initial state of zero to a prescribed nonzero terminal state. The extension is easy to achieve, by a time-reversal argument. It is then possible to characterize several properties of extremal solutions of inequalities involving Riemann-Stieltjes integrals, in the process linking various classes of problems whose analysis in terms of such integrals is feasible.

REFERENCES


CHAPTER III
LINEAR-QUADRATIC SINGULAR CONTROL: ALGORITHMS

1. INTRODUCTION

In this chapter we are concerned with the existence and computation of optimal controls in the singular, linear-quadratic control problem with no end-point constraints. To this end, we initially look at the slightly simpler problem of finding necessary and sufficient conditions for a quadratic cost functional to be bounded below independently of the control function, subject to linear differential equation constraints.

We are again interested in the system and cost defined by equations (II.3.1) and (II.3.2). For convenience, we rewrite these equations here. That is, we consider the cost

\[ V(x_0, t_0, u(\cdot)) \leq \int_{t_0}^{t_f} \left( x^T(t)Q(t)x + 2x^TR(t)u + u^T R(t)u \right) dt + x^T(t_f)Sx(t_f) \]

and dynamics

\[ \dot{x} = F(t)x + G(t)u, \quad x(t_0) = x_0. \] (1.1)

We make the same assumptions on the coefficient matrices and controls as in the previous chapter. However, we further assume that the various matrices \( F, G, \) etc., all have differentiability properties sufficient to allow the carrying out of certain transformations (involving differentiation) which are explained subsequently. The number of such transformations can vary from problem to problem and so consequently does the required degree of differentiability of the coefficient matrices. At each state of the development of the algorithm described in this chapter we shall state the degree of differentiability sufficient for the carrying out of that stage. (In case only one complete cycle of the algorithm is required, continuous differentiability of \( Q, R, F \) and \( G, \) and \( H \) twice continuously differentiable are sufficient.)

Besides this class of assumptions, we shall also on occasions need to assume further the constancy of rank on \([t_0, t_f]\) of certain matrices constructed from \( F, G, \) etc.

Denote the set of admissible controls by \( U. \) As in Chapter II, we are interested in the problem of finding necessary and sufficient conditions for

\[ V(0, t_0, u(\cdot)) \geq 0 \text{ for each } u(\cdot) \in U \] (1.3)

subject to (1.2). In case \( R(t) > 0 \) for all \( t \in [t_0, t_f] \) i.e. the problem is non-singular, it is easily solved. The interesting cases are those when \( R(t) \equiv 0 \) (the totally singular case), and \( R(t) \) is nonzero but singular somewhere in \([t_0, t_f]\) (the partially singular case).
We again stress the fact that this problem (1.3) has already been looked at in Chapter I where we were primarily interested in general existence conditions. Here, our interest is in how we might determine, by construction, whether or not (1.3) holds. To do this, we need to study a more restricted (in the sense that differentiability and constancy of rank conditions need to hold) version of the problem than that studied in Chapter II.

Historically this problem has proved to be important in several areas. It is the second variation problem of optimal control [1], and is closely connected with the linear-quadratic control problem [2], [3]. It also appears in the dual control problem of covariance factorisation [4], [5], [6], and finally, one definition of passivity leads to a similar problem in network synthesis [7].

Initially, however, it was as the second variation problem of optimal control that the question was studied. Stronger necessary conditions than the classical Legendre-Clebsch condition were needed to eliminate singular extremals from consideration as minimizing arcs for problems which arose in aerospace trajectory optimization. For more detailed information of the history of this problem see the surveys [8], [9] and the book [10], together with the references therein. Arising from these studies were the generalized Legendre-Clebsch conditions which in the totally singular case can be written

\[
\frac{d}{dt} \left( \frac{\partial H}{\partial u} \right) = 0 \quad \text{on} \ [t_0, t_f] \tag{1.4}
\]

\[
-\frac{d}{dt} \left( \frac{\partial H}{\partial \nu} \right) \geq 0 \quad \text{on} \ [t_0, t_f] \tag{1.5}
\]

where \( H \) is the Hamiltonian associated with (1.1) and (1.2), i.e.

\[
H = \pi(t)q + 2\pi(t)H(t)u + \lambda'(\pi(t))q + \xi(t)u
\]

\[
\lambda = \frac{\partial H}{\partial \pi}
\]

and \( \lambda \) is the costate vector. If (1.5) is met with equality, the procedure leading to (1.4) and (1.5) can be extended to give further necessary conditions. In general, the necessary conditions become

\[
\frac{d^2}{dt^2} \left( \frac{\partial H}{\partial u} \right) = 0 \quad \text{on} \ [t_0, t_f], \ q < 2p \tag{1.8}
\]

\[
(-1)^p \frac{d^2}{dt^2} \left( \frac{\partial H}{\partial u} \right) \geq 0 \quad \text{but nonzero on} \ [t_0, t_f] \tag{1.9}
\]

where \( \frac{d^2}{dt^2} \left( \frac{\partial H}{\partial u} \right) \) is the lowest order time derivative of \( \frac{\partial H}{\partial u} \) in which some component of the control \( u \) appears explicitly with a nonzero coefficient. The integer \( p \) is called the order of the singular arc for scalar \( u \); for vector \( u \) an extension of this definition is needed.

These conditions (1.8), (1.9) were initially derived by Kelley [12], [13] for
scalar controls only, in which case it is not difficult to show that, for odd $q$, (1.8)
is automatically satisfied. The original derivation [12] used the classical method of
constructing special variations and considering terms of comparable orders. In [13]
and [14], a transformation technique for deriving (1.9) is described and it is this
transformation which will be studied here for the general case of vector controls.

It should be noted that passing from the scalar case to the vector case is gener-
ally far from easy. To suggest why the extension is nontrivial, (1.5) can be examined.
In the scalar case, two possibilities arise, of equality (leading on to (1.8) and (1.9))
or inequality. In the vector control case, there are really three possibilities; (1.5)
can hold with equality (leading as before to (1.8) and (1.9), or at least some of these
equations) or it can hold with strict inequality (as with the scalar case), or it can
hold with a loose inequality, the matrix on the left side of (1.5) being singular and
nonzero. Some modification of (1.8) and (1.9) is called for to cope with this case.
In the dual problem of spectral factorization, the vector problem though now solved,
took much longer to solve than the scalar problem; this fact also suggests the non-
triviality of the scalar-to-vector extension. Nevertheless results have been obtained
for the vector control problem; a general form of the generalized Legendre-Clebsch
conditions [(1.8) and (1.9) being inadequate to cover all possibilities as just noted]has been derived by Robbins [11] and Goh [15]. Robbins' method was essentially variat-
ional, whereas Goh used a transformation of the states and controls in a treatment
which represents an application of work he had done on the singular Bolza problem in
the calculus of variations [16]. Though both Kelley and Goh used transformation
methods, there is a major difference in the style of the transformations. Kelley's
transformation procedure replaces the original performance index and linear system
equation by one involving a state variable of lower dimension than the original. Goh
retains the full state-space dimension and there arise as a result a number of extra
constraint conditions over and above those which might fairly be termed generalized
Legendre-Clebsch conditions. These extra constraint conditions have been examined at
length in [17].

As noted above, in this chapter we are interested in a set of necessary and suff-
cient conditions for (1.3) to hold subject to (1.2), of more limited applicability
than the necessary conditions and sufficient conditions of Theorems II.3.1 and II.3.2.
The limitation stems from the need to have certain differentiability of constancy of
rank conditions satisfied; the advantage gained is that the conditions are highly
pertinent to the problem of computing an optimal control and performance index for
the pair (1.1), (1.2) with free $x(t_0)$.

These conditions fit in with previous work in the following way. First, they
are an extension of conditions published in [18] applicable to scalar controls when
the singular arcs are of order 1. Second, the conditions are obtained using a vector
generalization of the Kelley transformation (which, it should be recalled, is limited
to the scalar control case). Third, the various steps required to obtain the con-
ditions are in large measure the dual of those arising in an algorithm of Anderson and Moylan, used in [5] and [7] for non-control purposes. We discuss the algorithm a little further.

The problem of constructing a $P$ matrix that appears in Riemann-Stieltjes inequalities as described in the previous chapter is central to the problems of covariance factorization and time-varying passive network synthesis. It was in the latter context that an algorithm suitable for the stationary case was developed [19], and then it was recognized that this algorithm with variations was also applicable to the time-varying synthesis problem [7], and with other variations to the covariance factorization problem [5]. An algorithm was in fact suggested in [5] for finding such a $P$ matrix under additional differentiability and constancy of rank assumptions. In this work, we show that the Anderson-Moylan algorithm is precisely the vector extension of Kelley's transformation executed in a particular co-ordinate basis and in showing this, we derive the generalized Legendre-Clebsch conditions in a reasonably straightforward manner.

In connection with the optimal control problem associated with free $x(t_0)$, the Anderson-Moylan algorithm, considered in isolation from the Kelley transformation procedure, can be shown to yield the optimal performance index. Linking it with the Kelley transformation procedure yields the optimal control as well.

Finally we mention the work in [20-23]. In [20], the solution of the singular regulator problem is studied by obtaining the asymptotic solution of the regularized problem (see comments prior to Theorem II.3.2) as $\varepsilon \to 0$. The methods used are those of singular perturbation theory of ordinary differential equations. The work in [21], of which we only became aware after completing the studies in this chapter, partially raises some of the problems raised here but is nowhere near as complete. References [22, 23] contain much of the work of this chapter.

We now outline the structure of this chapter. Section 2 is concerned with developing a standard form for the control problem and, in the process, possibly reducing the control space dimension. Section 3 develops the general Kelley transformation for the nonnegativity problem in standard form, producing the generalized Legendre-Clebsch conditions and a set of necessary and sufficient conditions for the existence of a solution to the nonnegativity problem. In the process, the state-space dimension is reduced. In Section 4, the results of Sections 2 and 3 are applied to the linear-quadratic optimal control problem. Using a sequence of control and/or state-space dimension reductions, minimizing (or infimizing) controls and the corresponding optimal cost are calculated. The results in Section 5 link an algorithm used in the dual problem of covariance factorization with the algorithm outlined in Section 4; here, heavy use is made of the Riemann-Stieltjes inequality of the preceding chapter. Section 6 contains summarizing remarks.
2. CONTROL SPACE DIMENSION REDUCTION AND A STANDARD FORM

In this section, our aim is to show how extraneous controls may be removed, and how, after their removal, a certain standard form may be assumed for the matrices $R$ and $G$. All this is done with the aid of coordinate basis changes of the input and state spaces; differentiability and constancy of rank assumptions need to be invoked.

We begin with some preliminary and simple observations.

1. The problem of minimizing (1.1) subject to (1.2) with initial condition $x(t_0)$ is equivalent to the problem of minimizing (1.1) subject to (1.2) with initial condition $x(t_0)$ and with $u(t)$, $R(t)$, $H(t)$ and $G(t)$ replaced by $\tilde{u}(t) = U^{-1}(t)u(t)$, $\tilde{H}(t) = U(t)H(t)U(t)$, $\tilde{R}(t) = U(t)R(t)U(t)$ and $\tilde{G}(t) = G(t)U(t)$ for any nonsingular matrix $U(t)$, continuous on $[t_0, t_f]$. This statement corresponds to a change of basis of the control space.

2. The problem of minimizing (1.1) subject to (1.2) with initial condition $x(t_0)$ is equivalent to the problem of minimizing (1.1) subject to (1.2) with initial condition $U(t_0)x(t_0)$ and with $x(t)$, $F(t)$, $R(t)$, $G(t)$, $Q(t)$ and $S$ replaced by $X(t) = U(t)x(t)$, $\tilde{F}(t) = U(t)F(t)U^{-1}(t)$, $\tilde{R}(t) = U(t)R(t)U(t)$, $\tilde{H}(t) = U(t)H(t)U(t)$, $\tilde{G}(t) = U(t)G(t)$, $\tilde{Q}(t) = [U^{-1}(t)]'Q(t)U^{-1}(t)$ and $\tilde{S} = [U^{-1}(t)]'SU^{-1}(t)$ for any nonsingular matrix $U(t)$, continuously differentiable on $[t_0, t_f]$. This statement corresponds to a change of basis of the state space. Note that in observation 1 continuity of $U(t)$ is sufficient whereas in observation 2 the stronger condition of continuous differentiability is required.

The transformation procedure now follows.

Step 1.

Assumption 1: $R(t)$ has constant rank $r$ on $[t_0, t_f]$.

With this assumption, an application of Dolezal's Theorem, see Appendix A, guarantees the existence of a matrix $U(t)$, nonsingular and continuous on $[t_0, t_f]$ such that

$$\tilde{R}(t) = U^{-1}(t)R(t)U(t) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

(2.1)

With this $U(t)$ we change the basis of the control space.

Step 2.

Partition $G$ as $[G_1 \ G_2]$ where $G_1$ is an $m \times r$ matrix.

Assumption 2: $G_2(t)$ has constant rank $s \leq n - r$. If $s = n - r$ pass to Step 3. Otherwise, another application of Dolezal's Theorem implies the existence of a matrix $V(t)$, nonsingular and continuous on $[t_0, t_f]$ such that $G_2(t)V(t) = [\tilde{G}_2(t) \ 0]$ with $\tilde{G}_2(t)$ having $s$ columns. Set $V(t) = I \oplus V_0(t)$, with the unit matrix of dimension $r$. With this $V(t)$ we change the basis of the control space. Having done
this we partition R, G and H as

\[
\begin{bmatrix}
  I_x & 0 \\
  0 & 0_{(m-x)\times(m-x)}
\end{bmatrix}
\]

\[
G(t) = [G_1(t) \ G_2(t) \ 0_{n\times(m-x)}]
\]

\[
H(t) = [H_1(t) \ H_2(t) \ H_3(t)].
\]

Writing \( u^*(t) = [\hat{G}(t) \ u_3(t)] \) where \( \hat{G}(t) \) has dimension \( m \times s \), one obtains from (1.1) and (1.2),

\[
V[x_0, t_0, u(\cdot)] = x^*(t_f)Sx(t_f) + \int_{t_0}^{t_f} \left( e^{x^*(t) R_x 2x^*H_3u_3 dt + 2 \int_{t_0}^{t_f} e^{x^*(t) R_x 2x^*H_3u_3 dt} \right)
\]

(2.2)

\[
x = \dot{x} + \hat{G}u
\]

(2.3)

where

\[
\hat{G}(t) = \begin{bmatrix}
  I_x & 0 \\
  0 & 0_{n\times s}
\end{bmatrix}
\]

\[
\hat{G}(t) = [G_1(t) \ G_2(t) \ 0_{n\times(m-x)}]
\]

\[
\hat{H}(t) = [H_1(t) \ H_2(t) \ H_3(t)].
\]

We now focus on the last term in (2.2):

**Lemma III.2.1**: A necessary condition for \( V[0, t_*, u(\cdot)] \) to be nonnegative for all \( u(\cdot) \) is that \( \xi^*(t)H_3(t) = 0 \) for all states \( \xi(t) \) reachable from \( x(t_0) = 0 \) for \( t \in [t_1, t_f] \). A necessary condition for \( V[x_0, t_0] \) to be finite for all \( x_0 \) is that \( H_3(t) = 0 \) for \( t \in [t_0, t_f] \).

**Proof**: Assume \( t < t_f \). Let \( \hat{u}_3 \) on \( [t, t_f] \) take \( \hat{u}_3 = 0 \) to \( x(t) = \xi(t) \). Set \( \hat{u}_3 = 0 \) on \( [t, t_f] \) and \( u_3 = v \) on \( [t, t_*] \) for small \( \epsilon > 0 \) and zero elsewhere. Then

\[
V[0, t_*, u(\cdot)] = \text{constant} + 2 \epsilon \xi^*(t)H_3(t)v + \text{terms of higher order in } \epsilon.
\]

Unless \( \xi^*(t)H_3(t) = 0 \), we can readily obtain a contradiction to the nonnegativity of \( V[0, t_*, u(\cdot)] \) by appropriate choice of \( v \). The second part of the lemma follows from the observation that at time \( t_0 \), all states are reachable from some \( x_0 \). Therefore \( H_3(t) = 0 \) on \( [t_0, t_f] \), and therefore \( [t_0, t_f] \) by continuity.

If the condition expressed in the lemma fails, no further computations are needed to check the nonnegativity or finite infimum condition. If the condition holds, then
in checking for nonnegativity there is no loss of generality in assuming \( H_2(t) = 0 \) on \([t_0, t_f]\) (since the performance index is unaltered). Then the \( u_2 \) component of \( u \) can be dispensed with, and, dropping the hat superscript, we obtain a problem of the same form as the given one, but with lower control space dimension. It should be clear that in the context of linear-quadratic control problems this step corresponds to the throwing away of those controls which have no effect on the states via (2.3) and do not appear directly in (2.2).

**Step 3.**

Using the fact that \( \hat{G}_2(t) \) has \( s \) columns and rank \( s \), Dolezal's theorem again guarantees the existence of a nonsingular matrix \( T_0(t) \), continuous on \([t_0, t_f]\) such that \( T_0(t) \hat{G}_2(t) = \begin{bmatrix} 0 & I_{n \times s} \end{bmatrix} \). Set \( T(t) = I_r \otimes T_0(t) \). Now continuity of \( T(t) \) is not sufficient for a state space change of basis. Dolezal's theorem guarantees the same degree of differentiability for \( T_0(t) \) as the matrix \( G_2(t) \); therefore we make the

**Assumption 3:** \( G(t) \) has continuously differentiable elements.

With \( T(t) \), we then change the basis of the state space.

The end result of these three steps, depending for their execution on Assumptions 1-3 and on a nonnegativity or finite infimum assumption, is that

\[
R = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \quad G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & I_s \end{bmatrix}
\]

where \( G_{21} \) has \( r \) columns. Notice that there is nothing special about the other matrices defining the problem. Notice also that, (2.4) results whether or not some of the controls are eliminated.

When \( R \) and \( G \) are as in (2.4), we shall say that the control problem is in standard form. There is then a natural partitioning of the state vector \( x \) as \( [x_1 \; x_2] \) where \( x_2 \) is \( s \)-dimensional, and the control vector \( u \) as \( [u_1 \; u_2] \) where \( u_1 \) is \( s \)-dimensional. We call \( u_1 \) and \( u_2 \) the nonsingular and singular controls respectively; this nomenclature arises because given the form of \( R \) in (2.4), the quadratic term \( u_1u_1 \) occurs in the performance index while the vector \( u_2 \) occurs at most linearly through the term \( x^Hu_2 \). Moreover, the singular controls all independently influence the state \( x_2 \) (as is clear from (2.4)) and hence the cost functional (1.1).

### 3. Vector Version of Kelley Transformation

In this section, we shall extend Kelley's transformation to the vector case, taking our problem in the standard form derived in the last section. We shall deduce the generalized Legendre-Clebsch conditions from the transformed problem, and shall then show that the application of Kelley's transformation to our problem leads to a
set of necessary and sufficient conditions for the solution of our problem. These consist of the existence of a solution to a problem of the same form but of lower state dimension, a set of end-point constraints and the relevant generalized Legendre-Clebsch condition corresponding to equation (1.4).

The development of this section will essentially follow that of [14] and [18], generalized to the vector case as in [22, 23]. Assume that we are given (1.1) and (1.2) with $R$ and $G$ given by (2.4), and that we are interested in finding necessary conditions for (1.3) to hold subject to (1.2). Construct the Mayer form of the problem by introducing the scalar variable $w_0$ defined by

$$
\dot{w}_0 = x'Qx + 2x'Ru + u'Ru, \quad w_0(t_0) = 0.
$$

Equations (1.2) and (3.1) now define a set of $(n+1)$ differential equations in the variables $w_0$ and $x$. Recalling the standard form of $R$ and $G$, and the resultant partitioning of $u$ and $x$, it is clear that (3.1) involves $u_2$ linearly but not quadratically. Moreover, from the partitioned form of (1.2) and (2.4), with obvious definitions of $F_1$, etc.,

$$
x_1 = F_1x_1 + F_2x_2 + G_1u_1
$$

$$
x_2 = F_1x_1 + F_2x_2 + G_2u_1 + u_2
$$

we see that (3.2) is influenced by $u_2$ only indirectly via (3.3). Clearly, if (3.1) did not contain a $u_2$ term at all, the original problem could intuitively be replaced by one with state $x_1$ (of lower dimension than $x$) and controls $x_2$ and $u_1$, since $u_2$ is essentially $x_2$ differentiated.

With this in mind, we attempt to find a transformation to new variables $z_0$, $z_1$ and $z_2$, with $z_0$ scalar and $z = [z_1' \ z_2']$ the corresponding partitioning of the $n$-dimensional vector $x$, such that the dynamics of $z_0$ and $z_1$ are independent of $u_2$. (Thus $z_0$ plays the role of the performance index, and $z$ the role of the state variable). Suppose that we set

$$
z_0 = h_0(w_0, x_1, x_2)
$$

$$
z_1 = h_1(w_0, x_1, x_2)
$$

$$
z_2 = h_2(w_0, x_1, x_2)
$$

and assume that all first order partial derivatives of $h_0$, $h_1$ and $h_2$ with respect to $w_0$, $x_1$ and $x_2$ exist; then the dynamics of $z_i$, $i = 0, 1, 2$, can be written

$$
z_i = \frac{\partial h_i}{\partial w_0} w_0 + \left( \frac{\partial h_i}{\partial x_1} \right)'x_1 + \left( \frac{\partial h_i}{\partial x_2} \right)'x_2
$$

where $\frac{\partial h_i}{\partial x}$ is a column vector and $\frac{\partial h_i}{\partial x}$ is a matrix with j-th column the gradient of the j-th component of $h_i$. 


Using (3.1), (3.2) and (3.3) in (3.5) for $i = 0, 1$, and setting the coefficient of $u_2$ to zero we obtain, as necessary and sufficient conditions for $\bar{z}_0$ and $\bar{z}_1$ to be independent of $u_2$.

\[
\frac{2h_1}{\bar{z}_0}(2\bar{x}_1^2H_{12} + 2\bar{x}_2^2H_{22}) + \left( \frac{2h_2}{\bar{z}_2} \right) = 0 \quad (3.6)
\]

\[
\frac{2h_1}{\bar{z}_0}(2\bar{x}_1^2H_{12} + 2\bar{x}_2^2H_{22}) + \left( \frac{2h_1}{\bar{z}_2} \right) = 0 \quad (3.7)
\]

Thus, we would like to find functions $h_0$ and $h_1$ satisfying the partial differential equations (3.6) and (3.7). A standard tool for tackling such problems is the method of base characteristics [27], which in this case proceeds as follows. Suppose $\theta$ is a vector variable such that $h_0$ and $h_1$ are constant on surfaces described by varying $\theta$, i.e.

\[
\frac{\partial h_0}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial h_1}{\partial \theta} = 0.
\]

Then we see from (3.6) and (3.7), that these equations will hold provided that

\[
\left( \frac{\partial h_0}{\partial \theta} \right) = 2\bar{x}_1^2H_{12} + 2\bar{x}_2^2H_{22} \quad , \quad \frac{\partial h_1}{\partial \theta} = 0 \quad , \quad \frac{\partial x_1}{\partial \theta} = 1.
\]

Noting the latter of these equations we set $\theta = x_2$ so that the remaining equations become

\[
\left( \frac{\partial h}{\partial x_2} \right) = 2\bar{x}_1^2H_{12} + 2\bar{x}_2^2H_{22} \quad , \quad \frac{\partial x_1}{\partial x_2} = 0.
\]

If $H_{22}$ is symmetric, these equations can be solved in closed form to give

\[
x_1 = c_1 \quad (3.8)
\]

\[
w_0 = 2\bar{x}_1^2H_{12}x_2 + \bar{x}_2^2H_{22}x_2 + c_0 \quad (3.9)
\]

where $c_0$ and $c_1$ are free parameters. But now, following standard procedure, we notice that $c_0$ and $c_1$ can be expressed using (3.8) and (3.9) as functions of $x_1$ and $x_2$, and that these functions form a set of mutually independent solutions to the equations (3.6) and (3.7). Altogether, we now have as our desired transformation

\[
x_0 = x_0 - 2\bar{x}_1^2H_{12}x_2 - \bar{x}_2^2H_{22}x_2 \quad (3.10)
\]

\[
x_1 = x_1 \quad (3.11)
\]

\[
x_2 = x_2 \quad (3.12)
\]

where (3.10) and (3.11) follow from (3.8) and (3.9), and $x_2$ is chosen arbitrarily. This transformation is nonsingular as the Jacobian determinant equals unity.

It is important to note that the closed form solution (3.9) exists only if $H_{22}$.
is symmetric. In the scalar control problem considered by Kelley, \( u_2 \) and \( x_2 \) are scalars, so that \( H_{22} \) is a scalar, and no difficulty arises. However, in general, there is no a priori symmetry constraint on \( H_{22} \) in (1.1) as it stands. Instead of (3.10), we consider the transformation

\[
x_0 = v_0 - 2x_1^S H_{12} x_2 - x_2^S H_{22} x_2
\]

(3.13)

where \( H_{22}^S \) is the symmetric part of \( H_{22} \), and in the subsequent discussion and Appendix B, we show that the nonnegativity requirement (1.3) forces \( H_{22} \) to be symmetric. (An alternative approach to proving the symmetry is presented in Section 5). The symmetry property is actually the relevant generalized Legendre-Clebsch condition corresponding to (1.8) for \( q = 1 \).

From (3.11) and (3.12), the dynamics of \( z_1 \) and \( z_2 \) are seen to be identical to those of \( x_1 \) and \( x_2 \). Calculation of the dynamics of \( z_0 \) is straightforward. Identifying \( x_1 \) and \( x_2 \) with \( \xi_1 \) and \( \xi_2 \), we obtain

\[
\begin{align*}
\dot{z}_0 &= \xi_0^T Q \xi_1 + 2x_1^S H_{12} x_2 + 2x_1^S H_{22} u_1 \\
&\quad + x_2^S \hat{H}_{12} x_2 + 2x_2^S \hat{H}_{22} u_1 + u_1^T u_1 + x_2^S \hat{H}_{22}^A u_2
\end{align*}
\]

(3.14)

where

\[
\begin{align*}
\hat{Q} &= Q_1 - H_{12} F_{21} - F_{21} H_{12} \\
\hat{H}_1 &= Q_1 - F_{11} H_{12} - H_{12} F_{22} - H_{22}^S H_{12} \\
\hat{H}_2 &= H_{11} - H_{12} G_{12} \\
\hat{H}_3 &= Q_{22} - F_{12} H_{12} - H_{12} F_{22} - F_{22} H_{22}^S - H_{22}^S F_{22} - H_{22}^S \\
\hat{H}_4 &= H_{21} - H_{12} F_{11} - H_{22}^S G_{12} \\
\hat{H}_{22}^A &= \text{anti-symmetric part of } H_{22}^S
\end{align*}
\]

(3.15)

It is now clear from (3.14) that \( z_0 \) is independent of the singular controls \( u_3 \), except for the final term \( x_2^S \hat{H}_{22}^A u_2 \).

This far we have considered only the dynamics of the new variables; it now remains to discuss the boundary conditions. We demand that the transformation (3.13) holds on the closed interval \([t_0, t_f]\), and we thus have at \( t_0 \) and \( t_f \) respectively,

\[
\begin{align*}
x_0(t_0) &= v_0(t_0) - 2x_1^S(t_0) H_{12}(t_0) x_2(t_0) - x_2^S(t_0) H_{22}(t_0) x_2(t_0) \\
x_0(t_f) &= v_0(t_f) - 2x_1^S(t_f) H_{12}(t_f) x_2(t_f) - x_2^S(t_f) H_{22}(t_f) x_2(t_f)
\end{align*}
\]

(3.16)

Noting that \( v[0, t_0, u(\cdot)] = x^*(t_f) S x(t_f) + v_0(t_f) - v_0(t_0) \) and that \( x_1(t_f) = 0 \) and \( x_2(t_f) = 0 \), we obtain from (3.14) and (3.16)
In Appendix B, Lemma B.1, we show that the nonnegativity requirement (1.3) on \( V[0, t_0, u(\cdot)] \) implies \( H_2^T \equiv 0 \) on the interval \([t_0, t_f]\). For continuous \( H_2 \), this equality can then be extended to the closed interval \([t_0, t_f]\).

Introduce the notation

\[
\hat{x} = x_1, \quad \hat{u} = [\hat{u}_1 \, \hat{u}_2] = [x_1 \, u_1]^T, \\
\hat{p} = F_{11}, \quad \hat{g} = [F_{12} \, G_{11}], \quad \hat{h} = [h_1 \, h_2].
\]

Now, (3.17) can be written as

\[
V[u(\cdot)] = \hat{x}'(t_0)x + 2\hat{x}'(t_0)B_{12}(t_0)u_1 + \hat{u}'(t_0)(B_{22} + R_{22})u_1 + \int_{t_0}^{t_f} [\hat{x}'(t)H_{22} + \hat{u}'(t)H_{21}]\,dt \\
+ \int_{t_0}^{t_f} \hat{x}'(t)x_2^AH_2x_2\,dt.
\]

(3.17)

Now we would like to replace the problem of finding necessary and sufficient conditions for (1.3) to hold, given (1.1) and (1.2) with \( x(t_f) = 0 \), by a problem of identical structure, save that the hat quantities only are involved. It is important to realize why we should want to do this. The state variable \( \hat{x} \) evidently has lower dimension than the state variable \( x \). Hence repetition of the cycle of reduction-to-standard form (with possible reduction of control space dimension) followed by state-space-dimension reduction via-Keller-transformation must terminate, in one of three possible ways: either a zero dimension control variable is encountered, or a zero dimension state variable, or a nonsingular problem. In either of these three cases, helpful necessary and sufficient conditions for (1.3) follow.

Let us then return to an examination of the replacement hat problem. For the replacement problem to be of identical structure, in particular \( \hat{u} \) is required to be a piecewise continuous control, and therefore \( u_2 \) constructed from (3.3) could contain delta functions at the discontinuities of \( u_1 \), i.e., \( x_2 \). Thus, the set of admissible control functions \( U \) for the original problem needs to be extended to
contain delta functions if the attainable performance indices are to be the same.
This is not a problem however; as a delta function can be constructed as a limit of continuous functions, the original nonnegativity requirement (1.3) is equivalent to

\[ V(0, t_0, u(\cdot)) \geq 0 \quad \text{for each } u(\cdot) \in U^* \tag{3.21} \]

where \( U^* \) is a suitably extended set of admissible controls \( u(\cdot) \).

From (3.19) it is clear that the nonnegativity requirement (3.21) implies that the end point term of (3.19) must be bounded below for each \( \hat{u}_1 \), independently of \( \hat{u}_1 \). Necessary and sufficient conditions for this are that

1) \( S_{12} + B_{12}(t_f) \geq 0 \) \tag{3.22}

2) \( N[S_{12} + B_{12}(t_f)] \subseteq N[S_{12} + B_{12}(t_f)]^\perp \tag{3.23} \)

where \( N \) denotes null space. Now using a completion of the square type argument, the end point term in (3.19) can be written as

\[ \hat{u}_1 + (S_{12}+B_{12})^\delta(S_{12}+B_{12})\left[(\hat{u}_1 + (S_{12}+B_{12})^\delta(S_{12}+B_{12})\hat{u}_1 + (S_{12}+B_{12})^\delta(S_{12}+B_{12})\right]_{t=t_f}^t \]

\[ + \hat{K}(S_{11} - (S_{12}+B_{12})(S_{12}+B_{12})^\delta(S_{12}+B_{12})^\perp)^2_{t=t_f} \tag{3.24} \]

where \( \delta \) denotes pseudo-inverse. With the notation

\[ \hat{S} = S_{11} - [S_{12} + B_{12}(t_f)]^\perp[S_{12} + B_{12}(t_f)] \]

\[ \hat{K} = -[S_{12} + B_{12}(t_f)]^\perp[S_{12} + B_{12}(t_f)] \]

(3.19) becomes

\[ V(0, t_0, u(\cdot)) = [\hat{\alpha}_1 - \hat{K}\hat{x}]^\perp[S_{12} + B_{12}(t_f)]^\perp[S_{12} + B_{12}(t_f)] \]

\[ + \int_{t_0}^{t_f} [x^\top \hat{K} + 2 x^\top \hat{H} + \hat{u}^\top \hat{R} \hat{u}] dt \tag{3.25} \]

Because we allow piecewise continuous controls \( \hat{u} \) and because \( u_1(t_f) \) appears in the end point term of (3.25) and has no effect on the value of the integral in (3.25), the minimization of (3.25) is carried out by separately minimizing the end point term involving \( \hat{u}_1(t_f) \) and the remaining integral-plus-terminal-cost term. We can now state the following theorem which summarizes what we have to this point.

**Theorem III.3.1:** Assume continuity of \( F, G, Q \) and \( R \), continuous differentiability of \( H \) and that the problem (1.1) through (1.3) with \( x(t_0) = 0 \) is in standard form. Further, with quantities \( \hat{x}, \hat{u}, \hat{F}, \hat{G}, \hat{Q}, \hat{R}, \hat{S} \) as defined earlier, set

\[ \hat{V}(0, t_0, \hat{u}(\cdot)) = x^\top \hat{K} + 2 x^\top \hat{H} + \hat{u}^\top \hat{R} \hat{u} \]

\[ + \int_{t_0}^{t_f} [x^\top \hat{K} + 2 x^\top \hat{H} + \hat{u}^\top \hat{R} \hat{u}] dt \tag{3.26} \]
Then
\[ V(0, t_r, u(\cdot)) \geq 0 \] for each \( u \in U \), subject to (1.2) with \( x(t_r) = 0 \)
if and only if

(a) \( V(0, t_r, \hat{u}(\cdot)) \geq 0 \) for each \( \hat{u} \in U \), subject to (3.20) with \( \hat{x}(t_r) = 0 \)

(b) \( H_{22}(t) \) is symmetric for each \( t \in [t_0, t_f] \)

(c) \( S_{11} + H_{32}(t_f) \geq 0 \)

(d) \( H[S_{22} + H_{22}(t_f) \leq H[S_{32} + H_{22}(t_f)] \).

**Remarks 3.1:**

1. Recall from the last section that to put the given problem into standard form, it is necessary to make some assumptions on the ranks and differentiability of matrices constructible from the coefficient matrices \( F, G \), etc.

2. This theorem is independent of any controllability assumption such as (II.3.5). However, for the reduction from nonstandard to standard form as set out in Section 2, (II.3.5) guarantees the retention of the controllability property and thus precludes the possibility of \( G \) and \( H \) being identically zero in the standard form.

3. A necessary condition for \( \tilde{V}(0, t_r, \hat{u}(\cdot)) \geq 0 \) for each \( \hat{u} \in U \) subject to (3.20) with \( \tilde{x}(t_r) = 0 \) is \( \tilde{\lambda}(t) \geq 0 \) on \( [t_0, t_f] \), the classical Legendre-Clebsch condition. For the original problem (without the hat superscripts) this becomes the generalized Legendre-Clebsch condition corresponding to (1.5).

Theorem 3.1 says that our original singular problem (1.3) is equivalent to an identical, though possibly nonsingular, problem of lower state dimension (condition (a) of Theorem 3.1) plus side conditions (b), (c) and (d) of Theorem 3.1. If \( R \) is singular on the interval \( [t_0, t_f] \) and the various differentiability and rank assumptions hold, the process of conversion to standard form with possible elimination of some controls, followed by application of Theorem 3.1 can be repeated to produce yet a lower dimensional problem and further side conditions. Now, since the state dimension is lowered at each application of Theorem 3.1, the process must end when either the state dimension shrinks to zero, or the problem becomes nonsingular, or \( G \) and \( H \) become zero in standard form. However, should the controllability assumption (II.3.5) be in force, this third possibility cannot occur [see Remark 3.1.2 above].

In case the state dimension shrinks to zero, necessary and sufficient conditions are trivial; in case a nonsingular problem is obtained, necessary and sufficient conditions are given by the classical Jacobi conjugate point condition in the form of a Riccati equation having no escape times on the interval \( [t_0, t_f] \) — see Cor. II.3.2.

Finally for \( G \) and \( H \) zero in standard form a necessary and sufficient condition is the nonnegativity of \( R(t) \) on \( [t_0, t_f] \).

In Section 4, Theorem 3.1 is extended to the general linear-quadratic control problem with no end-point constraints. Computation of optimal controls and the
corresponding optimal cost is also discussed.

4. COMPUTATION OF OPTIMAL CONTROL AND PERFORMANCE INDEX

In this section we are interested in the general linear-quadratic control problem which can be stated as:

Find necessary and sufficient conditions for (1.1) to be bounded below independently of \( u \in U \), subject to (1.2) with initial condition \( x(t_0) = x_0 \) not necessarily zero. Moreover, when the lower bound exists, find a minimizing (optimal) control and the corresponding minimal (optimal) cost.

Here we shall solve this problem by extending the results of the previous section. In Section 5 we will again derive the optimal control and the optimal cost using Theorem II.3.3 of the previous chapter and the Anderson-Moylan algorithm.

As in Section 2, we assume that (1.1) and (1.2) are in standard form, that the various differentiability assumptions hold and that the state and performance index transformation is described by equations (3.10)-(3.12). Recalling that \( x(t_0) = x_0 \) is now arbitrary but fixed, we obtain the additional fixed term (previously zero).

\[
-2x_2^2(t_0)H_{12}(t_0)x_2(t_0) - x_2^2(t_0)H_{22}(t_0)x_2(t_0)
\]  

(4.1)

in the computation of \( V[x_0, t_0, u(\cdot)] \) in (3.17). Now if the infimum of (1.1) subject to (1.2) is finite for all \( x(t_0) \), this implies \( V[0, t_0] = 0 \) and that \( V[0, t_0, u(\cdot)] \geq 0 \) for all \( u(\cdot) \). As argued in the last section, it follows \( H_{12}(t) \) is symmetric on \([t_0, t_f]\).

Arguing further as in the previous section, we have the following extension to Theorem 3.1.

Theorem III.4.1: With the same notation and assumptions as for Theorem 3.1 but allowing free \( x(t_0) \) (and again noting that controllability is not required), we obtain

\[
V[x_0, t_0, u(\cdot)] \text{ subject to (1.2) is bounded below for each fixed } x(t_0), \text{ independently of } u \in U \]

if and only if

(a) \( \hat{V}[x_0, t_0, \hat{u}(\cdot)] \text{ subject to (3.20) is bounded below for each fixed } \hat{x}_0, \text{ independently of } \hat{u} \in U \)

(b) \( H_{12}(t) \) is symmetric for each \( t \in [t_0, t_f] \)

(c) \( S_{22} + H_{22}(t_0) \geq 0 \)
Again extending the discussion of Section 3, we perform a series of such transformations and applications of Theorem 4.1 in conjunction with the transformation of the coefficient matrices to standard form until we obtain either a problem of zero state dimension, or a nonsingular problem or one with and being zero. For the first two possibilities, necessary and sufficient conditions for (4.1) are known. For the latter possibility, which would be ruled out by a controllability assumption (*3.5), a necessary and sufficient condition is that \( R(t) \geq 0 \) on \([t_0, t_f]\). The minimum value would then be \( x'(t_0)\{0(\bar{t}, t_0)S0(t_f, t_0) + \int_{t_0}^{t_f} \bar{p}(r)Q(r)\phi(t_0, t_0)dr \}x(t_0)\).

Now, to calculate the minimizing control and the corresponding minimal cost, we work backwards from either the nonsingular, zero state dimension or zero input dimension problem, minimizing at each successive stage. For the purpose of illustration, suppose first that after one transformation the problem is nonsingular, i.e. \( \hat{X}(t) > 0 \) on \([t_0, t_f]\). Then the necessary and sufficient condition for \( \hat{V}[x_0, t_0] \) to be finite is that the Riccati equation

\[
\dot{P} = \hat{P}^\top + \hat{P}P + \hat{Q} - (\hat{G}\hat{H})P^{-1}(\hat{G}^\top + \hat{H})^\top, \quad \hat{P}(t_f) = \hat{S}
\]  

(4.3)

where \( \hat{P} \) is a symmetric square matrix of appropriate dimension, has no escape times on the interval [\( t_0, t_f \)]. From standard linear regulator theory we know that the optimal control for the cost term \( \hat{V}[x_0, t_0] \) subject to (3.19) with \( \hat{X}(t_0) = \hat{S} \) not necessarily zero, is

\[
\hat{u}^*(t) - \hat{L}^\top(\hat{X}(t)) \hat{x}(t) \quad \text{for} \quad t \in [t_0, t_f]
\]  

(4.4)

where \( \hat{L} = -\hat{P}^{-1}(\hat{G}\hat{H} + \hat{S}) \) and the corresponding minimum cost is

\[
\hat{V}[x_0, t_0, \hat{u}^*(\cdot)] = \hat{x}^\top(t_0)\hat{P}(t_0)\hat{x}(t_0).
\]  

(4.5)

However, we also need to separately minimize a terminal point term occurring in \( \tilde{V}[x_0, t_0, u(\cdot)] \) but not in \( \hat{V}[x_0, t_0, \hat{u}(\cdot)] \); this terminal point term is shown in (3.25) for the case when \( x_0 = 0 \), but clearly takes the same form independently of the value of \( x_0 \). The separate minimization gives the optimal value for the control \( \hat{u}_1 \) at \( t = t_f \)

\[
\hat{u}_1(t_f) = \hat{X}(t_f),
\]  

(4.6)

and the corresponding minimal cost for the terminal point term of zero [see (3.25)]. The optimal value for \( \hat{u}_2(t_f) \) is seen to be indeterminate, the most convenient value being that defined by (4.4). Now considering the optimal control at \( t_0 \) we see that \( \hat{u}_1(t_0) \) is specified as \( x_0(t_0) \). Again, we also have \( \hat{u}_2(t_f) \) arbitrary, the most convenient value being that defined by (4.4).

We now combine the optimal cost and control from the separate optimization problems to obtain the optimal cost and control, in terms of the hat quantities, for the
problem (4.1). From (4.5) and (4.2), we can write the optimal value for
\[ V(x_0, t_0, u(\cdot)) \] as
\[
\begin{bmatrix}
  x_1(t_0) \\
x_2(t_0)
\end{bmatrix}
\begin{bmatrix}
  \hat{P}(t_0) & -H_{12}(t_0) \\
  -H_{21}(t_0) & -H_{22}(t_0)
\end{bmatrix}
\begin{bmatrix}
  x_1(t_0) \\
x_2(t_0)
\end{bmatrix}
\] (4.7)
while the optimal control constructed from (4.4) and (4.6) is
\[
\hat{u}^*(t) = \hat{L}(t)\hat{x}(t) \quad \text{for} \quad t \in (t_0, t_f)
\]
\[
\hat{u}_1(t_0) = x_2(t_0)
\]
\[
\hat{u}_1(t_f) = \hat{x}_2(t_f)
\]
\[
\hat{u}_2(t_0) \text{ and } \hat{u}_2(t_f) \text{ determined as discussed above.}
\]

The computation of the optimal control for the problem (4.1) is then completed by
using (3.3) to determine \( \hat{u}_2 \) from \( x_1, x_2 \) and \( u_1 \) (the last three quantities
being in hat notation, \( x, u_1 \) and \( u_2 \)); the \( u_1 \) part of the control \( u^* \) is
already determined by \( \hat{u}_2 \). The possible occurrence of delta functions in the optimal
control \( u^* \) at both the initial and final points of the interval \([t_0, t_f]\) is now
apparent since from (4.8) there is the possibility of jumps in the optimal control \( u^* \)
at the end points of \([t_0, t_f]\). To prevent the possibility of delta functions occurring
within the interval \((t_0, t_f)\), we demand that \( \hat{L}(t) \), which is constructed from
\( \hat{P}, \hat{X} \) and \( \hat{G} \), be continuously differentiable throughout the interval. Finally, the
arbitrary nature of \( \hat{u}_2(t_0) \) and \( \hat{u}_2(t_f) \) introduces nonuniqueness into the choice of
optimal control, in the form of nonuniqueness in the delta functions at \( t_0 \) and \( t_f \).

Above, we have discussed the procedure applying when the transformed problem is
nonsingular. Suppose now that the transformed problem has zero state dimension. Then
\( \hat{V}(x_0, t_0, u(\cdot)) \) is just
\[ \int_{t_0}^{t_f} \hat{R}^2 \hat{u}^2 \, dt; \] a necessary and sufficient condition for this
to be bounded below is that \( \hat{R} > 0 \) on \([t_0, t_f]\). For \( \hat{R} > 0 \), the optimal (and
clearly unique) control is \( \hat{u}^*(t) \equiv 0 \). However, for \( \hat{R} \) of rank \( s \) along \([t_0, t_f]\),
a transformation to standard form makes it clear that only the first \( s \) components of \( u \)
are required to be set to zero, the remaining components of \( u \) being arbitrary. Calculation of the optimal control and cost can now be carried out along the
lines of the procedure discussed for the case of the transformed problem being non-
singular.

Finally for \( G \) and \( H \) being zero in standard form, the minimum value was des-
cribed earlier. The corresponding optimal control is then calculated as in the prev-
ious paragraphs.

Any of the three cases discussed above could arise as the first step in the back-
ward procedure required to calculate the optimal control and cost for a problem where
more than one transformation is needed to obtain a nonsingular, zero state dimension,
or zero input dimension problem. To complete the discussion we therefore need to look
briefly at the procedure for calculating the optimal control and cost for a singular problem from the optimal quantities for a singular problem of lower (but nonzero) state dimension, as set out in Theorem 4.1. The optimal control for the lower dimensional problem is assumed known, and is continuously differentiable on \((t_0, t_f)\) with the possibility of delta functions and derivatives of delta functions at the end points, and the further possibility of jumps at the end points due to the minimization procedure at the stage under discussion. Noting (3.3), one sees that the optimal control for the higher dimensional problem will now contain derivatives of those delta functions and jumps in the optimal control for the lower dimensional problem. The calculation of the optimal cost would proceed along the lines of (4.7) with blocks of the matrix defining the quadratic performance index being uniquely identified by the end point conditions.

The optimal control is not necessarily unique though certain components of it are, such as the control derived from the nonsingular problem and the optimal value for the end point. However, nonuniqueness can arise in various ways, the main ones being: reduction of control dimension in bringing the problem to standard form, terminating with singular problem with zero state dimension, and certain end point controls not appearing in the cost. Note also, that as shown by looking at the optimal cost (4.7), part of the performance matrix is determined uniquely by parts of \(H(t_0)\), while the remaining part is determined uniquely by the Riccati equation (4.3).

In the next section, this will appear in the derivation more naturally than in the above.

Finally, in most discussions of problems in singular control there arises the question of the definition of singular strips, i.e. subspaces on which the state vector is concentrated when the control is optimal. In the derivation we have presented (in contrast to that in [19]), the singular strip is closely related to the subspace of the original state space described by the states in the terminating problem, whether it be a nonsingular problem of nonzero state dimension or a zero state dimension problem. In the former case, the definition of the singular strip in terms of the coordinates describing the original state space can be quite complicated since the various transformations performed in arriving at the terminating problem must be applied in reverse order. However, the latter case is simple; the singular strip is just the origin, the unique zero dimensional subspace of the original state space. It is also easy to interpret the occurrence of delta functions and their derivatives at the end points of the optimal control. They allow the instantaneous transfer of the initial state onto the singular strip at the initial time \(t_0\) and a jump off the singular strip at the final time \(t_f\).

3. SOLUTION VIA RIEMANN-STIELTJES INEQUALITY

In the solution of the linear-quadratic control problem presented in Section 4,
the reduction in state dimension and the calculation of the optimal control appear in a direct manner, with the computation of the optimal cost completing the solution of problem. Here we present an alternative derivation of the results of Section 4 employing the Anderson-Moylan algorithm in conjunction with Theorem II.3.3. By this method manipulations are made on the matrix measure involving $P$ in order to compute a matrix solution of the integral matrix inequality. In contrast to the method presented in Section 4, the state transformation and optimal control are not part of the main algorithm.

Recall that Theorem II.3.3 connects the linear-quadratic problem (4.1) and necessary and sufficient conditions involving the Riemann-Stieltjes inequality. In general, there can be many matrices $P(\cdot)$ satisfying the Riemann-Stieltjes inequality but as we have already shown in Theorem II.5.1, there is a maximal solution and this defines the performance index for the associated control problem.

For convenience, we rewrite the Riemann-Stieltjes inequality, namely,

$$\int_{t_1}^{t_2} \begin{bmatrix} v' & u' \end{bmatrix} \begin{bmatrix} dP + (FP + FP + Q)dt & (PG + B)dt \\ (PG + B)dt & R dt \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} \geq 0$$

(5.1)

for all continuous $v(\cdot)$, for all piecewise continuous $u(\cdot)$ and for all $[t_1, t_2]$ in $[t_0, t_f]$. In addition, the end point condition must be satisfied.

As in the previous sections, assume that $G$ and $R$ are given in standard form and that the corresponding partitioning of the various matrices and vectors hold. Substituting into (5.1), defining $w' = [v_1', v_2', u_2']$ and with the obvious definition of $dw$, we obtain

$$\int_{t_1}^{t_2} w'dw + 2 \int_{t_1}^{t_2} v_1'(P_{12} + B_{12})u_2 dt + 2 \int_{t_1}^{t_2} v_2'(P_{22} + B_{22})u_2 dt \geq 0$$

(5.3)

from which we are able to conclude that

$$P_{12}(t) + B_{12}(t) = 0 \quad \text{on} \quad (t_0, t_f)$$

(5.4)

$$P_{22}(t) + B_{22}(t) = 0 \quad \text{on} \quad (t_0, t_f).$$

(5.5)

To see that (5.5), holds, suppose that there exist $[t_1, t_2] \subset [t_0, t_f]$, $v_2'(\cdot)$ and $u_2(\cdot)$ such that

$$\int_{t_1}^{t_2} v_2'(P_{22} + B_{22})u_2 dt$$

(5.6)

is not zero. Then choose $v_1(\cdot) \equiv 0$ on $[t_1, t_2]$, so that the middle term in the inequality (5.3) is zero for all $u_2(\cdot)$. Since the first term of (5.3) is unaffected by $u_2(\cdot)$, it is then clear that for suitable scaling of $u_2(\cdot)$ we obtain a contra-
diction to the inequality (5.3). Thus, (5.6) must be zero for any \([t_1, t_2] \subset [t_0, t_f]\), for any continuous \(v_2(\cdot)\) and for any piecewise continuous \(u_2(\cdot)\). However, \(P_{22}(\cdot)\) is of bounded variation and therefore is continuous except at a countable number of points in the interval \([t_0, t_f]\). Thus, with \(H_{22}(t)\) continuous, we conclude that (5.5) holds at all points of continuity of \(P_{22}(t)\) in the interval \((t_0, t_f]\).

We can extend the validity of (5.5) to the entire interval \((t_0, t_f]\) in the following way. Suppose that \(t_2\) is a point of discontinuity of \(P(\cdot)\). Because \(P(\cdot)\) is monotone, it has left and right limits at \(t_2\); as noted in Chapter II, see Lemma II.3.5, jumps in \(P(\cdot)\) must be nonnegative matrices. Therefore jumps in \(P_{22}(\cdot)\) must likewise be nonnegative, i.e. \(\lim_{t \to t_2^-} P_{22}(t) \leq \lim_{t \to t_2^+} P_{22}(t)\). However, both these limits must be \(H_{22}(t_2)\), in view of the continuity of \(H_{22}(\cdot)\) and the almost everywhere equality of \(P_{22}\) and \(H_{22}\). Thus \(P_{22}(t_2) = H_{22}(t_2)\), with \(P_{22}(\cdot)\) possessing no jumps. This concludes the proof of (5.5). In a similar manner, we can argue that (5.4) must also hold.

We have now identified the blocks \(P_{12}\) and \(P_{22}\) of \(P\) uniquely for any \(P\) satisfying (5.1); any nonuniqueness can only occur in the \(P_{11}\) block. Moreover, \(P_0\) is symmetric on \((t_0, t_f]\), implying by (5.5) the symmetry of \(H_{22}\) as a necessary condition for the solvability of the optimal control problem.

We can also show that the equalities (5.4) and (5.5) extend to the point \(t_0\) in case \(P(t) = P^0(t)\), where \(P^0(\cdot)\) is as defined previously. Recall that the Riemann-Stieltjes integral inequality also implies – see Lemma II.3.5 – that all jumps in any \(P(\cdot)\) satisfying the inequality must be nonnegative, i.e. \(P(t^-) \leq P(t) \leq P(t^+)\) and so

\[
\begin{bmatrix} P_{11}(t_0) & -H_{12}(t_0) \\ -H_{21}(t_0) & -H_{22}(t_0) \end{bmatrix} \preceq \begin{bmatrix} P_{11}(t_0) & P_{12}(t_0) \\ P_{12}(t_0) & P_{22}(t_0) \end{bmatrix}
\]

It is clear that taking \(P_{12}(t_2) = -H_{12}(t_2)\), \(P_{22}(t_2) = -H_{22}(t_2)\) is consistent with the Riemann-Stieltjes integral inequality, and by the maximal property of \(F^\#(\cdot)\), and in particular \(F^\#(t_2)\), we must have \(P_{12}(t_2) = -H_{12}(t_2)\), \(P_{22}(t_2) = -H_{22}(t_2)\).

A study of the right hand end-point \(t_f\) leads to

\[
S = \begin{bmatrix} P_{11}(t_f) & P_{12}(t_f) \\ P_{12}(t_f) & P_{22}(t_f) \end{bmatrix} \preceq \begin{bmatrix} P_{11}(t_f^-) & P_{12}(t_f^-) \\ P_{12}(t_f^-) & P_{22}(t_f^-) \end{bmatrix}
\]

so that \(N[S_{22} + H_{22}(t_f)] \preceq [S_{12} + H_{12}(t_f)]\) and \(S_{22} + H_{22}(t_f) \succeq 0\).

Further, returning to (5.3) and the definition of \(dY\) we have
\[
\begin{bmatrix}
\frac{dy_{11}}{dt} & \frac{dy_{12}}{dt} & \frac{dy_{13}}{dt} \\
\frac{dy_{21}}{dt} & \frac{dy_{22}}{dt} & \frac{dy_{23}}{dt} \\
\frac{dy_{31}}{dt} & \frac{dy_{32}}{dt} & \frac{dy_{33}}{dt}
\end{bmatrix} = (5.7)
\]

where

\[
\begin{align*}
\frac{dy_{11}}{dt} &= \parallel dP_{11} + (Q_{11} + P_{11}F_{11} + F_{11}P_{11} + P_{12}F_{21} + F_{12}P_{21})dt \\
\frac{dy_{12}}{dt} &= \parallel dP_{12} + (Q_{12} + P_{12}F_{12} + P_{12}F_{22} + F_{12}P_{22})dt \\
\frac{dy_{22}}{dt} &= \parallel dP_{22} + (Q_{22} + P_{22}F_{22} + F_{22}P_{22} + P_{12}F_{12})dt \\
\frac{dy_{13}}{dt} &= \parallel (P_{11}G_{11} + P_{12}G_{21} + R_{11})dt \\
\frac{dy_{23}}{dt} &= \parallel (P_{12}G_{11} + P_{22}G_{22} + R_{22})dt.
\end{align*}
\]

(5.8)

Now, assuming the differentiability of \( f \), we combine the definitions (3.18) with the above to obtain

\[
\begin{bmatrix}
\frac{dy}{dt} = \parallel d\hat{P} + (\hat{P}P + \hat{P}Q)dt \\
& (\hat{P}G + \hat{R})dt
\end{bmatrix} dt
\]

(5.9)

where \( \hat{P} = P_{11} \).

Observing that the Riemann-Stieltjes integral of (5.9) has the same form as the original integral (5.1), we attempt to find the relevant minimization problem [of the same form as (4.1)] corresponding to (5.9). However, given our development of Sections 3 and 4 it is clear that with the transformation (3.10)-(3.12), the definitions of \( \hat{X} \) and \( \hat{u} \) as in (3.18) and \( \hat{S} \) as defined in Section 3, the minimization problem is just that described in part (a) of Theorem 4.1.

The above discussion leads us to

**Theorem III.3.5.1:** With the same assumptions as in Theorem 3.1, there exists an \( nxn \) matrix \( P(t) \), symmetric and of bounded variation of \( [t_0, t_f] \) such that with arbitrary \( [t_1, t_2] \subset [t_0, t_f] \), (5.1) and (5.2) hold if and only if there exists a matrix \( P(t) \) of appropriate dimension, symmetric and of bounded variation on \( [t_0, t_f] \) such that

\[
\begin{align*}
(a) \quad & \hat{P}(t_f) \preceq \hat{S} \\
(b) \quad & \int_{t_1}^{t_2} \left[ \begin{array}{c}
\hat{u}^T \\
\hat{u}
\end{array} \right] \left[ \begin{array}{cc}
\hat{d}P + (\hat{P}P + \hat{P}Q)dt & (\hat{P}G + \hat{R})dt \\
(\hat{P}G + \hat{R})^T dt & \hat{S} dt
\end{array} \right] \left[ \begin{array}{c}
\hat{u}^T \\
\hat{u}
\end{array} \right] \geq 0
\end{align*}
\]

(5.10)

(5.11)

for all continuous \( \hat{u}(\cdot) \) and for all piecewise continuous \( \hat{u}(\cdot) \)

(c) \( H_{22}(t) \) is symmetric on \( [t_0, t_f] \)

(d) \( S_{22} + H_{22}(t_f) \geq 0 \)
Again, as in the previous sections, the application of Theorem 5.1 and the transformation to standard form may need to be made a number of times, terminating with either a zero dimensional \( \hat{P} \) in which case the original \( P \) would be completely and uniquely identified by a series of equalities such as (5.4) and (5.5), or a problem with \( \hat{P} \) of positive dimension with \( \hat{P} \) nonsingular, or in transforming from nonstandard to standard form a problem with \( G \) and \( H \) zero may arise. For the second case, as we know, (5.10) and (5.11) have a solution \( \hat{P} \) if and only if the Riccati equation (4.3) has no escape times on \([t_0, t_f]\). Moreover, the unique solution \( \hat{P} \) of the Riccati equation is the maximal of many possible solutions of (5.10) and (5.11) - see Theorem II.5.1. Finally, \( \hat{P} \) as calculated from the Riccati equation is connected to the optimal cost via the standard quadratic form. Tracing back to the original control problem, the solution \( \hat{P} \) of (5.1) so generated defines the optimal cost for each \( x(t) \) for problem (4.1).

For the third case when \( G \) and \( H \) are zero, we have noted earlier what \( \hat{P} \) is. Again, one can trace back to a solution \( \hat{P} \) of the original control problem.

6. SUMMARIZING REMARKS

In this chapter, we have given an algorithmic procedure for computing a matrix, the existence of which is guaranteed by the nonnegativity of a certain functional. Indirectly, this gives a procedure for checking the nonnegativity of the functional. Second, we have shown how this algorithm can also be used in computing the optimal performance index and optimal control (the latter possibly not being unique) for linear-quadratic singular optimal control problems. Several key properties of the algorithm are: its capacity to handle vector control problems; its linkage with, on the one hand, other and possibly less complete approaches to the singular control problem, and on the other hand, with the singular time-varying covariance factorization problem; its illumination of singular strips; its disadvantage, viz., a requirement that the ranks of certain matrices remain constant over the interval of interest, and that certain matrices enjoy differentiability properties.

There is another possible approach to the optimal control problem which we have not mentioned to this point. By a standard completion of the square device, one can characterize the optimal control, if it exists, in open-loop form as the solution of a linear Fredholm integral equation which is only of the second kind in case the optimal control problem is nonsingular. A solution procedure for the dual singular problem (arising in detection theory) is studied in [28] and could presumably be modified to deal with the control problem.
APPENDIX III.A

DOLEZAL'S THEOREM

The following statement of Dolezal's Theorem is drawn from [24, 25].

**Theorem III.A.1** Let $A(t)$ be an $r \times r$ matrix with entries possessing continuous $p$-th order derivatives in $[a, b]$, and with rank $A(t) = h$ for all $t \in [a, b]$. Then there exists an $r \times h$ matrix $M(t)$ with entries possessing continuous $p$-th order derivatives in $[a, b]$ and with $M(t)$ nonsingular for all $t \in [a, b]$ such that $A(t)M(t) = B(t) : 0$ for all $t \in [a, b]$. Here, $B(-)$ is an $r \times h$ matrix with rank $B(t) = h$.

If $A(t) = A'(t)$ for all $t$ and $M$ is constructed as above, it is clear that $M'[B : 0]$ must be symmetric, from which it follows that

$$M'(t)A(t)M(t) = \begin{bmatrix} C(t) & 0 \\ 0 & 0 \end{bmatrix}$$

with $C(t)$ of dimension $h \times h$ and nonsingular for all $t$.

If in addition $A(t)$ is nonnegative definite, $C(t)$ is positive definite. There exists a triangular $D(t)$ with entries expressible in terms of the entries of $C(t)$ and inheriting their differentiability property, such that $C(t) = D'(t)D(t)$, [26]. Then with $N(t) = D(t)^{-1}B(t)$, possessing entries with continuous $p$-th order derivatives and nonsingular for all $t \in [a, b]$, one has

$$N'(t)A(t)N(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

APPENDIX III.B

SYMMETRY CONDITION

The result proved here is an extension of a similar result needed for the dual covariance factorization problem [5].

With notation independent of that in the main portion of this chapter, define

$$V(\xi, u) = \left[2z_1S_2z_2 + z_2S_3z_2\right]_{t=t_0}^{t_f} + \int_{t_0}^{t_f} z_2^2 du \quad dt$$

$$+ \int_{t_0}^{t_f} \left[z_1Qz_1 + 2z_1Rz_2 + z_2Rz_2\right] dt$$

(3.1)

with $z_1, z_2$ and $u$ related by

$$z_1(t_0) = \xi_1$$

(3.2)
Assume \( Q, H, R, F \) and \( A \) are continuous matrices on \([t_1, t_f]\). \( Q \) and \( R \) are symmetric and \( A \) is antisymmetric, and \( S_2, S_3, S_4 \) are constant. The dimensions of all quantities are arbitrary provided consistency is maintained.

**Lemma III.B.1:** If \( A(t) \neq 0 \) on \([t_1, t_f]\), there exists a bounded piecewise continuous control \( \bar{u}(\cdot) \) such that \( \forall [0, \bar{u}] < 0 \).

**Proof:** Since \( A \) is antisymmetric, the lemma does not apply for scalar \( u \). Also, if \( A(t) \equiv 0 \) on \([t_1, t_f]\) then by continuity of \( A(t) \) we have \( A(t_{f}) = 0 \).

Therefore, consider \( \sigma \in [t_1, t_f] \), \( k \) and \( j \) distinct indices such that \( a_{kj}(\sigma) \neq 0 \) where \( a_{kj}(\sigma) \) is the \( k-j \)th element of \( A(\sigma) \). Without loss of generality, assume \( a_{kj}(\sigma) > 0 \).

Let the transition matrix associated with \( (B.2) \) and \( (B.3) \) be \( \Phi(t, \tau) \). Then, for any given \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that for all \( t, \tau \) satisfying \( \sigma \leq t, \tau \leq \sigma + \delta \) we have \( ||\Phi(t, \tau) - I|| \leq \varepsilon \) where \( ||\Phi|| = \text{maximum of the norms of the elements of } \Phi \).

Let \( \delta \) be any positive number with \( \delta \leq \delta(\varepsilon) \), and choose the control \( \bar{u}(\cdot) \) to be identically zero except that

\[
\bar{u}_{k}(\tau) = -\cos \omega(\tau - \sigma) \\
\bar{u}_{j}(\tau) = -\sin \omega(\tau - \sigma)
\]

Using \( (B.2), (B.3) \) and \( (B.4) \) and taking \( \xi = 0 \) we can write

\[
\begin{bmatrix}
x_1(t) \\
x_2(t) - \int_{t_0}^{t} \bar{u}(\tau) d\tau
\end{bmatrix} = \begin{bmatrix}
\int_{t_0}^{t} (\Phi(\tau, t) - I) \\
0
\end{bmatrix} dt.
\]

From \( (B.5) \), we conclude that on the interval \([\sigma, \sigma + \delta]\) we have

\[
\begin{align*}
|z_{2k}(t) + \sin \omega(t - \sigma)| & \leq 2\varepsilon n p \\
|z_{2j}(t) + 1 - \cos \omega(t - \sigma)| & \leq 2\varepsilon n p \\
|z_{2k}(t)| & \leq 2\varepsilon n p & \forall k,j \\
|z_{1r}(t)| & \leq 2\varepsilon n p & 1 \leq r \leq \dim z_1
\end{align*}
\]

where

\[
p = \dim z_1 + \dim z_2 \\
n = \text{smallest integer greater than or equal to } \frac{2\varepsilon d}{\pi}.
\]

Now consider \( z_1 \) and \( z_2 \) on the interval \([\sigma + \delta, t_f]\). Set \( z' = [z_1' z_2'] \) and
\text{Let } K_1 = \max_{t, \tau \in [t_0, t_f]} |\Phi(t, \tau)|; \text{ this quantity is well defined by the continuity of } \Phi(t, \tau). \text{ If we choose } \omega \text{ such that }

\omega^2 = 2\pi

\text{then from (B.6) we see that } |z_x(\sigma, \delta)| \leq 8\epsilon \delta \text{ for every component of the vector } z(\sigma, \delta). \text{ Finally, since } z(t) = \Phi(t, \sigma, \delta)z(\sigma, \delta) \text{ on } [\sigma, t_1], \text{ we have }

|z_x(t)| \leq 8\epsilon \delta K_1 \text{ on } [\sigma, t_1]. \tag{B.8}

\text{Just as we derived the inequalities (B.6) from (B.5) we have }

\left| \int_{t_0}^{t} z_x^2 \omega^2 \, dt + z_{k_j}(0)[\omega^2 - \sin \omega \delta] \right| \leq K_0 \epsilon \omega \delta \tag{B.9}

\text{where } K_0 \text{ is a constant. Some care needs to be taken with the bound in (B.9). In particular, } \delta \text{ needs to be chosen so that } |a_{k_j}(t) - a_{k_j}(\sigma)| \leq \delta \text{ for } \sigma \leq t \leq \sigma + \delta; \text{ this causes no problems since } A \text{ is continuous and } \delta \text{ is arbitrary. Setting } \omega^2 = 2\pi \text{ (B.9) becomes }

\left| \int_{t}^{t_f} z_x^2 \omega^2 \, dt + 2\pi a_{k_j}(\sigma) \right| \leq 2\pi \epsilon \delta K_0 \tag{B.10}

\text{From (B.9) the first term on the right side of (B.1) is upper bounded by } K_0 \epsilon^2 \text{ for some constant } K_0; \text{ from (B.10) the first integral on the right side of (B.1) is upper bounded by } -2\pi a_{k_j}(\sigma) + 2\pi \epsilon \delta K_0; \text{ and from (B.8) and (B.6) the final integral in (B.1) is upper bounded by } K_0 + K_0 \epsilon^2 \text{ for constants } K_0 \text{ and } K_1. \text{ Combining we have }

V[0, \bar{u}] \leq K_0 \epsilon^2 - 2\pi a_{k_j}(\sigma) + 2\pi \epsilon K_0 + K_0 + K_0 \epsilon^2. \tag{B.11}

\text{Because } a_{k_j}(\sigma) > 0 \text{ and } \delta \text{ and } \epsilon \text{ are arbitrarily small, we see that } V[0, \bar{u}] \text{ can be guaranteed negative. This proves the lemma.}

\text{REFERENCES}


CHAPTER IV

DISCRETE-TIME LINEAR-QUADRATIC SINGULAR
CONTROL AND CONSTANT DIRECTIONS

1. INTRODUCTION

In continuous-time, the general linear-quadratic control problem is called singular if the weighting matrix of the controls in the integrand of the cost is singular anywhere within the time interval of interest. In such cases, the Riccati equation naturally associated with this problem is not well defined, so an alternative approach to solving the problem is required.

In Chapter III, it is shown that a singular control problem can (subject to the satisfaction of certain structural and differentiability assumptions) be solved, using the solution of another control problem, again of a linear-quadratic nature and possibly singular, but of lower state and/or lower control space dimension. This reduction procedure is continued until one of three possible terminating problems is obtained: a nonsingular problem, or one with zero state dimension, or one with zero control dimension, each of which can be solved in a straightforward manner. The solution of the originally given control problem can be simply constructed from the solution of the terminating problem and knowledge of the reduction procedure.

Moreover, in carrying out any state dimension reduction, it becomes evident that part of the optimal cost depends on the coefficient matrices of the problem, excluding the terminal weighting matrix, in an identifiable way.

In contrast, it is well known that for any discrete-time linear-quadratic optimal control problem, regardless of the singularity or otherwise of any matrices in the cost, the associated Riccati equation is well defined and may be solved in a straightforward manner to yield a solution of the optimal control problem. As such, the concept of singularity is apparently meaningless for discrete-time problems.

As mentioned above, one possible (though not necessary) consequence of singularity in continuous-time is the identification of blocks of the matrix defining the performance index from knowledge of the coefficient matrices only. It is this idea that has been studied in the context of the discrete-time linear-quadratic control problem and the associated Riccati equation in [1, 2]. There the concept of a constant direction is introduced, being a direction in which the solution of the Riccati equation is, first, constant (except for a finite transient period), and second, independent of the terminal weighting matrix. In [1], for the multi-input constant coefficient case for a restricted class of coefficient matrices, (the nature of the restriction being described in Section 3), the approach is to identify all constant directions of all orders and then to reduce the dynamic order of the Riccati equation away from the final transient period. Also introduced in [1] is the idea of a state being taken to zero optimally on an interval as a way of characterizing a constant direction. In
this paper, this idea also proves to be useful. Work related to that in [1] was reported in [3] where the dual of the control problem, that of covariance factorization, is studied. These authors considered only scalar covariances, which corresponds to assuming a scalar input in the control problem formulation. Nonstationary covariances are studied, and to handle the time-varying case, the idea of a constant direction is extended and called a degenerate direction.

The contributions of this chapter are two-fold. Primarily, we study in detail the question of characterizing all constant directions for an arbitrary discrete-time linear-quadratic control problem. This generalizes corresponding results in [1] though our methods are somewhat different. Secondarily, we obtain results analogous to those for the general continuous-time singular linear-quadratic control problem. In particular, from the coefficient matrices we construct a matrix which plays the role of the control weighting matrix in continuous-time. That is, singularity of this matrix allows the originally given problem to be solved in terms of the solution of another control problem but with lower state and/or lower control dimension. Moreover, a reduction procedure which terminates in one of three control problems can be defined. Each of these terminating problems, a nonsingular problem, a zero state dimension problem and a zero control dimension problem can be solved simply; the solution to the originally given problem can be computed by tracing back through the reduction procedure.

An outline of the chapter is as follows. Section 2 reviews the general discrete-time control problem. In Section 3, we characterize j-constant directions completely in terms of being taken to zero optimally in j steps for some terminal weighting matrix and as vectors in the range of a certain matrix. Section 4 considers the problem of control space dimension reduction while in Section 5, state dimension reduction is demonstrated provided j-constant directions exist. The results of the preceding sections are combined in Section 6 to derive an algorithm which solves any singular discrete-time control problem, making use of computational simplifications arising from constant direction existence. Section 7 contains brief concluding remarks, including mention of connections with the Silverman structure algorithm [4], and the recently developed "fast" methods for Riccati equation solution of Kailath and his coworkers, [5, 6].

The majority of the results in this chapter appeared in [7].

2. LINEAR-QUADRATIC CONTROL IN DISCRETE TIME

In this section, we review basic results concerning the discrete-time, time-invariant, linear-quadratic control problem. On the interval \([K, N]\) consider the dynamic system

\[
\begin{align*}
x(i+1) &= Ax(i) + Bu(i), & i = K, \ldots, N-1 \\
x(K) &= x_K
\end{align*}
\]  

(2.1)
where $x(i)$ is an n-dimension state vector, $u(i)$ is an m-dimension control vector, and $A$ and $B$ are matrices with dimensions consistent with $x$ and $u$. Let $u^{N-1}_K$ be a control sequence $u(K), u(K+1), \ldots, u(N-1)$ and assign to the initial state $x_K$, control sequence $u^{N-1}_K$ and terminal weighting matrix $S$, the cost functional

$$V_{N-K}[x_K, u^{N-1}_K, S] = x^T(N)Sx(N) + \sum_{i=K}^{N-1} \{x^T(i)Qx(i) + 2x^T(i)Cv(i) + v^T(i)Hv(i)\}$$

(2.2)

where $x(K), x(K+1), \ldots, x(N)$ is the trajectory of (2.1) generated by $u^{N-1}_K$ and $x_K$. The matrices $Q, C, R$ and $S$ are of appropriate dimension, with $Q$, $R$ and $S$ symmetric. (No definiteness assumptions are yet made). Finally, define

$$V^*_N[x_K, S] = \inf_{u^{N-1}_K} V_{N-K}[x_K, u^{N-1}_K, S].$$

(2.3)

Consider a family of dynamics (2.1) and cost functionals (2.2) indexed by $K = 0, 1, \ldots, N-1$. The discrete-time linear-quadratic control problem can now be stated in two parts.

1. Find necessary and sufficient conditions for $V_{N-K}[x_K, u^{N-1}_K, S]$ to be bounded below, independently of $u^{N-1}_K$, for each $x_K$ and each $K = 0, \ldots, N-1$.

2. If these conditions hold, determine $V^*_N[x_K, S]$ and, if it exists, a control sequence $u^{N-1}_K$ depending on $x_K$ such that

$$V^*_N[x_K, S] = V_{N-K}[x_K, u^{N-1}_K, S]$$

for each $K = 0, \ldots, N-1$.

As a matter of terminology, we shall say that the control problem has a solution on $[0, N]$ for terminal weighting matrix $S$ whenever (2.4) has a solution.

The solution to this problem is well-known, and depends largely on the following lemma which we shall not prove here. The notations $N(X), X^\theta, X \geq 0 (> 0)$ denote respectively null space, Moore-Penrose pseudo-inverse, nonnegative (positive definiteness of the matrix $X$).

**Lemma IV.2.1:** Consider the quadratic form $q(y, v) = y^T\Gamma y + 2y^T\eta v + v^T\eta v$ for matrices $\Gamma = \Gamma^T$, $\eta = \eta$ and $H$ and vectors $y$ and $v$ of arbitrary but consistent dimensions, and define $q^*(y) = \inf_v q(y, v)$. The following three conditions are equivalent:

(a) $q^*(y) > -\infty$ for each $y$

(b) $G \geq 0, \eta(v) \in \eta(H)$
(c) there exists a symmetric matrix $X$ such that

$$
\begin{bmatrix}
X & H \\
H^T & C
\end{bmatrix} \succ 0.
$$

Moreover, if any one of the above conditions holds, then (c) is satisfied by $X^* = F - H^*H$. In addition, $X^* \succeq X$ for any other $X$ satisfying (c). Finally, if for each $y$ we set $v = -H^*y$, then

$$
q^*(y) = q(y, v^*) = y^*X^*y.
$$

Using this lemma and the next definitions, we shall be able to state the solution of (2.4).

**Definition:** The set of allowable weighting matrices, denoted by $S$, is the set of $n \times n$ symmetric matrices $P$ such that $B^*PB + R \succeq 0$ and $N(B^*PB + R) \subseteq N(A^*PB + C)$. (The reason for this definition will become clear below).

**Remark 2.1:** The set $S$ is in a sense open at one end. Specifically, if $S_0 \in S$ and $S \supset S_0$, then $S \in S$. To show this we must verify two properties of $S$,

(a) $B^*SB + R \succeq 0$ and (b) $N(B^*SB + R) \subseteq N(A^*SB + C)$. Now (a) follows readily by writing $B^*SB + R$ as $(B^*S_0B + R) + B^*(S - S_0)B$. For (b), assume $u \in N(B^*SB + R)$. Then since $B^*S_0B + R \succeq 0$ and $S \supset S_0$, we conclude that $u \in N(B^*SB + R)$, and $(S - S_0)Bu = 0$. However $S_0 \in S$. Therefore $u \in N(A^*SB + C)$, and the result follows by writing $(A^*SB + C)u = (A^*S_0B + C)u + A^*(S - S_0)Bu$.

We can now write down the solution to problem (2.4) as

**Theorem 2.2.1:** The control problem has a solution on $[0, N]$ for terminal weighting matrix $S$ if and only if the $n \times n$ symmetric matrix $P(i+1) \in S$ for each $i = 0, \ldots, N-1$, where $P(i)$ is defined by the recursion

$$
$$

$$
i = 0, \ldots, N-1
$$

(2.5)

$$
P(N) = S.
$$

If $P(i)$ is so defined, then the control sequence $u^{N-1}_x$ defined by

$$
u^*_x(i) = -[B^*P(i+1)B + R]^{1/2}[A^*P(i+1)B + C]^*u(i),
$$

$$
i = K, \ldots, N-1
$$

(2.6)

achieves the infimum for each $K = 0, \ldots, N-1$. That is, with $u^{N-1}_x = [u^*_x(K), \ldots, u^*_x(N-1)]$ we have
A simple application of Lemma 2.1 and the Principle of Optimality gives the result.

Remarks 2.2: 1. Should \( P(i+1) \not\in S \) for some \( i = 0, \ldots, N-1 \) then \( v^*_K[x_i, S] = \infty \) for at least one \( x_i \).

2. Note that \( P(i) \) is also a function of the terminal weighting matrix \( S \in S \). When we wish to make this dependence explicit, we shall write \( P(i, S) \) for \( P(i) \).

3. If \( B^TP(i+1)B + R \) is nonsingular for each \( i = 0, \ldots, N-1 \) then the optimal control sequence \( u^{N-1} \) defined by (2.6) is unique and so, therefore, is the corresponding optimal trajectory of (2.1) for \( K = 0 \) generated by \( u^{N-1}_0 \) and \( x_0 \). If for some \( i = 0, \ldots, N-1 \), \( B^TP(i+1)B + R \) is singular, then the optimal control at time \( i \) is not uniquely defined, \( \text{i.e., alternatives to (2.6) are possible} \) and the optimal state \( x^*(i+1) \) of the trajectory is also not unique. However, the definition of \( u^*(i) \) in (2.6) is the optimal control at time \( i \) of minimum norm, and as such is uniquely defined. For later reference, we call a control sequence \( u^{N-1}_K \) with controls defined by (2.6) the minimum norm optimal control sequence.

4. If \( P(i+1) \in S \), then the matrix \( P(i) \) defined in equation (2.5) may also be characterized as the maximal symmetric solution of

\[
\begin{bmatrix}
A^TP(i+1)A + Q - P(i) & A^TP(i+1)B + C \\
B^TP(i+1)A + C^T & B^TP(i+1)B + R
\end{bmatrix} \geq 0.
\]  

This fact will be used in the development of the state-space reduction procedure later in the chapter.

The preceding discussion amounts to the standard approach for solving the linear-quadratic control problem, in that \( P(N-1, S) \) is determined by separate sequential minimizations over each of the controls \( u(N-1), u(N-2), \ldots, u(N-1) \) respectively. However, it is possible to calculate \( P(N-1, S) \) directly by minimizing with respect to an extended control vector \( u^{(N-1)}_K \triangleq [u^{(N-1)} \ldots u^{(N-1)}] \), provided we suitably define the dynamics and the cost.

In particular, consider \( i = 2 \) and the interval \([N-2, N]\). From (2.1) and (2.2) for \( K = N-2 \) we can write the dynamics as

\[
x(N) = A^{(2)}x(N-2) + B^{(2)}u^{(2)}(N-2)
\]  

where \( A^{(2)} \triangleq A^2 \) and \( B^{(2)} \triangleq [A B] \) and the cost as

\[
v^{(2)}[x(N-2), u^{N-1}_{N-2}, S] = x^{*}(N)Sx(N) + \{x^{*}(N-2)Q^{(2)}x(N-2) + 2x^{*}(N-2)C^{(2)}u^{(2)}(N-2) + u^{*}(N-2)B^{(2)}u^{(2)}(N-2)\}
\]  

(2.10)
where
\[ Q(2) \triangleq A^\top Q A + Q \]
\[ C(2) \triangleq [A^\top Q B + C] A^\top C \]
\[ R(2) \triangleq \begin{bmatrix} B^\top Q B + R & B^\top C \\ C^\top B & R \end{bmatrix}. \]

Consequently, if the control problem associated with (2.1) and (2.2) has a solution on \([0, N]\) for terminal weighting \(S\), then
\[ P(N-2, S) \]

is given by
\[ P(N-2, S) = A^\top(2) S A(2) + Q(2) - [A^\top(2) S B(2) + C(2)] [B^\top(2) S B(2) + R(2)]^{1/2} [A^\top(2) S B(2) + C(2)]^\top. \]

(2.11)

For convenience, we call the above the 2nd stage control problem, with terminal weighting \(S\).

In general, suppose the \(j\)th stage control problem with terminal weighting \(S\) has been defined in terms of quantities \(A_j, B_j, C_j, R_j\) and \(u_j(i); \) then the \((j+1)\)th stage control problem with terminal weighting \(S\) is defined in terms of the quantities
\[ A_{j+1} \triangleq A_j A, \quad B_{j+1} \triangleq [A_j B] B_j, \quad Q_{j+1} \triangleq \tilde{A} Q_j \tilde{A} + Q \]
\[ C_{j+1} \triangleq [A^\top Q_j B + C_j A^\top C_j] A_{j+1}, \quad R_{j+1} \triangleq \begin{bmatrix} B^\top Q_j B + R & B^\top C_j \\ C^\top B_j & R_j \end{bmatrix} \]

(2.12)

\[ u_{j+1}(i+1) \triangleq [u_j(i)]^* + u^*(i+j). \]

Now it is clear that if the control problem associated with (2.1) and (2.2) has a solution on \([0, N]\) then the \(j\)th stage control problem with terminal weighting \(P(i,j, S)\) defined by
\[ x(i+j) = A_j x(i) + B_j u_j(i) \]

(2.13)

and
\[ v_j[i, u_j(i), P(i+j), S] = x^*(i+j) P(i+j, S) x(i+j) + [x^*(i) Q_j x(i) + 2x^*(i) C_j u_j(i)] + u^*(i+j) R_j u_j(i) \]

(2.14)

has a solution for all \([i, i+j] \subset [0, N]\) and conversely. Moreover,
\[ P(i, S) = A'(i, S)A(i, S) + Q(i, S) \]
\[ - [A'(i, S)B(j, S)C(j) + R(j)]^\# [A'(i+j, S)B(j) + C(j)]^\# \]

(2.15)

with an optimal control \( u^*(i) \) obviously equivalent to the optimal sequence \( u^*_{i+j-1} \). Moreover, existence of the optimal control implies the following replacement of the requirement that \( P(i+1) \in S \):

\[ P'(i+j, S)B(j) + R(j) \leq 0 \]

(2.16)

Existence also implies that \( P'(i+j, S)B(j) + R(j) \geq 0 \) but we shall make more use of (2.16).

By setting \( i + j = N \), (2.15) becomes an equation for \( P(N-j, S) \). Of course, calculation of \( P(N-j, S) \) using the \( j \)-th stage formulation is highly inefficient. Nevertheless, it is of theoretical interest, in that, as will later be evident, the \( 1 \)-constant directions of the \( j \)-th stage control problem are identical to the \( j \)-constant directions of the original control problem. (See the next section for a definition of \( j \)-constant directions).

3. CONSTANT DIRECTIONS - BASIC PROPERTIES

As stated in the introduction, we are interested in those directions in which the Riccati equation (2.5) is completely determined by the coefficient matrices (save for a possible transient period) and at the same time is independent of the terminal weighting matrix \( S \). If such directions exist, the Riccati equation is really of lower dynamic order than it appears and solving (2.5) recursively involves carrying out certain redundant calculations at each step of the recursion.

As in [1] we formalize the notion of constant directions in the following way:

**Definition:** Suppose \( 1 \leq j \leq N-1 \). The \( n \)-vector \( \alpha \) is called a \( j \)-constant direction of (2.5) on \([0, N]\) if and only if \( P(i, S)\alpha \) is independent of those \( S \) in \( S \) for which a solution to the optimal control problem exists and of \( i \) for \( i = 0, 1, ..., N-j \).

Denote the set of \( j \)-constant directions by \( I_j \). Clearly, each \( I_j \) is a finite-dimensional linear space and is therefore completely described by a finite set of linearly independent vectors. Moreover, for \( 1 \leq j \leq N-2 \), it is immediate that \( I_{j+1} \supset I_j \).

It is of interest to have available various alternative characterizations of \( j \)-constant directions, and our main task in this section is to develop these alternative characterizations. One such characterization will involve the notion of a state being taken to zero optimally in \( j \) steps, which we now define precisely.

For each \( j \) with \( 1 \leq j \leq N-1 \), consider the control problem with dynamics and
cost as in (2.1) and (2.2), restricted to the interval \([N-j, N]\), with coefficient matrices \(A, B, Q, C\) and \(R\) and terminal weighting matrix \(S\). Suppose that this control problem has initial state \(x(N-j) = \alpha\).

**Definition:** We say that \(\alpha\) can be taken to zero optimally in \(j\) steps for \(S = \tilde{S}\) if and only if there exists a control sequence \(u_{N-j}^{N-1}\) such that the corresponding trajectory of (2.1) satisfies \(x(N-j) = \alpha\) and \(x(N) = 0\), and \(v_j[\alpha, u_{N-j}^{N-1}, \tilde{S}] = v_j^*[\alpha, \tilde{S}]\). (An equivalent definition in terms of the \(j\)th stage control problems of (2.13) and (2.14) is obviously possible).

In [1], it is shown that the \(j\)-constant directions are completely characterized by the property of being taken to zero optimally in \(j\) steps for zero terminal weighting matrix when \(C = 0, R = 0, Q > 0, S > 0\) and \(A\) is invertible. Moreover, the \(1\)-constant directions can not only be characterized in this form but also simply as the range of the matrix \(A^{-1}B\), while the \(j\)-constant directions are related to the null space of a matrix \(W_j\) (defined in [1]). The remainder of this section is concerned with the extensions of these results to control problems with coefficient matrices which are arbitrary, save that a solution to the control problem exists on \([0, N]\) for some terminal weighting matrix \(S \in S\). Theorem 3.1 below contains the main result; it is preceded by several lemmas.

Define the \((n+jm) \times (n+jm)\) dimensional matrix

\[
A_{(j)} = \begin{bmatrix} A_{(j)} & B_{(j)} \\ C_{(j)} & R_{(j)} \end{bmatrix}.
\tag{3.1}
\]

This matrix proves a major tool in analyzing constant directions, and we show first that vectors in its nullspace define constant directions.

**Lemma IV.3.1:** Suppose \(w_{(j)} \in N(A_{(j)})\) and partition \(w_{(j)}^*\) as \([\alpha^* \beta_{(j)}^*]\)

where \(\alpha\) is \(n\)-dimensional and \(\beta_{(j)}\) is \(jm\)-dimensional. Then if the control problem has a solution on \([0, N]\) for terminal weighting matrix \(S\), we have

\[
P(i, S)\alpha = Q_{(j)}\alpha + C_{(j)}\beta_{(j)} \text{ for each } i = 0, 1, \ldots, N-j
\tag{3.2}
\]

so that \(\alpha\) is a \(j\)-constant direction.

**Proof:** Since \(w_{(j)} \in N(A_{(j)})\) we have

\[
A_{(j)}\alpha + B_{(j)}\beta_{(j)} = 0 \text{ and } C_{(j)}\alpha + R_{(j)}\beta_{(j)} = 0.
\tag{3.3}
\]

Postmultiply (2.15) by \(\alpha\), and use (3.3) to obtain
By hypothesis, the control problem has a solution for terminal weighting $S$. Therefore conditions (2.16) hold for each $[i, i+j]$ and consequently does the identity

$$
A'_{(j)} P(i+j, S) B(j) + C(j) = A'_{(j)} P(i+j, S) B(j) + C(j) \beta(j)
$$

By hypothesis, the control problem has a solution for terminal weighting $S$. Therefore conditions (2.16) hold for each $[i, i+j] \in [0, N]$ and so consequently does the identity

$$
A'_{(j)} P(i+j, s) B(j) + C(j) B(j) P(i+j, s) B(j) + C(j) \beta(j)
$$

Substituting (3.4) in the above we obtain (3.2).

Remarks 3.1: 1. Suppose that $x(N-j) = a$ and $u(N-j) = \beta(j)$. Then this control causes $x(N) = 0$, in view of the first equation of (3.3). Further, the performance index $V_j(x(N-j), u_{N-j}, S)$ can be evaluated, using (2.14) with $i+j = N$ and the fact that $x(N) = 0$, as

$$
v_j(x(N-j), u_{N-j}, S) = \alpha^0 q(j) \alpha + 2 \alpha^0 C(j) \beta(j) + \beta^0 \Gamma(j) \beta(j)
$$

The second equality comes from (3.2) and (3.3). Thus not only does $\alpha$ have the significance of being a constant direction, but $\beta(j)$ has the significance of being a control sequence which is optimal, and which drives $x(N-j)$ to $x(N) = 0$.

2. It is possible that there could exist two different vectors $\tilde{u}(j) = [\alpha^0 \beta^0(j)] \beta(j)$ and $\tilde{u}(j) = [\alpha^0 \beta^0(j)] \beta(j)$ both in $H[B^0(j)]$. Then $\beta(j) - \tilde{\beta}(j) \in H[B^0(j)]$. In such a circumstance, the same performance index results from using $\beta(j)$ and $\tilde{\beta}(j)$.

As it turns out, this nonuniqueness implies that there exist superfluous controls, a point we explore more fully in the next section.

If the control problem has a solution on $[0, N]$ for terminal weighting $S_0$, it also has a solution on $[0, N]$ for each terminal weighting $S$ satisfying $S \geq S_0$. Thus, from Lemma 3.1 and under the hypothesis of this lemma, we can assert that $V_k^*(a, S) = V_k^*(a, S_0)$ for each $S \geq S_0$ and any $k = j, \ldots, N$, with the optimum being achievable using the control sequence defined by $\beta(j)$. The next lemma explores the consequences of such an equality of performance indices for a state $a$ and some $j$.

Lemma IV.3.2: Suppose that $V_j^*(a, S) = V_j^*(a, S_0)$ for a state $a$ and all $S \geq S_0$, for some $S_0 \in S$. Then for $S > S_0$, with the overbar indicating the
terminal weighting matrix $\mathcal{S}$ is being considered, we have

(a) $x^+(N) = 0$ for any optimal control sequence $u^*_{N-j}$ associated with $\mathcal{S}$,

(b) $v^*_j[a, S] = v^*_j[a, u^*_{N-j}, S] = v^*_j[a, S_0]$ for all $S \geq S_0$.

**Proof:** By hypothesis, $v^*_j[a, S] = v^*_j[a, S_0 \leq u^*_{N-j}, S]$ for any control sequence $u^*_{N-j}$, and therefore for any optimizing control sequence $u^*_{N-j}$ associated with terminal weighting $\mathcal{S}$. Now,


Consequently, $x^+(N) = 0$ since $\mathcal{S} > S_0$. This proves (a). Part (b) follows trivially.

Part (a) and (b) of the above lemma yield:

**Corollary 3.1:** Suppose that $v^*_j[a, S] = v^*_j[a, S_0]$ for a state $a$ and all $S \geq S_0$ for some $S_0 \in \mathcal{S}$. Then $a$ can be taken to zero optimally in $j$ steps for all $S \geq S_0$.

It is important to note that in corollary 3.1 the minimum norm optimal control is not guaranteed to take $a$ to zero optimally in $j$ steps for $S \geq S_0$ but $S \nexists S_0$. This does not invalidate the result of Corollary 3.1, however, because Definition 3.2 only requires some optimal control to take $a$ to zero in $j$ steps, not necessarily the minimum norm control.

By way of a converse to this corollary, we have the following:

**Lemma IV.3.3:** Suppose that $a$ can be taken to zero optimally in $j$ steps for some $S_0 \in \mathcal{S}$. Then $a$ can be taken to zero optimally in $j$ steps for all $S \geq S_0$.

**Proof:** It is easily checked that the control taking $a$ to zero optimally for $S_0 \in \mathcal{S}$ also is optimum for all $S \geq S_0$.

We have shown that the null vectors of $A_{(j)}$ determine states which can be taken to zero optimally. Let us now check the reverse of this idea.

**Lemma IV.3.4:** Suppose that $a$ can be taken to zero optimally in $j$ steps for all $S \geq S_0$ where $S_0$ is some element of $\mathcal{S}$. Then there exists a vector $\mathcal{S}$ such that $v'_{(j)} = [a' \beta'_{(j)}]$.

**Proof:** With terminal weighting matrix $\mathcal{S} > \mathcal{S}$, Lemma 3.2 implies that $a$ can be taken to zero optimally in $j$ steps with any optimizing control $u^*_{N-j}$ or equivalently $v^*_{(j)}$. In particular, one such optimizing control is the minimum norm control

$$u^*_{(j)}(N-j) = -[R_{(j)} \mathcal{S}^*_{(j)} + R_{(j)} \beta_{(j)} \mathcal{S}^*_{(j)} + C_{(j)}] a.$$  

and therefore since $x^+(N) = 0$ we have

$$A_{(j)} a - R_{(j)} \beta_{(j)} \mathcal{S}^*_{(j)} + R_{(j)} \beta_{(j)} \mathcal{S}^*_{(j)} + C_{(j)} a = 0.$$  

Now, $\bar{S} \in S$ and so the identity (3.4) holds for $P(i+j, S) = \bar{S}$, which together with (3.6) implies

$$C(j)^a - R(j)^a [B(j)^a \bar{S}_j(j) + R(j)]^a [A(j)^a \bar{S}_j(j) + C(j)]^a = 0. \quad (3.7)$$

Hence, with $\beta(j) = \alpha(j)(N-j)$, (3.6) and (3.7) become $\Lambda(j)^a + B(j)^a \beta(j) = 0$ and $C(j)^a = R(j)^a \beta(j) = 0$, completing the proof of the lemma.

We can summarize the preceding results as follows:

**Theorem IV.3.1:** Suppose that the solution to the control problem exists on $[0, N]$ for some terminal weighting $S$. Then the following statements are equivalent.

(a) $a$ is a $j$-constant direction of (2.5) on $[0, N]$.

(b) $a$ can be taken to zero optimally in $j$ steps for all $S \in S$.

(c) There exists $w(j) \in N(A(j))$ with $w^a(j) = [a^a S^j(j)]$.

(d) [restricted form of (a)]. For some $S_0 \in S$ and all $S \supset S_0$, $V^a_j[a, S] = V^a_j[a, S_0]$.

(e) [restricted form of (b)]. $a$ can be taken to zero optimally in $j$ steps for some $S_0 \in S$.

Moreover, should any of the above hold, any optimal control associated with an $S \in S$ such that $S - n^j \in S$ for some $n > 0$ takes $a$ to zero.

**Proof:** The implications (a) $\Rightarrow$ (b), (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a) follow from Corollary 3.1, Lemma 3.3, Lemma 3.4 with Corollary 3.1, and Lemma 3.1 respectively. Finally, (b) $\Rightarrow$ (a) is trivial. The final part of the theorem is a consequence of Lemma 3.2. This completes the proof.

We also have the following simple consequence of parts (a) and (c) of Theorem 3.1 and Remark 3.1.2.

**Theorem IV.3.2:** Suppose the control problem has a solution on $[0, N]$ for some terminal weighting $S$. Then the space of $j$-constant directions $I_j$ is the range of $V^a_j$, where $V^a_j = [V^a_{j1}, V^a_{j2}]$ is a basis matrix for $N(A(j))$. Moreover, the dimension of $I_j$ equals $s_j - p_j$ where $s_j$ is the nullity of $A(j)$ and $p_j$ is the nullity of $[B^a(j), \bar{B}^a(j)]^a$.

In this section, we have not singled out 1-constant directions for special attention. This we shall do in later sections, since it is the 1-constant directions that are most easily found [clearly $N(A(j))$ is easier to compute than $N(A(j))$] and, as it turns out, $j$-constant directions can be found by computing 1-constant directions for a collection of problems.

Our next immediate task however is to analyse the issue of superfluous controls.
In this and the next section, we shall concentrate on the situation where 1-
constant directions exist. In the previous section we showed in Theorem 3.2 that the
number of linearly independent 1-constant directions \( k \) is equal to the dimension
\( n \) of \( N(A) \) less the dimension \( p \) of the null-space of \( \begin{bmatrix} \mathcal{S} & R' \end{bmatrix} \). In this section,
we argue that if \( k < s \) or \( p > 0 \), then we can find a basis of the control space such
that the dynamics (2.1) and the cost (2.2) are independent of \( p \) components of the
control vector \( u \). Consequently the solution of the control problem on \([0, N]\) can
be shown to be equivalent to the solution of a control problem on \([0, N]\) with the
same state basis but of lower control space dimension.

For \( p = 0 \), no reduction of the control space dimension will be possible. For
the present, assume that \( 0 < p < m \). We show that the dimension of the control space
can be reduced by \( p \). (The case \( p \geq m \) will be treated subsequently). Let \( W \) be a
basis matrix for \( N(A) \) and partition \( AW = 0 \) as

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} A & B_1 & B_2 \\ C_1 & R_{11} & R_{12} \\ C_2 & R_{21} & R_{22} \end{bmatrix} \\
W &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}
\end{align*}
\]

with the dimensions of the various submatrices as shown.

With a basis change of \( N(A) \) via column operations and a basis change of the
control space via row operations on the final \( n \) rows of \( W \), we transform \( W \) to a
more suitable form. Specifically, since \( [W_{11}, W_{12}] \) is an \( m \times s \) matrix of column
rank \( k \) there exists an \( m \times s \) nonsingular matrix \( T \) such that \( [W_{11}, W_{12}]T = [\tilde{W}_{11}, 0] \)
with \( \tilde{W}_{11} \) of full column rank \( k \). With \( T \) change the basis of \( N(A) \) to \( \tilde{W} = WT \)
where

\[
\begin{bmatrix}
\tilde{W}_{11} \\
\tilde{W}_{21} \\
\tilde{W}_{31} \\
\tilde{W}_{32}
\end{bmatrix}
\]

Since \( \tilde{W} \) has full column rank, so must \( [\tilde{W}_{21}, \tilde{W}_{32}] \). Therefore there exists an \( m \times m \)
nonsingular matrix \( U \) such that \( U[\tilde{W}_{21}, \tilde{W}_{32}] = [0, I_p] \). With \( U \), change the
control space basis. Dropping the bar notation, we have a basis of the control space
such that a basis matrix \( W \) of \( N(A) \) is given by

\[
W = \begin{bmatrix} W_{11} & 0 \\ W_{21} & 0 \\ W_{31} & I_p \end{bmatrix}
\]
This result might also be obtained by directly constructing a basis of \( N(A) \) as
\[
\begin{bmatrix}
  v_1^T, & \ldots, & v_s^T, & 0, & \ldots, & 0
\end{bmatrix}.
\]
That such a basis exists follows from the fact that \( s = 2 + p \).

From the special form of \( W \), we see that \( AW = 0 \) implies
\[
B_2 = 0, \quad R_{12} = 0, \quad R_{22} = 0.
\]

Assume that the control problem has a solution on \([0, N]\) for terminal weighting matrix \( S \). Then because \( \mathbb{N}[B'SB + R] \subset \mathbb{N}[A'SB + C] \) and every \( m \)-vector of the form \( [0, v^T] \) with \( v \) a \( p \)-vector lies in \( \mathbb{N}[B'SB + R] \), it follows that \( C^Tv = 0 \). Because \( v \) is arbitrary, \( C_2 = 0 \).

Now partition the controls \( u \) as \( [u_1^T, u_2^T]^T \) where \( u_2 \) has dimension \( p \). The dynamics (2.1) then become
\[
x(i+1) = Ax(i) + B_1u_1(i) \quad i = 0, \ldots, N-1
\]
\[
x(0) = x_0
\]
and the cost (2.2) becomes
\[
V[x_N, u_{N-1}, s] = x^*(N)Sx(N)
+ \sum_{i=0}^{N-1} \{x^*(i)Qx(i) + 2x^*(i)C_1u_1(i) + u_1^2(i)R_{11} + u_2^2(i)R_{22}\}.
\]

Hence, the control problem with dynamics (2.1) and cost (2.2) has a solution on \([0, N]\) if and only if the control problem with dynamics (4.4), cost (4.5) and control space dimension \( m-p \) has a solution on \([0, N]\). Moreover, the solution of these two control problems are simply related, see Lemma 4.1 below.

It remains to consider the possibility \( p = s - 2 \geq m \). Since \( p \) is the dimension of the nullspace of \( [B' R] \) and \( [B' R] \) has \( m \) columns, we must have \( p = m \) and \( B = 0, R = 0 \). Minor modification of the argument applying for \( p < m \) yields \( C = 0 \) (assuming an optimal solution exists). We see then that the control vector \( u \) has no effect on the dynamics (2.1) or the cost (2.2); the control problem is trivial on \([0, N]\).

The results of the section are summarized as

Lemma IV.4.1: Assume that the dimension of \( N(A) \) is \( s \), that \( N(A) \) has basis matrix \( W \) with first \( a \) rows \( W_1 \), that the rank of \( W_1 \) is \( a \); define \( p = s - a \geq 0 \). Then the control problem has a solution on \([0, N]\) for terminal weighting matrix \( S \) if and only if a control problem of identical form but with control dimension \( m-p \) has a solution on \([0, N]\) for terminal weighting matrix \( S \). Moreover, the solutions \( P(i, S) \), \( i = 0, \ldots, N-1 \) of the associated Riccati equations are identical while there exists a basis of the original control space such that any optimizing control at time \( i \), \( u^*(i) \) is given by \( [u_1^*(i), u_2^*(i)] \).
where \( u^*(i) \) is any optimizing control of the lower control dimension problem and \( u^*_1(i) \) is chosen arbitrarily. In particular, for \( u^*_1(i) \) the minimum norm optimal control, \( u^*_1(i) \) is the corresponding minimum norm optimal control for the problem of lower control dimension and \( u^*_2(i) = 0 \).

It is clear from this lemma that when \( p > 0 \), the amount of computation involved in solving the Riccati equation for the original control problem can be reduced to that involved in solving one of the same dynamic order but of lower control dimension. This lemma is analogous to the result that holds for continuous-time singular linear-quadratic control problems, as discussed in the previous chapter.

5. STATE SPACE DIMENSION REDUCTION

In this section, we assume that the matrix \([B' \ R]'\) has full column rank or, equivalently, all superfluous controls have been eliminated. Then we know that the dimension of \( I_1 \) equals the nullity of \( A = A(I) \); let this number be \( k \). We will now show that if there are nontrivial \( i \)-constant directions, i.e. \( L > 0 \), then the state space dimension can be reduced by \( k \), and a related control problem can be defined on the interval \([0, N-1]\) rather than \([0, N]\). Moreover, the solution of the Riccati equation and the optimal controls on \([0, N]\) are simply related to those for the reduced state dimension problem on \([0, N-1]\).

The first stage in the procedure is to choose a basis of the state space to display the constant parts of the matrices \( F(i, S), \ i = 0, \ldots, N-1 \). Choose as a basis of the state space \( \{\alpha_i, \ldots, \alpha_n\} \) arbitrarily, save that \( \alpha_{n-k+1}, \ldots, \alpha_n \) spans \( I_1 \). With this state space basis, we have

\[
P(i, S) = \begin{bmatrix} P_{11}(i, S) & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad i = 1, \ldots, N-1
\]

(5.1)

where \( P_{12} \) and \( P_{22} \) are constant matrices independent of \( S \) and \( i = 1, \ldots, N-1 \).

By our assumption that \([B' \ R]'\) has full column rank, we know that for each \( \alpha_i, \ i = n-k+1, \ldots, n \), there exists a unique \( \beta_i \) such that \( w_i = [\alpha_i' \beta_i]' \in W(A) \). Let \( Z \) be the matrix \([\alpha_{n-k+1} \ldots, \alpha_n]' \). Partition \( Q \) and \( C \) conformably with \( P(i, S) \), i.e., set

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}
\]

(5.2)

In this state space basis the result of Lemma 3.1 can now be restated as

\[
P_{12} = Q_{12} + C_1Z \quad \text{and} \quad P_{22} = Q_{22} + C_2Z
\]

(5.3)

Thus, we have completely identified \( P_{12} \) and \( P_{22} \) as constant parts of \( P(i, S), \ i = 0, \ldots, N-1 \). Moreover, the theory developed in Section 3 says that no part of
If \( P_{11}(i, S) \) is independent of \( i = 0, \ldots, N-1 \) and \( S \in \mathcal{S} \), though some part may be for \( i = 0, \ldots, N-2 \). (If some part of \( P_{11}(i, S) \) were independent for \( i=0, \ldots, N-1 \) \( S \in \mathcal{S} \), there would exist a vector in \( \mathcal{N}(A) \) not in \( I_1 \) and this would be a contradiction).

By virtue of (5.3), the evaluation of \( P(i, S) \) via the Riccati equation (2.5) on \([0, N]\) clearly involves a substantial amount of unnecessary calculation. It would be of interest if we could show that \( P_{11}(i, S), i = 0, \ldots, N-1 \) could be computed via a Riccati equation for \( P_{11} \) rather than \( P \), presumably involving different \( A, B, Q, C \) and \( R \). Recall that the solution to the Riccati equation (3.5) is the maximal symmetric matrix \( P(i, S) \) such that

\[
\begin{bmatrix}
A^*P(i+1, S)A + Q - P(i, S) & A^*P(i+1, S)B + C \\
B^*P(i+1, S)A + C^* & B^*P(i+1, S)B + R
\end{bmatrix} \geq 0
\] (5.6)

for each \( i = 0, \ldots, N-1 \) with \( P(N, S) = S \). With \( A \) partitioned as \( A = [A_1 \ A_2] \) this can then be written as

\[
\begin{bmatrix}
A_1^*P(i+1, S)A_1 + Q_{11} - P_{11}(i, S) + C_1 & A_1^*P(i+1, S)A_2 + A_2^*P(i+1, S)B \\
A_2^*P(i+1, S)A_1 & A_2^*P(i+1, S)A_2 + A_2^*P(i+1, S)B \end{bmatrix} \geq 0.
\] (5.5)

Since the vectors \( w_i \) form a basis matrix of \( \mathcal{N}(A) \) we also have the relations

\[ A_2 + EZ = 0 \] and \[ C_2^* + EZ = 0. \] (5.6)

Premultiply (5.5) by the nonsingular matrix

\[ T = \begin{bmatrix}
I & 0 & 0 \\
0 & I & Z \\
0 & 0 & I
\end{bmatrix}\]

and postmultiply it by \( T^* \). Using (5.3) and (5.6), we obtain in this way another matrix inequality which is equivalent to (5.5):

\[
\begin{bmatrix}
A_1^*P(i+1, S)A_1 + Q_{11} - P_{11}(i, S) & 0 & A_1^*P(i+1, S)B + C_1 \\
0 & 0 & 0 \\
B^*P(i+1, S)A_1 + C_1 & 0 & B^*P(i+1, S)B + R
\end{bmatrix} \geq 0
\]
which is of course in turn equivalent to
\[
\begin{bmatrix}
A_l^T P(i+1, S) A_l + Q_{i1} - P_{11}(i, S) & A_l^T P(i+1, S) B + C_1 \\
B^T P(i+1, S) A_l + C_i^T & B^T P(i+1, S) B + R
\end{bmatrix} \succeq 0. \tag{5.7}
\]

It is now evident that \( P_{11}(i, S) \) is the maximal solution of the inequality (5.7).
However we know that an equivalent definition is provided by a matrix Riccati equation
\[
\hat{P}(i, S) = \hat{L}^T \hat{P}(i+1, S) \hat{L} + \hat{Q}
- [\hat{L}^T \hat{P}(i+1, S) \hat{B} + \hat{C}] [\hat{B}^T \hat{P}(i+1, S) \hat{B} + \hat{R}]^{-1} [\hat{L}^T \hat{P}(i+1, S) \hat{B} + \hat{C}]',
\]
i = 0, \ldots, N-2 \tag{5.8}

where \( \hat{P}(i, S) = P_{11}(i, S), \hat{L} = \hat{A}_{i1} \) (supposing \( A_1 \) is partitioned as \( [A_{i1} \ A^T_{i1}] \)) and the other hat quantities are defined in terms of \( P_{22}, P_{12} \) and the original coefficient matrices \( A, B, Q, C \) and \( R \). (The precise definitions are contained in the Appendix). Finally, we initialize (5.8) with
\[
\hat{P}(N-1, S) = P_{11}(N-1, S).
\]

Let us summarize what we have shown so far. If the Riccati equation (2.5) has a solution on \([0, N]\) for terminal weighting matrix \( S \), and if the null space of \( L \) has dimension equal to \( i \), (or equivalently, surplus controls have been eliminated) then, modulo state and control space basis changes, the solution of (2.5) on \([0, N]\) with \( P(N) = S \) is equivalent to the solution of (5.8) on \([0, N-1]\) with \( \hat{P}(N-1, S) = P_{11}(N-1, S) \) and with \( P_{12} \) and \( P_{22} \) defined by (5.3) for \( i = 0, \ldots, N-1 \).

To complete this section, we point out that (5.8) can be associated with a control problem, closely related to and of the same form as that originally given, but now involving hat quantities and defined on \([0, N-1]\) rather than on \([0, N]\). (This observation allows the relation of optimal controls for the two problems). Suppose that the state and control bases are chosen as described at the start of this section, and partition the state variable \( x \) as \( [x^1 \ x^2] \) with \( x^2 \) of dimension \( i \).

Define new state and control variables
\[
\hat{x} = x^1 \tag{5.9}
\]
\[
\hat{u} = u - Zx^2.
\]

With this notation, it follows from the dynamics of the original system (2.1), from the definitions of \( \hat{A}, \hat{B} \) and from (5.3) and (5.6) that
\[
\hat{x}(i+1) = \hat{A} \hat{x}(i) + \hat{B} \hat{u}(i), \quad i = 0, \ldots, N-2 \tag{5.10}
\]
\[
\hat{x}(0) = \hat{x}_1(0) = \hat{x}_0.
\]
Equation (5.10) constitutes the dynamics of a reduced system on \([0, N-1]\).

In terms of the hat quantities, it is also possible after some manipulation to write

\[
\mathcal{V}[\hat{x}_a, \hat{u}_0^{N-2}, \hat{s}] = \hat{\mathcal{V}}[\hat{x}_a, \hat{u}_0^{N-2}, \hat{s}] + x^r(0)p_x(0) + [u(N-1) + (B'SB + R)^\beta(A'SB + C)x(N-1)]^\gamma(B'SB + R) \times [u(N-1) + (B'SB + R)^\beta(A'SB + C)x(N-1)]
\]  

(5.11)

where

\[
\hat{\mathcal{V}}[\hat{x}_a, \hat{u}_0^{N-2}, \hat{s}] \triangleq \hat{x}^r(N-1)p_x(n-1) + \sum_{i=0}^{N-2} \left( \hat{x}^r(i)q_{i2} + 2\hat{x}^r(i)p_{i2} + \hat{x}^r(i)p_{i2} \right)
\]  

(5.12)

and

\[
\bar{p} = \begin{bmatrix}
0 & Q_{12} + C_sZ \\
Q_{12} + Z_1C_s & Q_{22} + C_sZ
\end{bmatrix}
\]  

(5.13)

Now the right side of (5.11) is the sum of three terms, the first \(\hat{\mathcal{V}}[\hat{x}_a, \hat{u}_0^{N-2}, \hat{s}]\) depends on \(\hat{u}(1), i = 0, \ldots, N-2\) and \(x_0\), the second is constant and depends on \(x_0\) alone, while the third is a function of \(u(N-1)\) and \(x(N-1)\). The independence of these terms together with the nonnegativity of \(B'SB + R\) allows us to conclude that \(\mathcal{V}^{*}[x_0, S]\) is finite if and only if \(\hat{\mathcal{V}}[\hat{x}_a, \hat{s}]\) is finite, with these quantities satisfying

\[
\mathcal{V}^{*}[x_0, S] = \mathcal{V}^{*}[\hat{x}_a, \hat{s}] + x^r(0)p_x(0).
\]  

(5.14)

Altogether then we can solve the control problem on the interval \([0, N]\) in terms of another control problem of lower state space dimension on the interval \([0, N-1]\), whenever there exist nontrivial \(1\)-constant directions. [Knowing \(\hat{u}(0), x_1(0), x_2(0), u(-1)\), we obtain \(u(0)\) from (5.9) and \(x_1(1), x_2(1)\) from \(Ax(0) + Bu(0)\); then knowing \(\hat{u}(1), x_1(1), x_2(1)\), we obtain \(u(1)\) from (5.9), etc.].

We summarize the main result as

**Theorem IV.5.1:** Assume that the dimension of \(N(A)\) is \(k\) and that for any basis matrix \(W\), the rank of the matrix \(W_i\) from the first \(n\) rows of \(W\) is \(k\). Then the original control problem has a solution on \([0, N]\) for terminal weighting matrix \(S \in S\) if and only if the control problem with dynamics (5.10) and cost (5.12) has solution on \([0, N-1]\) for terminal weighting matrix \(\hat{S}\), with the state dimension \(n-k\). Moreover, the optimal costs are related as in (5.14) and any optimizing control sequence \(u_0^{N-1}\) can be related to an optimizing sequence \(u_0^{N-2}\), using \(x_0\) and noting (5.9).
6. TOTAL REDUCTION OF THE PROBLEM

In the last two sections, we have shown how the complexity of a linear-quadratic problem can be reduced in case there are 1-constant directions. In this section, we shall study what happens when there are \( j \)-constant directions for \( j > 1 \). Our principle conclusion is that repeated application of the reduction procedures applicable for 1-constant directions will ultimately eliminate all constant directions of any index. In establishing this conclusion, we shall draw heavily on the general theory of Section 3, as well as the procedures of the last two sections.

To begin with, we shall assume that the set of constant directions of a problem includes 1-constant directions. (Later, we shall show that if there are \( j \)-constant directions for \( j > 1 \), there must also be 1-constant directions. So it transpires that there is really no loss of generality in this assumption). Also, we shall assume that redundant controls have been eliminated. These assumptions mean that we can carry out the reduction procedure of the last section. The first lemma considers the effect of this procedure on the \( j \)-constant directions for \( j > 1 \).

**Lemma IV.6.1:** Suppose that the solution to the control problem exists on \([0, N] \) for some terminal weighting \( S \). Assume that the state coordinate base is chosen such that the reduction procedure of Section 5 may be applied. Then with \( j \geq 2 \), \( \alpha \) is a \( j \)-constant direction of the Riccati equation (2.5) if and only if \( \alpha_1 \) is a \((j-1)\)-constant direction of the Riccati equation (5.8), where \( \alpha = [\alpha_1; \alpha_2]^T \), \( \alpha_1 \) having dimension 1.

**Proof:** Suppose that \( \alpha_1 \) is a \((j-1)\)-constant direction of the Riccati equation (5.8) for \( j \geq 2 \). Then, from the definition of a constant direction, and noting that (5.8) is defined on \([0, N-1] \), we have

\[
\hat{P}(N-1, S)\alpha_1 = \text{constant} \tag{6.1}
\]

independent of the weighting matrices \( S \in S \), for which a solution to a reduced problem exists and all \( i \geq j \). Then for \( \alpha = [\alpha_1; \alpha_2]^T \) for any \( \alpha_2 \) and any \( S \) for which the original problem has a solution

\[
P(N-1, S)\alpha = \begin{bmatrix} P_{11}(N-1, S) & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} I & \hat{P}(N-1, S) \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{6.2}
\]

where \( \hat{S} \) is of the form \( P_{11}(N-1, S) \). Therefore from (6.1), which applies to all \( \hat{S} \in \hat{S} \) and fortiori to those of the form \( P_{11}(N-1, S) \),

\[
P(N-1, S)\alpha = \text{constant} \tag{6.3}
\]
for all \( S \in \mathcal{S} \) and \( i \geq j \). Thus, \( \alpha \) is a \( j \)-constant direction of (2.5).

Conversely, suppose that \( \alpha \) is a \( j \)-constant direction of (2.5) for \( j \geq 2 \). Write \( \alpha = [\alpha_1 \alpha_2] \). Then (6.3) holds for any \( i \geq j \) and \( S \in \mathcal{S} \). Therefore, (6.1) holds for any \( i \geq j \) and \( S \in \mathcal{S} \) of the form \( P_{11}(N-1, S) \). By Theorem 3.1, it is sufficient to show that (6.1) holds for all \( i \geq j \) and all \( S \geq S_0 \), for some \( S_0 \in \mathcal{S} \).

First, we show that if \( S > S_0 \in \mathcal{S} \), then \( P_{11}(N-1, S) > P_{11}(N-1, S_0) \). We argue by contradiction. Suppose that there exists an \( x \neq 0 \) such that

\[
P_{11}(N-1, S)x = P_{11}(N-1, S_0)x.
\]

For any \( \bar{S} \) satisfying \( S_0 < \bar{S} \leq S \), we have \( P_{11}(N-1, S_0) \leq P_{11}(N-1, \bar{S}) \leq P_{11}(N-1, S) \) and therefore \( 0 \leq x'[P_{11}(N-1, \bar{S}) - P_{11}(N-1, S_0)]x = x'[P_{11}(N-1, S) - P_{11}(N-1, S_0)]x \). Hence, \( P_{11}(N-1, \bar{S})x = P_{11}(N-1, S_0)x \) for all \( \bar{S} \) such that \( S_0 \leq \bar{S} \leq S \). Now since \( S > S_0 \), an argument similar to that in Lemmas 3.2 and 3.3 shows that \( P_{11}(N-1, S_0)x = P_{11}(N-1, S_0)x \) for any \( \bar{S} \leq S_0 \), even if \( \bar{S} \leq S \) does not hold. Therefore (6.2) for \( i=1 \) and Theorem 3.1 imply that \([x' \alpha_1]\) is a \( 1 \)-constant direction, which is a contradiction. (In view of the basis chosen, all \( 1 \)-constant directions have the form \( \alpha = [0 \alpha_2] \).)

Thus, (6.1) holds for \( \bar{S} = P_{11}(N-1, S_0) \) and \( \bar{S} = P_{11}(N-1, S) \) with \( \bar{S} > S_0 \). Again, an argument as in Lemmas 3.2 and 3.3 implies that (6.1) holds for all \( \bar{S} \geq S_0 \).

Therefore, by Theorem 3.1, \( \alpha_1 \) is a \( (j-1) \)-constant direction of (4.7) for \( j \geq 2 \). This completes the proof of the lemma.

Hence, if the space of \( 1 \)-constant directions is non-zero, the state space reduction procedure holds and the \( j \)-constant directions, \( j \geq 2 \), of the originally given problem become \( (j-1) \)-constant directions of the reduced state dimension problem. Consider now the repeated application of the idea of the above lemma. Suppose that a reduced problem is obtained via the procedures of the last two sections.

Now if this new problem has \( N(A) \) nonempty, we first eliminate any unnecessary controls by the procedure of Section 4. Then if \( I_1 = \{0\} \), with \( I_1 \), the space of \( 1 \)-constant directions for the reduced state dimension problem, we can again reduce according to Section 5. Clearly, if at some stage in the above procedure we obtain \( I_1 = \{0\} \) for one of the reduced problems, we cannot proceed any further. Now by Lemma 6.1 this is equivalent to having \( I_1 = I_{k+1} \) for some \( k \) in the original problem. We will now show that \( I_1 = I_{k+1} \) for some \( k \) implies that \( I_1 = I_{k+k} \) for every \( k \geq 2 \) in the original problem. This means that there is no way, possibly using some other algorithm than that presented, of eliminating further constant directions.

Since \( I_{k+1} \leq I_{k+k} \) for all \( k \geq 2 \) we could only have \( I_{k+1} = I_{k+k} \) if there were \((k+1)\)-constant directions which were not \((k+1)\)-constant directions for the original problem, or \( k \)-constant directions which were not \( 1 \)-constant directions for the reduced problem. Since the reduced problem has no \( 1 \)-constant direction, the result will follow from the following lemma:
Lemma 6.2: If $I_k = \{0\}$, then $I_k = \{0\}$ for all $k > 1$.

Proof: We argue by contradiction. Let $j$ be the least value of $k > 1$ for which $I_j \neq \{0\}$. Let $w(j) = [\alpha^* \beta(j)^*] \subseteq N(A(j))$ with $\alpha \neq 0$. By (2.12),

\[
\begin{bmatrix}
A(j-1)^A & A(j-1)^B & B(j-1) \\
B'^*(j-1)A + C' & B'^*(j-1)B + R & B'^*(j-1)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = 0
\] (6.5)

where $\beta(j) = [\beta^* \gamma^*]^*$. Immediately, we see that

\[
A(j-1)
\begin{bmatrix}
\alpha + BS \\
Y
\end{bmatrix} = 0.
\]

If $\alpha + BS \neq 0$, there exists a $(j-1)$-constant direction, which is a contradiction. So $\alpha + BS = 0$. Then $\gamma \in N[(j-1) \cap R(j-1)]$ and so by Remark 3.1.2, $\gamma \in N[C(j-1)]$. From the middle block row of (6.5), we have then

\[
0 = B'^*(j-1)[\alpha + BS] + C'\alpha + BS + B'^*(j-1)\gamma = C'\alpha + BS.
\]

This shows that $[\alpha^* \beta^*]^* \subseteq N[A]$, again a contradiction.

We now complete the program set out at the start of this section, of showing that by successive removals of superfluous controls and 1-constant directions, ultimately all $j$-constant directions, for any $j$, are eliminated. Motivated by this result, it is evidently sensible to make the following definition.

Definition: The optimal control problem is called singular whenever $A$ is singular. Otherwise, it is called nonsingular.

The procedure for solving a singular problem can be outlined as follows:

1. Determine the nullity of $A$, say $s$. If $[\beta^* \beta]^*$ has full rank proceed to 2. If not, eliminate any unnecessary controls by the procedure of Section 4.

2. Let $k = \dim I_1$. If $k = 0$, we have a nonsingular problem. If $k > 0$, reduce the state dimension by $k$ as described in Section 4. If $k = n$, we have a zero state dimension problem. If $k < n$, return to 1.

3. Cycle through 1 and 2 until the procedure terminates. This is guaranteed by the above theory, and moreover, it is guaranteed that all of the constant directions of the original problem are determined in at most $n$ applications of 1 and 2.

4. Determine the solution of the terminating control problem and trace back through the reduction procedure to construct the solution of the originally given problem.

In the commonly occurring case of $[A, B]$ completely reachable, we cannot terminate with a zero control dimension problem. This follows from the fact that complete reachability is preserved under the reduction procedure.
7. TIME-VARYING PROBLEMS, MISCELLANEOUS POINTS AND SUMMARIZING REMARKS

In Chapter III, results for the time-varying linear-quadratic control problem are presented in detail for the continuous-time case. However, for the derivation of these results, certain constancy of rank assumptions are required on the interval of interest. These assumptions might be thought of as a constant structure requirement.

A similar idea applies to the extension of the constant coefficient results obtained in Sections 2 - 5 to the time-varying case. For example, suppose that we consider the interval \([M, N]\); we can define the matrix \(A(i)\) for each \(i = M, \ldots, N-1\). Let \(a(i)\) be the nullity of \(A(i)\), let \(W(i)\) be a basis matrix for \(\mathcal{N}(A(i))\) and let \(\lambda(i)\) be the rank of \(W(i)\), with \(\overline{W(i)}\) the first \(n\) rows of \(W(i)\). Then \(a(i)\) and \(\lambda(i)\) must be constant on the interval \([M, N-1]\) for the reduction of the control and state dimensions previously described to be valid. Essentially, for this reason we have for the time-varying case the following definition.

**Definition:** Suppose \(1 \leq j \leq N-M-1\). The \(n\)-vector \(a(i)\) is called a \(j\)-degenerate direction at \(i\) of (2.5) on the interval \([M, N]\) if and only if \(P(i, S)a(i)\) is the same function of \(A(\cdot), B(\cdot), Q(\cdot), C(\cdot)\) and \(R(\cdot)\) for all \(M \leq i \leq N-j\) and all \(S\) for which (2.5) has a solution.

For a number of reasons we have chosen not to set out the time-varying results in detail. First, the notation becomes complex and, save for the idea of constancy of structure, no principles other than those already considered for the time-invariant case are needed. Second, the principle that it can be done has already been established in [31], though only for the special case of scalar covariances. Perhaps it is worthwhile to point out at this stage that the constancy of structure requirement in [3] appears via the condition that the covariances possess what is termed a definite relative order. Third, one of the major reasons for studying constant directions and the associated reduction in order of the Riccati equation is the computational saving.

The derivations in this paper require basis changes of both the state and control spaces. For the time-varying case, these basis changes need to be carried out for each \(i = M, \ldots, N-1\) so for \(N-M\) large and no functional relation between the coefficient matrices at each time instant, it is not unlikely that the hoped for computational advantage could be nullified.

**Other Miscellaneous Points**

1. If there are any constant directions at all, there must be \(1\)-constant directions. Therefore, the testing for existence of any constant directions is easily executed by looking at the matrix \(A\).

2. Let \(a\) be any constant direction, and let \(\Pi_1, \Pi_2\) be any two constant solutions of the Riccati equation (2.5) with \(B'\Pi_1 B + R > 0\) and \(N[B'\Pi_2 B + R] \leq N[A'\Pi_2 B + C]\). Since \(P(N-j, \Pi_1) = \Pi_2\) for all \(j\) and \(i = 1, 2\), one has \(\Pi_1 a = \Pi_2 a\).
3. In case \( R > 0 \), \( C = 0 \) (which is a common situation in regulator and filtering theory), and if also \( A \) is nonsingular, there are no constant directions. (This follows easily from an examination of \( A \)). On the other hand, \( R > 0 \), \( C = 0 \) and \( A \) singular implies that any vector in the nullspace of \( A \) is a 1-constant direction. (This is easy to see intuitively; use of the zero control ensures that the next state is zero).

4. The condition that \( N(A) \) be nonempty is equivalent to the condition that \( R + C'(zI-A)^{-1}B \) have a transmission zero at \( z = 0 \). If \( A \) is nonsingular, this is equivalent to demanding that the matrix \( R + C'(zI-A)^{-1}B \) be singular at \( z = 0 \). In turn this is equivalent to \( R + C'(zI-A)^{-1}B + B'(z^{-1}I-A)^{-1}C \) having this property.

5. In one interesting case, all directions are \( j \)-constant directions for some \( j \). Let \([A, B]\) be completely reachable, suppose the optimal control problem has a solution on \([0, 2n]\) for some \( S \), and suppose that \( R + C'(zI-A)^{-1}B \) have a transmission zero at \( z = 0 \). Let \( x(0) \) be arbitrary and fixed, \( x(0) = 0 \). Let \( u^{n-1} \) be a control sequence taking \( x(0) = 0 \) to the prescribed \( x(n) \) and let \( u_{2n-1} \) be arbitrary. Because the control problem has a solution on \([0, 2n]\), it follows that

\[
0 \leq V_n[0, u_{2n-1}, S] = V_n[0, u_{2n-1}, 0] + V_n[x(n), u_{2n-1}, S].
\]

[If the inequality failed, \( V_n \) could be made as negative as desired by scaling \( u_n \).] Now suppose that \( u_{2n-1} \) is such that \( x(2n) = 0 \). Define \( u(k) = 0 \) for \( k \neq [0, 2n-1] \). Use Parseval's theorem to evaluate

\[
\sum_{k=0}^{2n-1} |x^{*}(k)Qx(k) + 2x^{*}(k)Cu(k) + u^{*}(k)Ru(k)| = V_n[0, u_{2n-1}, S].
\]

The frequency domain equality then yields \( V_n[0, u_{2n-1}, S] = 0 \). (A similar argument has been used in the continuous-time linear-quadratic problems in the proof of Theorem 4 of [9]). Hence \( V_n[x(n), u_{2n-1}, S] \) attains its lower bound, via., \(-V_n[0, u_{2n-1}, 0]\). Any \( u_{2n-1} \) causing \( x(2n) = 0 \) is optimal, and since \( x(n) \) is arbitrary, all directions are constant. This means incidentally that the transient solution of (2.5) will agree with the steady state solution after at most \( n \) steps. Finally, note that it is not necessarily the case that if all directions are constant, the frequency domain relation holds.

6. The matrix \( A \) is square, so that if \( N(A) \) is nonempty, so is \( N(A') \), and the question then arises as to what significance, if any, attaches to vectors in \( N(A') \).

One can make the trivial observation that nonzero vectors in \( N(A') \) are associated with 1-constant directions for a dual problem [where \( x(i+1) = A'x(i) + Cu(i) \) and \( C \) in the performance index is replaced by \( B \)], but beyond this, not much can be said. In particular, it does not seem to be possible to make statements relevant to the
primal control problem.

7. Suppose that for some \( j, a, S \) and \( S > S_0 \), one has \( V_j^0[a, S] = V_j^0[a, S_0] \). Thus \( a \) is a \( j \)-constant direction. It is immediate that the control sequence minimizing \( V_j[a, S] \) also minimizes \( V_j[a, S_0] \). The point of this remark is that a control sequence minimizing \( V_j[a, S_0] \) need not minimize \( V_j[a, S] \); this is shown by the example below. On the other hand, if \( V_j[a, S_0, \eta] \) exists for some \( \eta > 0 \), the control sequence minimizing \( V_j[a, S_0] \) must carry \( a \) optimally to zero, by Lemma 3.2, and such a control sequence also minimizes \( V_j[a, S] \). For the example, take \( A = B = C = R = I, Q = 0 \). Then \( V_1[x(N-1), u(N-1), S] = (S+1)u^2(N-1) + 2(S+1)u(N-1)x(N-1) + x^2(N-1) \); if \( S > -1 \), \( u(N-1) = -x(N-1) \) is the unique optimal control, while if \( S = -1 \), any value of \( u(N-1) \) is optimal. The minimum norm value is of course zero, and this is certainly not optimal for \( S > -1 \) if \( x(N-1) \neq 0 \).

8. For time-invariant linear-quadratic problems, the so-called Chandrasekhar algorithms [5, 6] appear very attractive computationally. It would therefore be interesting to connect the ideas of this chapter, at the computational level, with the Chandrasekhar algorithms. Conceptually, it is fairly clear that a connection should be fruitful. This is because the Chandrasekhar algorithms work with first differences \( P(i+1, S) - P(i, S) \) of Riccati equation solutions, and are advantaged by these quantities having low rank; the more linearly independent constant directions there are, the lower the rank of the first difference. At a more detailed level, some care would however be required in separating the control dimension reduction step from the state dimension reduction step, and one would need to vary the proofs of existence of certain orthogonalizing transformations in [6] to cope with various possible singularities. Finally, extensions to the time-varying case could pose problems.

9. In [4], a "structure algorithm" is presented which bears on linear quadratic problems in which \( S \geq 0 \) and \( C \) \( S \) \( C \geq 0 \). The manipulations for eliminating superfluous controls, characterizing and eliminating \( 1 \)-constant directions, and identifying them with states which can be taken to zero optimally in one step are equivalent to manipulations in [4]. The full extent of the parallels, and the point at which they break down because of the nonnegativity requirement of [4] has yet to be explained.

10. Though we have confined our discussions to control problems, we could equally well have worked within a framework of filtering and covariance factorization, as in [3]. Combination of the ideas here and of [3] will readily yield the results.

**Summarizing Remarks**

There have been two main themes of this chapter. First, we have discussed properties of constant directions in the context of the most general linear-quadratic control problem. The most important constant directions, viz. \( 1 \)-constant directions, are characterized via the null-space of a certain simply constructed matrix, and all
constant directions are characterized in terms of optimal controls yielding trajectories which terminate in the zero state. Second, we have shown how the existence of constant directions may be exploited in solving an optimal control problem. They may be eliminated to yield a lower dimension problem, the solution of which determines the solution of the original problem, with the adjunction of certain quantities computed in the construction of the lower dimension problem.

APPENDIX IV.A

DEFINITIONS OF COEFFICIENT MATRICES

The matrices used in equation (5.8) are defined as follows:

\[
\hat{A} = A_{11}
\]

\[
\hat{B} = [\hat{B}_1 \ \hat{B}_2] = [B_{12} \ B_{11}]
\]

\[
\hat{Q} = A_{11}' (Q_{12} + C_{12} Z) A_{21} + A_{21}' (Q_{12} + C_{12} Z) A_{11} + A_{21}' (Q_{22} + C_{22} Z) A_{21} + Q_{11}
\]

\[
\hat{C} = [\hat{C}_1 \ \hat{C}_2]
\]

\[
\hat{C}_1 = A_{11}' (Q_{12} + C_{12} Z) B_{12} + A_{21}' (Q_{12} + C_{12} Z) B_{12} + A_{21}' (Q_{22} + C_{22} Z) B_{22} + C_{11}
\]

\[
\hat{C}_2 = A_{11}' (Q_{12} + C_{12} Z) B_{22} + A_{21}' (Q_{12} + C_{12} Z) B_{22} + A_{21}' (Q_{22} + C_{22} Z) B_{22} + C_{11}
\]

\[
\hat{R} = \begin{bmatrix}
\hat{R}_{11} & \hat{R}_{12} \\
\hat{R}_{21} & \hat{R}_{22}
\end{bmatrix}
\]

\[
\hat{R}_{11} = B_{11}' (Q_{12} + C_{12} Z) B_{11} + B_{12}' (Q_{12} + C_{12} Z) B_{12} + B_{21}' (Q_{22} + C_{22} Z) B_{21} + B_{22}' (Q_{22} + C_{22} Z) B_{22}
\]

\[
\hat{R}_{12} = B_{11}' (Q_{12} + C_{12} Z) B_{22} + B_{12}' (Q_{12} + C_{12} Z) B_{22} + B_{21}' (Q_{22} + C_{22} Z) B_{22} + B_{22}' (Q_{22} + C_{22} Z) B_{22}
\]

\[
\hat{R}_{21} = B_{11}' (Q_{12} + C_{12} Z) B_{21} + B_{12}' (Q_{12} + C_{12} Z) B_{21} + B_{21}' (Q_{22} + C_{22} Z) B_{21} + B_{22}' (Q_{22} + C_{22} Z) B_{21}
\]

\[
\hat{R}_{22} = B_{11}' (Q_{12} + C_{12} Z) B_{22} + B_{12}' (Q_{12} + C_{12} Z) B_{22} + B_{21}' (Q_{22} + C_{22} Z) B_{22} + B_{22}' (Q_{22} + C_{22} Z) B_{22}
\]
REFERENCES


CHAPTER V

OPEN QUESTIONS

There are a number of open questions arising from the work discussed in the previous chapters. Some problems have already been mentioned in those chapters; we will not repeat these here.

Perhaps one of the more obvious questions is: how can one cope with a problem for which there are a finite number of structural changes along the interval of interest? (Trying to cope with an infinite number of structural changes seems impossibly difficult). Structural changes correspond to the matrix \( P^* \) possessing jumps at the points of structural change, and the problem is to determine the magnitude of the jumps in the \( P^*(*) \) matrix at the points of structural change or, equivalently, to match the \( P^*(*) \) matrices on the individual intervals at the junction points.

A question which is clearly related is that of joining up controls and trajectories between singular and nonsingular regimes, or between dissimilar singular regimes. There are both quantitative issues and qualitative issues involved; some results are surveyed in [1].

Recently "high order" maximum principles [2] have been applied to singular problems. It would be of interest to closely relate the methods of Chapter III to the results achievable by the high order maximum principle; to the extent that the Euler-Lagrange equations are likely to figure prominently, recent work on their use in singular problems [3] is likely to be relevant.

The possibility of reduction in computational complexity of a general singular discrete-time problem has been demonstrated in the previous chapter. For particular classes of problems there are a number of efficient computational algorithms, e.g. the square root filtering algorithm and the Chandrasekhar algorithm mentioned in Section 7 of Chapter IV. It would be of interest to set up the connections between these algorithms and our own.

Finally, we draw attention to what is almost a problem of logic. Often a singular linear-quadratic control problem (or nonnegativity problem) is the result of linearization of a nonlinear system about a nominal control and trajectory. It is therefore clear that there are, in effect, implied constraints on both the controls and states in the second variation linear-quadratic problem. This would obviously preclude the appearance of delta functions in an optimal control problem (delta functions being the limit of continuous functions with upper bound approaching infinity). We must then ask to what extent the calculation of singular controls is a valid exercise. Of course, in a specific problem, regularization of any singular second variation problem [though the addition of a term \( \varepsilon u'(t)u(t) \) to the loss function for some \( \varepsilon > 0 \)] will determine control gains which are usable to give a valid approximation to the adjustment in the optimal control stemming from a small enough adjustment in the initial state.
One approach to solving singular control problems is to regularize them, i.e. to add a term \( \varepsilon u'u \) with \( \varepsilon > 0 \) and small to the loss function, thereby obtaining a nonsingular problem with a solution in some way close to that of the singular problem. The resulting nonsingular problem—a "cheap control" problem—is normally numerically ill-conditioned, and special approaches are being developed to solve such problems, see e.g. [4]. It would be of interest to check whether the reduction procedures proposed for singular problems could be profitably used also on nonsingular, cheap control, problems.

A further problem is to tidy up some of the results presented here for problems with end-point constraints. The robustness results for the case when the final state is partially but not completely constrained have not been fully developed. An algorithm for constructing optimal controls and the optimal performance index has not yet been given, but almost certainly, it should be straightforward to obtain as an extension of the free end-point algorithm.

Another area left untouched relates to allowing semi-infinite intervals \([t_0, \infty)\) instead of the finite intervals \([t_0, t_f]\) considered throughout this book. There is of course a fairly extensive theory, see e.g. [5], for time-invariant nonsingular problems and for the time-varying linear regulator problem applicable to semi-infinite intervals, and much constitutes non-trivial extension of the finite interval results. All this suggests that it might be fruitful to study the general singular problem on the semi-infinite interval.

REFERENCES


