Brief paper

Pose localization of leader–follower networks with direction measurements

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\begin{abstract}
A distributed pose localization framework based on direction measurements is proposed for a type of leader–follower multi-agent system in $\mathbb{R}^3$. The novelty of the proposed localization method lies in the elimination of the need for using distance measurements and relative orientation measurements for the network pose localization problem. In particular, a network localization scheme is developed based directly on the measured directions between an agent and its neighboring agents in the network. The proposed position and orientation localization algorithms are implemented through differential equations which simultaneously compute poses of all followers by using locally measured directional vectors and angular velocities, and actual pose knowledge of some leader agents, allowing some tracking of time-varying orientations. Further, we establish almost global asymptotic convergence of the estimated positions and orientations of the agents to the actual poses in the stationary case.

\end{abstract}

1. Introduction

Networked cooperative pose localization tackles the determination of positions and orientations of a networked set of agents in an underlying three dimensional space through various interagent measurements. It may well be done in order to perform further coordination control or distributed estimation tasks (Oh, Park, & Ahn, 2015; Zhao & Zelazo, 2019). Distances and directions are the two most commonly used measurements that are widely used in position localization literature (Aspnes et al., 2006; Mao, Fidan, & Anderson, 2007). In a three dimensional ambient space, direction is characterized by a unit length vector, and this can often be obtained by visual imaging, see Ma, Soatto, Kosecka, and Sastry (2004). However, in three-dimensional space, additional relative orientation measurements\footnote{A relative orientation is effectively the rotation matrix linking a local coordinate frame of one agent to the local coordinate frame of another agent.} between neighboring agents are often required for estimating orientations (as opposed to positions) of the agents in a network, a process which is called orientation localization (Piovan, Shames, Fidan, Bullo, & Anderson, 2013; Tron & Vidal, 2014). Nevertheless, there are not many works that study simultaneous localization of positions and orientations, a process which is called pose localization, in a distributed setup. Motivated by these facts, this work attempts to provide a distributed pose localization framework for a type of leader–follower networks based on direction measurements and pose knowledge of some leader agents.

For a two-dimensional (2-D) ambient space, network localization laws using angles of arrival between triplets of nodes are proposed in Zhu, Huang, and Jiang (2008) and an orientation localization method utilizing orientation knowledge of a few nodes is presented in Rong and Sitichiu (2006). The authors in Piovan et al. (2013) further proposed a least-squares optimization problem to achieve orientation localization by exploiting kinematic relationships among the orientations of nodes. A least-squares algorithm for position localization using bearing-only information is proposed in Bishop and Shames (2011). In 3-dimensional space (3-D), it is often required that relative orientation measurements are available for estimating the orientations of the agents. For example, some necessary and sufficient conditions...
are provided for orientation localizability of triangular sensing networks of relative orientation measurements in Piovan et al. (2013), without providing a distributed orientation localization law. Network localization schemes using relative poses (relative orientations and relative positions), which are measured by a vision-based technique, are investigated in Thunberg, Bernard, and Gonçalves (2017), Tron and Vidal (2014). The estimation of relative poses, however, generally requires the agents to have views of a common scene. By using relative orientation measurements, our recent works in Tran, Ahn, and Anderson (2018) and Tran, Trinh, Zelazo, Mukherjee, and Ahn (2019) propose distributed orientation estimation laws which guarantee almost global convergence of the estimated orientations up to a common orientation. 

Zaho and Zelazo (2016) proposes a direction-only position localization law for bearing rigid structures with two anchor nodes. However, Zhao and Zelazo (2016) further assumes that the agents know their actual orientations. There is no framework for direction-only network localization and formation control in 3-D when agents lack knowledge of a global frame. The orientation localization problem is challenging and requires sophisticated estimation algorithms. In 2-D, it is straightforward to see how two neighboring agents observing each other might determine a common view of their relative orientation (i.e., a scalar angle), within an unknown constant rotation common to both, see e.g. Oh and Ahn (2014), as is now described. Each agent maintains a (possibly body-fixed) coordinate frame and measures the orientation angle of its neighboring agent (assuming direction sensing technology). In any common frame, the measured angles of the two neighboring agents must differ by precisely π radians. Hence a rotation of the coordinate axes of one agent can be made to ensure that after rotation, the angle difference is compensated. For an n agent network, one has to put together in a distributed fashion a collection of such calculations. How to do something like this in a 3-dimensional ambient space is less clear. For example, with only a pair of direction measurements between two neighboring agents i and j (bj_i, bj_j) ∈ R^3 × R^3 (see Fig. 1(b)), it is insufficient for the agents i and j to determine their relative orientation, i.e., R_j = R^T_i R_j ∈ SO(3), where R_i and R_j ∈ SO(3) are the orientation matrices of agents i and j, respectively, due to the ambiguity of the rotation along the common direction vector, bj_i. This difficulty can be overcome by examining additional direction constraints of each of the two agents to a third agent k that they both observe. Indeed, as shown in Tran et al. (2018), by exploiting the triangle sensing network agents i and j can compute R_j. The orientations of all agents then can be computed up to a common orientation bias by using a consensus protocol (Tran et al., 2018). This method, however, relies on the existence of triangle networks and requires pre-definition of a complicated computation sequence.

This paper proposes a distributed pose localization scheme for a type of leader–follower network that uses continuous-time directional vectors and two or more anchor agents which know their absolute poses. A distributed orientation localization protocol in SO(3) that estimates orientations of all followers is proposed. Under the proposed orientation localization protocol, estimated orientations converge to the true orientations of agents almost globally and asymptotically. By using the estimates of orientations and direction measurements, we investigate a position localization law for the leader–follower network. Under the proposed position localization law, positions of all followers are also globally and asymptotically determined. The proposed network pose localization scheme can work exclusively with inter-agent directional vectors.

The rest of this paper is organized as follows. Section 2 presents some preliminaries and the problem formulation. The orientation localization problem is studied in Section 3. We propose a position localization law in Section 4. Finally, Section 5 concludes this paper.

2. Preliminaries and problem formulation

2.1. Directional vector and orientation of agent

Consider a network of n nodes in R^3. Each node corresponds to an agent, and an agent is defined by the position of its centroid and the orientation of a body-fixed coordinate frame Σ relative to a global frame Σ. In the sequel, the position of an agent will be taken to be the position of its centroid. Let p_i and p_j ∈ R^3 be the position of agent i expressed in Σ and Σ, respectively. We define the unit directional vector (expressed in Σ) pointing from agent i toward its neighbor j along the direction of p_i − p_j as b_ij = p_i − p_j ∥ ∥p_i − p_j∥∥. The directional vector with the reverse direction is b_ij = −b_ij. The directional vector b_ij measured locally in Σ is denoted as b_ij. The orientation of agent i in R^3 can be characterized by a square, orthogonal matrix R_i ∈ SO(3). The pair (R_i, p_i) ∈ SE(3) characterizes the pose of agent i in the global Cartesian space.

2.2. Graph theory

An interaction graph characterizing an interaction topology of a multi-agent network is denoted by G = (V, Σ), where, V = {1, . . . , n} denotes the vertex set and Σ ⊆ V × V denotes the set of edges of G. An edge is defined by the ordered pair e_k = (i, j), k = 1, . . . , m, m = |Σ|. The graph G is said to be undirected if (i, j) ∈ Σ implies (j, i) ∈ Σ, i.e., if j is a neighbor of i, then i is also a neighbor of j. If the graph G is directed, (i, j) ∈ Σ does not necessarily imply (j, i) ∈ Σ. The set of neighboring agents of i is denoted by N_i = {j ∈ Σ : (i, j) ∈ Σ}. 

2.3. Problem formulation

Consider a leader–follower network in R^3 with at least two non-collocated leader agents 1 and 2 which know their actual poses (position and orientation in a global coordinate frame). The leader–follower network studied in this work is defined as follows (See also Fig. 1(a)).

Definition 1 (Twin-Leader–Follower Network). A twin-leader–follower network is a directed network in which agents are ordered such that (a) all leader agents appear first, there are two (or more) leaders 1 and 2 which know their absolute poses (R_1, p_1) and (R_2, p_2), respectively (b) a follower agent i, 3 ≤ i ≤ n, has at least two neighboring agents j’s in the set {1, . . . , i−1}, i.e., |N_i| ≥ 2, where N_i denotes the set of neighboring agents.
of $i$. Agent $i$ knows the direction $\mathbf{b}_{ij}$ to the neighbor $j$, while its neighbor knows the direction $\mathbf{b}_{ji}$.

Note importantly that with only one leader, it is impossible to compute the actual agent poses due to the translational and scale ambiguities in networks with direction-only measurements (Trinh, et al., 2019; Zhao & Zelazo, 2019). Further, without pose knowledge of the two leaders, the two agents can arbitrarily select the translation, rotation, and scale factors of the pose estimation (Tran, Anderson, & Ahn, 2019, Remark 3). We remark that the first listed nonleader agent is known as a first follower and any leader agents beyond the first two are known as redundant leaders. To streamline nomenclature, we number the agents as $\{1, 2, 1', 2', \ldots, 3, 4, \ldots, n\}$, where the follower agents are $3, 4, \ldots, n$; also $\mathcal{V}_j = \{1, 2, 1', 2', \ldots\}$, where $1', 2'$ are redundant leaders, and $\mathcal{V}_j = \{3, 4, \ldots, n\}$ will denote the sets of leader and follower agents, respectively.

Each agent $i \in \mathcal{V}_j$ in the network aims to estimate its actual pose, i.e., $(\mathbf{R}_i, \mathbf{p}_i) \in \text{SO}(3) \times \mathbb{R}^3$, based on the direction constraints to its neighboring agents and the actual poses of the leader agents. At each time instant $t$ agent $i$ holds an estimate of its pose, denoted as $(\hat{\mathbf{R}}_i, \hat{\mathbf{p}}_i) \in \text{SO}(3) \times \mathbb{R}^3$.

**Assumption 1.** Agent $j$ estimates its orientation at time $t$ by $\hat{\mathbf{R}}_i$, and transmits the information $\hat{\mathbf{R}}_i \mathbf{b}_{ij}$ to agent $i, j \in \mathcal{N}_i$ (see Fig. 1(b)).

We assume that the agents in the network do not translate but they might rotate according to the kinematics $\dot{\mathbf{R}}_i = \mathbf{R}_i(\omega_i')$, for $i \in \mathcal{V}$, where $\omega_i'$ is the angular velocity of agent $i$ measured locally in $\Sigma_i$. We assume that $\omega_i'$ and its derivative are bounded, i.e., $||\omega_i'|| \leq \tilde{\omega}_i, ||\omega''_i|| \leq \tilde{\omega}_i$, for positive constants $\tilde{\omega}_i, \tilde{\omega}_i > 0$, and each agent $i$ can measure $\omega_i'$ without noise. The angular velocity expressed in the global coordinates is $\omega_i = \mathbf{R}_i \omega_i'$. This kind of system might represent a visual sensor network (Tron & Vidal, 2014) or a system of autonomous agents in a desired formation (Tran, Trinh, Mukherjee, & Ahn, 2019) where the agents might rotate to track objects. Moreover, to secure uniqueness of the localized poses of the agents, we have the following assumption.

**Assumption 2.** No two agents are collocated and each follower $i \in \mathcal{V}_j$ has at least one pair of neighbors with which it is not collinear.

We first address the problem of calculating the orientation $\hat{\mathbf{R}}_i$ for all follower agents.

**Problem 1.** Considering a twin-leader-follower network of $n$ agents, under Assumptions 1–2, compute $\hat{\mathbf{R}}_i$ for each follower $i \in \mathcal{V}_j$ based on the directional measurements $(\mathbf{b}_{ij}, \mathbf{b}_{ji})$, estimated orientations of its neighbors $\hat{\mathbf{R}}_j, j \in \mathcal{N}_i$, and the knowledge of the true orientations of the two or more leaders, i.e., $\mathbf{R}_k \in \text{SO}(3), k \in \mathcal{V}_l$.

Assuming solvability of Problem 1, the second problem investigated is to determine the locations of agents.

**Problem 2.** Consider a twin-leader-follower network of $n$ non-translating but possibly rotating agents with at least two leaders. Under Assumptions 1–2, for each follower $i$, determine its actual position, $\mathbf{p}_i \in \mathbb{R}^3$, based on the estimate $\hat{\mathbf{R}}_i$, the direction constraints $\mathbf{b}_{ij}, j \in \mathcal{N}_i$, and absolute positions of some leaders, i.e., $\mathbf{p}_k \in \mathbb{R}^3, k \in \mathcal{V}_l$.

### 3. Orientation localization

In this section, we present a differential equation constituting a continuous-time orientation localization law in SO(3) that computes time-varying orientations of agents simultaneously using continuous-time directional vectors to multiple neighboring agents, angular velocity measurements, and actual orientations of some leaders. Further, the equilibrium set of the differential equation is first characterized and almost global asymptotic convergence of the estimated orientations is established.

#### 3.1. Error function and critical points

Consider an agent $i \in \mathcal{V}_j$ which senses the local directions, $\mathbf{b}_{ij} \in \mathbb{R}^3$, to its neighboring agents $j \in \mathcal{N}_i$. If $|\mathcal{N}_i| = 2$, the third direction constraint is defined by the normalized cross product of the first two directions, for positive definiteness of $\mathbf{K}_i$ in (3). The objective is to find an estimate, $\hat{\mathbf{R}}_i \in \text{SO}(3)$, of the true orientation, $\mathbf{R}_i$, that is a minimum of the following error function

$$
\Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) = 1/2 \sum_{j \in \mathcal{N}_i} k_j ||\hat{\mathbf{R}}_i \mathbf{b}_{ij} - \mathbf{b}_{ij}||^2
$$

which is sum of squared norms of all direction constraint errors. We do not assert that $\Phi_i$ can be evaluated from the measurements, but we shall show that it can be minimized from the measurements. In (2), positive constant gains, $k_j \in \mathbb{R}$, are used to impose different weights on error terms in the error function. The above configuration error function is in the form of Wahba’s cost function (Wahba, 1965) and used for attitude tracking control (Lee, 2015) or attitude estimation of a rigid body (Izadi & Sanyal, 2016; Mahony, Hamel, & Pilgrim, 2008). In the sequel, we follow techniques similar to those in Bullo and Lewis (2005, Chap. 1), (Lee, 2015) to design our orientation localization law.

Let $\varphi_j = 1 - \hat{\mathbf{R}}_i \mathbf{b}_{ij} \cdot \mathbf{b}_{ij} = 1 - \text{tr}(\hat{\mathbf{R}}_i \mathbf{b}_{ij} \mathbf{b}_{ij}^T) = 1 - \text{tr}(\hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T \mathbf{b}_{ij} \mathbf{b}_{ij}^T)$, where we use the relations $\mathbf{x}^T \mathbf{y} = \text{tr}(\mathbf{x} \mathbf{y}^T)$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, and $\mathbf{b}_{ij} \mathbf{b}_{ij}^T$. Let $\hat{Q}_j \triangleq \hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T$ and hence $\varphi_j = 1 - \text{tr}(\hat{Q}_j \mathbf{b}_{ij} \mathbf{b}_{ij}^T)$. Consider a vector in the tangent space of SO(3) at the point $\hat{\mathbf{R}}_i$ (resp. $\mathbf{R}_i$) as $\delta \hat{\mathbf{R}}_i = (\hat{\mathbf{R}}_i \mathbf{n}_j, \mathbf{n}_j \in \mathbb{R}^3$ (resp. $\delta \mathbf{R}_i = (\mathbf{R}_i \mathbf{\xi}_j, \mathbf{\xi}_j \in \mathbb{R}^3$) (Bullo & Lewis, 2005). Then, the following straightforwardly established lemma can be proved (Tran, Anderson, & Ahn, 2019).

**Lemma 1.** The derivative of the error function $\Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i)$ with respect to $\hat{\mathbf{R}}_i$ (resp. $\mathbf{R}_i$) along the direction of $\mathbf{R}_i \mathbf{\xi}_j$ (resp. $\mathbf{R}_i \mathbf{n}_j$) is given by

$$
\mathbf{D}_{\hat{\mathbf{R}}_i} \Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) \cdot \mathbf{R}_i \mathbf{\xi}_j = \eta_j \sum_{j \in \mathcal{N}_i} \mathbf{e}_{ij}j \in \mathcal{N}_i
$$

$$
\mathbf{D}_{\mathbf{R}_i} \Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) \cdot \mathbf{R}_i \mathbf{\xi}_j = -\zeta_j \sum_{j \in \mathcal{N}_i} \mathbf{e}_{ij}j \in \mathcal{N}_i
$$

where $\mathbf{e}_{ij} \triangleq k_j (\hat{\mathbf{R}}_i \mathbf{b}_{ij} \times \mathbf{b}_{ij}) \in \mathbb{R}^3, j = 1, \ldots, |\mathcal{N}_i|$. We now study the critical points of $\Phi_i(\hat{\mathbf{R}}_i)$. To proceed, we rewrite the error function as
The fact that the zeroset of the discriminant of the characteristic equation of \( K \), which is a polynomial of the real entries of \( K \), is a set of measure zero. Since \( \text{Range}(k_q b_j^T b_j) = \text{span}(b_j) \), it can be verified that \( K \) is positive definite if and only if \( [b_j]_{j \in N} \) are non-coplanar. Thus, \( K \) can be decomposed as \( K = U G U^T \) where \( G = \text{diag}(\lambda_1(K)), \lambda_1(K) > 0, k = 1, 2, 3, \) and \( U \in O(3) \). Also note that \( \text{tr}(G) = \text{tr}(K) = \text{tr}(\sum_{j \in N} k_q b_j^T b_j) = \sum_{j \in N} k_q b_j^T b_j \). Consequently, one has \( \Phi_i = \text{tr}(G) - \text{tr}(\tilde{Q} U G U^T) = \text{tr}((I - U^T \tilde{Q} U)) \), whose critical points are given as follows.

**Lemma 2** (Bullo & Lewis, 2005, Prop. 11.31). Let \( G \) be a diagonal matrix with distinct positive entries and \( U \in O(3) \). Then, \( \Phi_i(Q) = \text{tr}(G(I - U^T \tilde{Q} U)) \) has four critical points given by \( \tilde{Q} \in \{I_3, UD, U^T, UD, U^T, UD, U^T \} \), where \( D_i = 2[I_3]_{i=1}^3 - I_3 \) and \( I_3 \) is the ith column vector of \( I_3 \).

Those critical points are clearly isolated in which \( \tilde{Q} = \hat{R}_i \hat{R}_i^T = I_3 \) is the desired point and \( \text{tr}(\tilde{Q}) = -1 \) for the three undesired points.

### 3.2. Orientation localization law

We now propose orientation localization law for each follower agent \( i \) as

\[
\dot{\hat{R}}_i = \tilde{\dot{\hat{R}}}_i, \quad \dot{\hat{R}}_i = \hat{\Omega}_i^e, \quad (4)
\]

where the control vector \( \hat{\Omega}_i \in \mathbb{R}^3 \) will be designed later and \( \tilde{\dot{\hat{R}}}_i \) is initialized arbitrarily in \( SO(3) \). Let \( \hat{\Omega}_i = \hat{\omega}_i - \hat{\Omega}_i \); we have the following lemma, which can be proved using techniques similar to those in Lee (2015, Prop. 1).

**Lemma 3.** The vector, \( e_i \triangleq \sum_{j \in N} \hat{Q}_{ij} e_p \), and error function, \( \Phi_i \) in (2) satisfy the following properties

(i) \( \|e_i\| \leq \sum_{j \in N} k_q |\hat{\omega}_j| \|\hat{\Omega}_i\| + \hat{\omega}_i \|e_i\| \), where the positive constant \( \hat{\omega}_i > 0 \) satisfies \( \|\hat{\omega}_i\| \leq \hat{\omega}_0 \),

(ii) \( \Phi_i(\hat{R}_i, \hat{R}_i) = -\dot{\hat{\omega}}_i \cdot e_i \),

(iii) There exist constants \( \sigma_i, \gamma_i > 0 \) such that \( \sigma_i \|e_i\|^2 \leq \Phi_i(\hat{R}_i, \hat{R}_i) \leq \gamma_i \|e_i\|^2 \), where the upper bound holds when \( \Phi_i < 2 \min[\lambda_1, \lambda_2, \lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3] \), \( (\lambda_3 = \lambda(K), k = 1, 2, 3) \).

The control vector \( \hat{\Omega}_i = \hat{\omega}_i - \hat{\Omega}_i \), where \( \hat{\Omega}_i \in \mathbb{R}^3 \) is designed via

\[
\dot{\hat{\Omega}}_i = -k_q \hat{\omega}_i + \sum_{j \in N} k_q (\hat{\tilde{R}}_i^T \hat{R}_j b_j^T b_j^T), \quad (5)
\]

where \( k_q > 0 \) is a positive constant. The orientation localization law (4)–(5) is distributed since only directional vectors, i.e., \( b_j^T \), and information communicated from neighboring agents, i.e., the estimate of direction in global coordinates, \( \hat{R}_j b_j^T \), are utilized. Since the right hand side of (5) is linear in \( \hat{\Omega}_i \) and the second term is bounded, \( \hat{\Omega}_i \) is uniformly continuous in \( t \).

### 3.3. Stability and convergence analysis

We rewrite (5) as

\[
\dot{\hat{\Omega}}_i = -k_q \hat{\omega}_i + \sum_{j \in N} k_q (\hat{\tilde{R}}_i^T \hat{R}_j b_j^T b_j),
\]

which is clearly bounded and converges to zero asymptotically since \( \hat{R}_i \to R_i \), \( \forall j = 1, \ldots, k - 1 \). Note that \( \dot{\hat{\omega}}_i(t) \) can be considered as an additive input to the nominal system

\[
\dot{\hat{R}}_i = \hat{\omega}_i - \hat{\Omega}_i, \quad \dot{\hat{\Omega}}_i = -k_q \hat{\omega}_i + e_i, \quad \text{for } k = 3.
\]

It follows from the above theorem that \( \hat{R}_i \to R_i \) almost globally asymptotically as \( t \to \infty \). For induction, we now suppose that the corresponding result holds for agents \( k - 1, k - 1 \geq 3 \), i.e., \( R_{k-1} \to R_{k-1} \) as \( t \to \infty \) almost globally. We show that it is also true for the agent \( k \) as follows.

#### 3.3.2. Follower \( k \)

Using (4) and (6), we have

\[
\dot{\hat{R}}_k = \hat{\omega}_k, \quad \dot{\hat{\Omega}}_k = -k_q \hat{\omega}_k + e_k + \hat{h}_k(t),
\]

where \( \hat{h}_k(t) = \sum_{j \in N} k_q (\hat{\tilde{R}}_j^T (\hat{R}_j - R_j) b_j^T b_j) \) which is clearly bounded and converges to zero asymptotically since \( \hat{R}_j \to R_j \), \( \forall j = 1, \ldots, k - 1 \). Note that \( \hat{h}_k(t) \) can be considered as an additive input to the nominal system

\[
\dot{\hat{R}}_k = \hat{\omega}_k - \hat{\Omega}_k, \quad \dot{\hat{\Omega}}_k = -k_q \hat{\omega}_k + e_k.
\]
It is noted that the above system is in a similar form to (7) and hence the following result follows directly.

**Lemma 4.** Consider the nominal system (10) under the Assumptions 1–2, then:

(i) The equilibrium points of (10) are given as \( \{(\bar{Q}_i, \bar{\Delta}_k)\} \) \( Q_i \in \{I, UD, U^T, UD^T, UD^2\} \), \( \bar{\Delta}_k = \delta \), where \( D \) and \( U \) are defined in Lemma 2.

(ii) The desired equilibrium, \( \{\bar{Q}_i, \bar{\Delta}_k = 0\} \) is aGAS while the three undesired equilibria are unstable.

The perturbed system (9) is linear in \( \bar{\Delta}_k \) and \( e_i + h_i \) is bounded. Thus \( \bar{\Delta}_k \) is bounded. Define the set \( S_\delta = \{Q_i| \Phi_i(Q_i) < \phi_k\} \), where \( \phi_k = 2 \min(\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3) \), \( \{\lambda_i\}_{i=1,2,3} = \lambda(K_k) \), or, i.e., the minimum value of \( \Phi_k \) evaluated at the three undesired critical points.

**Lemma 5.** Suppose that Assumptions 1–2 hold. The perturbed system (9) is input-to-state stable (ISS) with respect to \( h_i(t) \).

**Proof.** Consider the Lyapunov function \( V_k = 1/2 \bar{\Delta}_k^T + \Phi_k - k_k \bar{\Delta}_k \).

\[ V_k (z) \leq 1/2 \bar{\Delta}_k^T + \Phi_k - k_k \bar{\Delta}_k \]

It can be shown that the time derivative of \( V_k \) along the trajectory of (9) satisfies \( V_k (z) \leq -1/2 \bar{\Delta}_k^T \bar{\Delta}_k^T + d(h_i) \), where \( d = \sup(\|\bar{\Delta}_k - k_k \bar{\Delta}_k\|) \) and \( C_k \in \mathbb{R}^{2 \times 2} \) is a positive definite matrix (and so are \( A_k \) and \( B_k \) if \( k_k \) is sufficiently small). Therefore, it follows from (11) and the boundedness of \( \bar{\Delta}_k \) we have that

\[ V_k (z) \leq -\min(C_k)/\max(B_k) \|\bar{\Delta}_k\|^2 + d(h_i) \]

which shows ultimate boundedness of the system (9) and input-to-state stability of the system (9) w.r.t. \( h_i(t) \) according to Angeli and Praly (2011, Prop. 3).

From (12) that \( V_k < 0 \) it follows that

\[ V_k (z) \leq -\min(C_k)/\max(B_k) \|\bar{\Delta}_k\|^2 + d(h_i) \]

The uniqueness of the target configuration (the actual position of agents) is a key property of the network that allows us to localize the network. In the sequel, under the nonlocalization and non-collinearity conditions of the true positions of the agents in Assumption 2, we show that the target configuration is uniquely defined using the direction constraints, estimate of orientation of agent \( i \), and the absolute positions of some leaders. The following lemma is similar to Trinh, et al. (2019, Lem. 1).

**Lemma 6.** Consider the twin-leader–follower network with two or more leaders and locally measured directions \( b_j^i \), the estimated orientation \( \hat{R}_i \), of agent \( i \) and the absolute positions of some leaders. For this, we first study the uniqueness of the target positions of the followers and present a distributed localization law for each agent. Under the proposed position localization law, estimated positions of all followers converge globally and asymptotically to the true positions.

**4. Position localization**

This section investigates position localization using locally measured directions \( b_j^i \), the estimated orientation \( \hat{R}_i \), of agent \( i \) and the absolute positions of some leaders. For this, we first study the uniqueness of the target positions of the followers and present a distributed localization law for each agent. Under the proposed position localization law, estimated positions of all followers converge globally and asymptotically to the true positions.

**4.1. Unique target configuration**

The uniqueness of the target configuration (the actual positions of agents) is a key property of the network that allows us to localize the network. In the sequel, under the nonlocalization and non-collinearity conditions of the true positions of the agents in Assumption 2, we show that the target configuration is uniquely defined using the direction constraints, estimate of orientation of agent \( i \), and the absolute positions of some leaders. The following lemma is similar to Trinh, et al. (2019, Lem. 1).

**Lemma 6.** Consider the twin-leader–follower network with two or more leaders and locally measured directions \( b_j^i \),\( i,j \in \mathbb{N} \). Suppose that Assumptions 1–2 hold, and the orientation of agent \( i \), \( R_i \in \mathbb{SO}(3) \), is available to \( i \) or otherwise can be estimated, e.g. Problem 1. Then the actual position of the agent \( i \), \( i \geq 3 \), i.e., \( p_i \in \mathbb{R}^3 \) is uniquely determined from its direction constraints \( b_j^i \) and \( p_j \) and the positions of its neighbors \( (p_j)_{j \in \mathbb{N}} \). Furthermore, \( p_i \) is uniquely computed as

\[ p_i = \left( \sum_{j \in \mathbb{N}} b_j^i \right)^{-1} \sum_{j \in \mathbb{N}} b_j^i p_j \]

where \( b_j^i = R_i b_j^i \) and \( p_j \in \mathbb{R}^{3 \times 3} \) denotes the projection matrix as defined in (1).

**4.2. Proposed position localization law**

Each follower agent \( i \) holds an initial estimate of its position \( \hat{p}_i(0) \in \mathbb{R}^3 \), and updates the estimate as follows

\[ \hat{p}_i = -\hat{R}_i \sum_{j \in \mathbb{N}} k_{p_{ij}} \hat{b}_j \hat{R}_i^{-1} (\hat{p}_i - \hat{p}_j) \]

where, \( k_{p_{ij}} > 0 \) is a positive gain, the local projection matrix \( P_{ij} = I - b_j^i (b_j^i)^T \), \( R_i = (b_i^j)^T P_{ij} R_i P_{ij} \), and \( \hat{p}_i(0) \) is initialized arbitrarily. The localization law (16) is implemented using only local direction measurements \( b_j^i \), estimate of orientation \( \hat{R}_i \), and estimates of its neighbors’ positions \( \hat{p}_j \), which are communicated by agents \( j \in \mathbb{N} \); in the case of leaders, \( \hat{p}_i = p_i \), \( i \in \mathbb{N} \).

The estimation law (16) is linear in the estimated state \( \hat{p}(t) := [\hat{p}_1^T(t), \ldots, \hat{p}_n^T(t)]^T \), and so the right side is globally Lipschitz in \( \hat{p}(t) \).
4.3. Stability analysis

We rewrite the localization law (16) as follows

\[ \dot{p}_i = f_i(p, R) - h_i(p, R), \]

where \( f_i(p, R) := -\sum_{j \in N_i} k_{ij} p_{ij}(\hat{p}_j - p_i) \) and \( h_i(p, R) := -(R_i - R_i^T) \sum_{j \in N_i} k_{ij} p_{ij}(\hat{p}_j - p_i) R_i^T (\hat{p}_j - p_i) - R_i \sum_{j \in N_i} k_{ij} p_{ij} R_i^T (\hat{p}_j - p_i). \) The above dynamics can be written in a more compact form

\[ \dot{\hat{p}} = f(\hat{p}) + h(\hat{p}), \quad (17) \]

where the stack vectors \( f(\hat{p}) = [f_1^T, \ldots, f_n^T]^T \) and \( h(\hat{p}) = [h_1^T, \ldots, h_n^T]^T. \) Due to the cascade structure of the system (17), we will study (17) using the stability theory for cascade systems (Angeli, 2004). Consider \( h(\hat{p}, t) \) in (17) as an input to the following nominal system

\[ \dot{\hat{p}} = f(\hat{p}). \quad (18) \]

We showed above that the position estimation via (16) using unidirectional communications is also almost globally convergent using the input-to-state stability theory.

**Corollary 2.** Suppose that Assumptions 1–2 hold and the sum in (16) is taken over all \( j \) to which agent \( i \) measures directions \( b_{ij}. \) Then, under the estimation law (16), \( \hat{p}(t) \rightarrow p \) almost globally and asymptotically as \( t \rightarrow \infty. \)

5. Conclusion

In this paper, a network pose localization scheme was proposed for twin-leader–follower networks by using direction measurements in \( \mathbb{R}^3. \) In particular, an orientation localization law in \( SO(3) \) and a position localization protocol were presented. We showed that the actual orientations and positions of all follower agents can be estimated almost globally and asymptotically. An extension of this work to systems with more general graph topologies is left as future work.

References


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