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Almost global convergence for distance- and area-constrained hierarchical formations without reflection

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Abstract— **This paper discusses formation shape control systems with both distance and signed area constraints, which aim to avoid the flipping or reflection ambiguity in a target formation shape. We prove an almost global convergence result for a triangle formation with general shape, by choosing a particular value of the control gain associated with the formation area term. This result extends the recent paper [1] on formation shape stabilization from equilateral triangles to general triangles, with an almost global convergence from almost all initial conditions. We then consider a hierarchical formation system comprised of multiple triangular shapes, and prove almost global convergence of a shape control algorithm with appropriate gains. Several simulations are provided to validate the theoretical results.**

I. Introduction

This paper focuses on a formation shape control problem that aims to design distributed controllers to stabilize a group of autonomous agents to reach a predefined target formation shape. Two popular approaches in formation control are the displacement-based approach and the distance-based approach [2]. The former one inherits ideas from linear consensus dynamics with an offset vector of desired relative positions that encodes information of desired formation shapes, while the latter describes a target formation shape by inter-agent distance constraints, and often employs a gradient control law associated with some potential functions for formation shape stabilization. The distance-based formation approach (in contrast to the displacement-based approach) does not require a common orientation alignment of each agent's local coordinate frame, which may be a substantial advantage in practice and therefore is the focus of this paper. For more recent advances and novel approaches on formation control, we refer readers to [2], [3].

In distance-based formation control, since the target formation shape is defined by a certain set of inter-agent distances, the uniqueness and existence of a target formation shape (up to translations and rotations) are often ensured by graph rigidity theory, at least locally. Recent years have witnessed much progress on shape stabilization and convergence analysis of rigid formation control systems. In general, a rigid formation constrained only by distances may not have a uniquely defined shape, since shape flipping or reflection ambiguity remains for the whole formation. Some recent papers [4], [5], [1] aim to resolve the flipping or reflection ambiguity by imposing additional geometric constraints for target formation shapes. The paper [5] considered both distance and planar (or volume) constraints in defining a formation potential function, which can exclude the reflection shape of a target formation with a gradient control law derived from formation potential functions. The paper [4] introduced the idea of incorporating signed areas in formation shape constraints to resolve reflection ambiguity in distance-based formation systems. More recently, a hierarchical formation control framework with both distance and area constraints was introduced in [1], which aims to stabilize a formation system comprising multiple equilateral triangular shapes into a desired shape without reflection.

This paper is built on [1], [4], and we resolve some outstanding issues in these papers on a hierarchical formation system that consists of multiple triangular sub-formations. We develop a distributed control law with a particular control gain $K = 4$ associated with the signed area term to ensure an almost global convergence of any triangular formation shape with both distance and signed area constraints. Since the result is independent of particular formation shape, it improves the result in [1] which considered hierarchical formations with multiple *equilateral* triangles. Based on the developed stabilization law for triangular shapes, we then prove an almost global convergence for hierarchical formations comprising general triangular sub-formations. The formation shape is uniquely defined by distance and area constraints (up to translations), and the reflection ambiguities are avoided.

This paper is organized as follows. Background and problem formulation are presented in Section II. Section III presents the main result of this paper. We will show an almost global convergence for triangular shapes with both distance and area constraints, and for hierarchical formations comprising multiple triangular shapes without reflection. Simulations are provided in Section IV to validate the theoretical results. Concluding remarks in Section V close this paper.

II. Problem setup

We consider a multi-agent formation system consisting of n agents and m edges in a 2D space that aims to achieve a predefined rigid formation shape. For background on graph rigidity theory and its applications in formation shape control, we refer the reader to [6]. The formation control

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structure is described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} = \{1, 2, \cdots, m\} \subset \mathcal{V} \times \mathcal{V}$.

Assume that the underlying target formation graph is a triangulated graph, which can be constructed by vertex extensions in a Henneberg construction sequence (see [7]). One example is shown in Fig. 1: the whole formation graph is constructed by several triangular cliques (i.e., complete subgraphs). We use (i, j, k) to define a triangular clique, which has to satisfy $\{(i, j), (j, k), (k, i)\} \in \mathcal{E}$. For example, in Fig. 1, we have nine triangular cliques. In the formation control problem under discussions, the index order of i, j, k in every clique also matters and should be exactly the same as that in the desired formation to avoid flipping or order reflection.

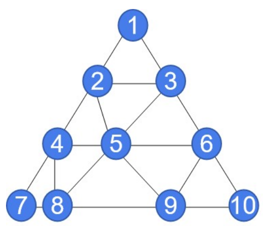

Fig. 1. An example of triangulated formation graph.

Let $p = [p_1^T, p_2^T, \cdots, p_n^T]^T \in \mathbb{R}^{2n}$ denote the aggregated positions of all agents with $p_i = [p_{ix}, p_{iy}]^T \in \mathbb{R}^2$. We introduce the *signed* area $Z_{i,j,k}$ for the clique (i, j, k) by (see also [4], [1])

$$Z_{i,j,k} := \frac{1}{2}\det \begin{bmatrix} 1 & 1 & 1 \\ p_i & p_j & p_k \end{bmatrix}. \quad (1)$$

The signed area represents both the area of the triangular sub-formation and the order of its vertexes (i, j, k). Obviously, if the three agents are not collinear and $Z_{i,j,k}$ is positive, the three agents' positions, p_i, p_k and p_j, are located in a clockwise ordering; otherwise they are in an opposite ordering. For example, in Fig. 1, $(1, 2, 3)$ has a negative signed area and $(2, 1, 3)$ has a positive signed area.

The formation control system is described by

$$\dot{p}_i(t) = u_i(t) \qquad i \in \mathcal{V}, \quad (2)$$

where $u_i(t) \in \mathbb{R}^2$ is the control input for agent i. The control objective for the multi-agent formation stabilization without reflection is to realize a target formation that satisfies

- $||p_i - p_j|| = d_{i,j}^*, \quad \forall (i, j) \in \mathcal{E}$
- $Z_{i,j,k} = Z_{i,j,k}^*, \quad \forall$ cliques(i, j, k)

where $d_{i,j}^*$ is the given desired distance between agents i, j, and $Z_{i,j,k}^*$ is the given desired signed area for the triangular clique (i, j, k). Of course, when the interagent distances of a clique of three agents are known, the magnitude of the area is defined, but the sign is not defined. Evidently, the distance and signed area constraints for a triangular clique specify a target triangular formation shape which needs to be stabilized. Obviously, translation or rotation of a target formation shape is permissible while a reflection is not.

Remark 1: One recent paper [7] solved the almost global stabilization of multi-agent triangulated formations, while formation reflection was not considered in its target formations. In [1] and [4], flipping elimination was resolved by adding an additional term of signed areas in formation potential functions. The results in [1] have been based on equilateral triangular formations. In this paper, we will develop control laws for general triangulated formations with an almost global convergence and without reflection.

III. Main results

In this section, by following [1], we propose a distributed and hierarchical approach to solve the stabilization control problem for triangulated formations with both distance and area constraints. We will prove an almost global convergence for general triangulated formations under a special value of control gain associated with the area control term.

A. Hierarchical control strategy

We use a gradient-based control law for a multi-agent formation control system to reach a target rigid formation with triangulated graph structure. The choice of potential functions is based on a hierarchical control strategy. In this strategy, the position of one agent (say agent i) is fixed. A potential function $V_{(i,j)}$ is constructed for the second agent j to stabilize its position to reach a given formation. By following the same procedure, another potential function $V_{(i,j,k)}$ is chosen for the third agent k, where the three agents i, j, k should be in a clique. Similarly, the positions of all other agents can be controlled.

Take the formation graph in Fig. 1 as an example. If we fix the position of agent 1, we can construct potential functions for each agent (except the stationary agent 1) for stabilizing a target formation shape. Following [1] and employing a cascade system modelling, we can decompose the hierarchical control layer of the formation graph in Fig. 1 as below:

Layer 1: $V_1 = 0$ (Agent 1 is stationary).
Layer 2: $V_2 = V_{(1,2)}$.
Layer 3: $V_3 = V_{(1,2,3)}$.
Layer 4: $V_5 = V_{(2,3,5)}, V_4 = V_{(2,4,5)}, V_6 = V_{(3,5,6)}$.
Layer 5: $V_8 = V_{(4,5,8)}, V_7 = V_{(4,7,8)}, V_9 = V_{(5,6,9)}$, $V_{10} = V_{(6,9,10)}$.

The potential functions $V_{(i,j)}$ and $V_{(i,j,k)}$ are defined in the following subsections. Note that the procedure is divided into several layers, in which the lower layer cannot influence the upper layer.

B. Analysis with a two-agent case

We first consider a two-agent formation. Suppose the formation system consists of two agents i and j. Keep agent i's position p_i fixed and let agent j with its position p_j be governed by

$$\dot{p}_j = -\frac{\partial V_{(i,j)}}{\partial p_j}, \quad (3)$$

with the potential function defined by (see [1])

$$V_{(i,j)} := \frac{1}{4}((||p_i - p_j||^2 - (d_{ij}^*)^2)^2, \quad (4)$$

where d_{ij}^* is the desired distance between agents i and j. We denote the distance error for edge (i,j) as $e_{ij} = ||p_i - p_j||^2 - (d_{ij}^*)^2$. The stability and convergence of (3) has been analyzed in [1] which shows that p_j converges to an equilibrium and all stable equilibria of p_j satisfy

$$||p_i - p_j|| = d_{ij}^*. \quad (5)$$

***Lemma** 1:* With the control law (3), agent j's trajectory asymptotically converges to a desired equilibrium point exponentially fast under almost all initial positions.

Proof: Since agent i is stationary, agent j's motion with the control law (3) is *constrained in the line* defined by its initial positions $p_2(0) - p_1$. Therefore, the 2-D dynamics for $\dot{p}_2$ is reduced to a 1-D dynamical system confined in the line of $p_2(0) - p_1$. Without loss of generality, we assume that the line $p_2(0) - p_1$ is parallel to the x-axis. According to [1, Theorem 1], agent j's trajectory constrained in the x-axis direction exponentially converges to an equilibrium point with a correct distance to agent i. Except the initial position collocated with agent i's position, which can be verified to be an equilibrium point of (3), albeit an unstable one, almost all initial positions ensure the convergence. ■

C. Analysis with a three-agent case

Now we consider a formation control system consisting of three agents, i, j and k. Assume that agent k's position p_k is governed by

$$\dot{p_k} = -\frac{\partial V_{(i,j,k)}}{\partial p_k}, \quad (6)$$

The potential function $V_{(i,j,k)}$ is defined as the same as that in [4] and [1],

$$V_{(i,j,k)} = \frac{1}{4}\Big((\|p_i - p_j\|^2 - {d_{ij}^*}^2)^2 + (\|p_j - p_k\|^2 - {d_{jk}^*}^2)^2 + (\|p_k - p_i\|^2 - {d_{ki}^*}^2)^2\Big) + \frac{1}{2}K(Z_{i,j,k} - Z_{i,j,k}^*)^2, \quad (7)$$

where d_{ij}^*, d_{jk}^* and d_{ki}^* denote the desired distances between the three pairs of every two agents, K denotes a positive control gain associated with the signed area term and $Z_{i,j,k}$ defined in (1).

Without loss of generality, we consider a triangular formation shown in Fig. 2, with the positions of the three agents described by

$$p_i = \begin{bmatrix} -a \\ 0 \end{bmatrix}, p_j = \begin{bmatrix} a \\ 0 \end{bmatrix}, p_k = \begin{bmatrix} x \\ y \end{bmatrix}, \quad (8)$$

where $a := d_{ij}^*/2 > 0$, d_1^* and d_2^* denote the desired distances between agents i,k and j,k, respectively. The desired signed area is defined by $Z_{i,j,k}^* > 0$ (and the sign constraint imposes no real loss of generality). By substituting

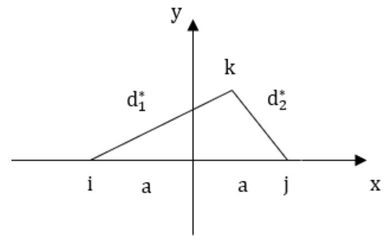

Fig. 2. A configuration of the three-agent triangular formation shape

the given data in (7), one obtains the following control for agent k

$$\begin{aligned}\dot{p_k} = &-(\|p_i - p_k\|^2 - {d_1^*}^2)(p_k - p_i) \\ &-(\|p_j - p_k\|^2 - {d_2^*}^2)(p_k - p_j) \\ &-\frac{1}{2}K(Z_{i,j,k} - Z_{i,j,k}^*)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(p_i - p_j),\end{aligned} \quad (9)$$

which is equivalently written as

$$\dot{p_k} = \begin{bmatrix} -x\big(2x^2+2y^2+6a^2-{d_1^*}^2-{d_2^*}^2\big)+a\big({d_1^*}^2-{d_2^*}^2\big) \\ -y\big(2x^2+2y^2+2a^2-{d_1^*}^2-{d_2^*}^2\big)+Ka\big(Z_{i,j,k}^*-ay\big) \end{bmatrix}. \quad (10)$$

We now analyze the equilibrium points and stability property of the system (9). The stability of each equilibrium can be analyzed by determining the signs of the corresponding eigenvalues of the Hessian matrix for the potential $V_{(i,j,k)}$, provided the Hessian is non-singular. [1] By substituting (8), we have the Hessian matrix as

$$H = \begin{bmatrix} 6x^2+2y^2+6a^2-{d_1^*}^2-{d_2^*}^2 & 4xy \\ 4xy & 6y^2+2x^2+2a^2-{d_1^*}^2-{d_2^*}^2+Ka^2 \end{bmatrix}. \quad (11)$$

The control gain K in the signed area term plays an important role in stabilizing a desired formation shape without reflection. In [1] the authors showed that, under the example of equilateral triangles, convergence to an undesired equilibrium can occur when K is small. The theorem below gives a single value of K applicable to all triangles which ensures the desired equilibrium point exists and is almost globally stable. The reason for choosing the special value $K = 4$ will become evident in the proof of the main theorem. We anticipate that a large set of K for ensuring almost global convergence is available, which will be discussed in future research.

***Theorem** 1:* Consider a three-agent formation control system with both distance and area constraints, while agents i and j are fixed and agent k is governed by the control in (9). Set $K = 4$ in the formation control law. Then with almost all initial conditions for p_k, the system (9) converges to the desired equilibrium at which the correct distances and signed area are attained.

Proof: From the equilibrium condition of the system $\dot{p_k} = 0$, an equilibrium point $[x,y]$ should satisfy the following formula

$$KZ_{i,j,k}^*x - ({d_1^*}^2 - {d_2^*}^2)y = (Ka - 4a)xy. \quad (12)$$

[1]The Hessian formula can be found in [8].

The control gain $K = 4$ facilitates the analysis of the above equilibrium formula (12). With $K = 4$, the right-hand side of (12) is zero, which then enables a clear characterization of different equilibrium points. Now we consider two cases, viz., (a). $d_1^* \neq d_2^*$ and (b). $d_1^* = d_2^*$.

(a). The case with $d_1^* \neq d_2^*$. For all equilibrium points of p_k, there necessarily holds

$$y = \frac{4Z_{i,j,k}^*}{{d_1^*}^2 - {d_2^*}^2}x, \quad x = \frac{{d_1^*}^2 - {d_2^*}^2}{4Z_{i,j,k}^*}y, \tag{13}$$

which will be used to analyze the stability property of equilibrium points. By substituting (13) into the equilibrium equation $\dot{p_k} = 0$ as in Eq. (9), we have

$$2\left(\frac{{d_1^*}^2 - {d_2^*}^2}{4Z_{i,j,k}^*}\right)^2 y^3 + 2y^3 + (4a^2 - m)y - 4aZ_{i,j,k}^* = 0, \tag{14}$$

where $m = {d_1^*}^2 + {d_2^*}^2 - 2a^2$. Note that the item m can be rewritten as $m = (x^* - a)^2 + {y^*}^2 + (x^* + a)^2 + {y^*}^2 - 2a^2$, where (x^*, y^*) is the desired position of agent k. It is obvious that $m > 0$. From ${d_1^*}^2 - {d_2^*}^2 = (x^* + a)^2 + {y^*}^2 - (x^* - a)^2 - {y^*}^2 = 4ax^*$, we have $x^* = \frac{{d_1^*}^2 - {d_2^*}^2}{4a}$ and $Z_{i,j,k}^* = ay^*$. Therefore, one can show

$$m = 2\left(\frac{{d_1^*}^2 - {d_2^*}^2}{4a}\right)^2 + 2\left(\frac{Z_{i,j,k}^*}{a}\right)^2, \tag{15}$$

which implies

$$\left(\frac{a}{Z_{i,j,k}^*}\right)^2 m = 2\left(\frac{{d_1^*}^2 - {d_2^*}^2}{4Z_{i,j,k}^*}\right)^2 + 2. \tag{16}$$

The above equality will often be used in later analysis to simplify several equations.

Furthermore, it holds that

$$\left(\frac{a}{Z_{i,j,k}^*}\right)^2 my^3 + (4a^2 - m)y - 4aZ_{i,j,k}^* = 0,$$

which implies

$$\left(y - \frac{Z_{i,j,k}^*}{a}\right)\left(\left(\frac{a}{Z_{i,j,k}^*}\right)^2 my^2 + \frac{am}{Z_{i,j,k}^*}y + 4a^2\right) = 0. \tag{17}$$

Hence, we can obtain three equilibrium points of p_k. They are

$$p_1 = \begin{bmatrix} \frac{{d_1^*}^2 - {d_2^*}^2}{4a} \\ \frac{Z_{i,j,k}^*}{a} \end{bmatrix}, \tag{18}$$

which is the correct equilibrium, and

$$p_2 = \begin{bmatrix} \frac{{d_1^*}^2 - {d_2^*}^2}{4}\left(-\sqrt{\frac{1}{4a^2} - \frac{4}{m}} - \frac{1}{2a}\right) \\ Z_{i,j,k}^*\left(-\sqrt{\frac{1}{4a^2} - \frac{4}{m}} - \frac{1}{2a}\right) \end{bmatrix},$$

$$p_3 = \begin{bmatrix} \frac{{d_1^*}^2 - {d_2^*}^2}{4}\left(\sqrt{\frac{1}{4a^2} - \frac{4}{m}} - \frac{1}{2a}\right) \\ Z_{i,j,k}^*\left(\sqrt{\frac{1}{4a^2} - \frac{4}{m}} - \frac{1}{2a}\right) \end{bmatrix}.$$

which correspond to undesired shapes with incorrect distances or area, and are termed undesired equilibrium points. Note that p_2 and p_3 exist if and only if $m \geq 16a^2$ (when $m = 16a^2$ the equilibrium points p_2 and p_3 reduce to a single equilibrium point that still corresponds to an undesired shape).

The correct equilibrium p_1 is the global minimum of the potential function and is asymptotically stable by the nature of the gradient system (9). For the undesired equilibrium points p_2 and p_3, we now analyze their stability and convergences by substituting them into (11) to see if the Hessian matrix is positive definite.

1) For the equilibrium p_2, we have

$$p_{2x}^2 + p_{2y}^2 = \left(\left(\frac{{d_1^*}^2 - {d_2^*}^2}{4}\right)^2 + {Z_{i,j,k}^*}^2\right)\left(-s - \frac{1}{2a}\right)^2,$$

where $s = \sqrt{\frac{1}{4a^2} - \frac{4}{m}}$.

From (16), we can show $\left(\frac{{d_1^*}^2 - {d_2^*}^2}{4}\right)^2 + Z_{i,j,k}^{*2} = \frac{1}{2}ma^2$. Therefore, it holds

$$\begin{aligned} p_{2x}^2 + p_{2y}^2 &= \frac{1}{2}ma^2\left(-s - \frac{1}{2a}\right)^2 \\ &= \frac{m}{4} - 2a^2 + \frac{s}{2}ma. \end{aligned}$$

To ensure p_2 is a stable equilibrium, the Hessian matrix at p_2 should be positive definite, or equivalently

$$\begin{aligned} &f(p_{2x}, p_{2y}) + 4p_{2x}^2 > 0, \\ &det(H) = \left(f(p_{2x}, p_{2y}) + 4p_{2x}^2 + 4p_{2y}^2\right) f(p_{2x}, p_{2y}) > 0, \end{aligned} \tag{19}$$

where $f(p_{2x}, p_{2y}) = 2p_{2x}^2 + 2p_{2y}^2 + 6a^2 - {d_1^*}^2 - {d_2^*}^2$.

We can simplify $f(p_{2x}, p_{2y})$ by substituting $m = {d_1^*}^2 + {d_2^*}^2 - 2a^2$, from which we have

$$\begin{aligned} f(p_{2x}, p_{2y}) &= 2p_{2x}^2 + 2p_{2y}^2 - (m - 4a^2) \\ &= \frac{m}{2} - 4a^2 + sma - (m - 4a^2) \\ &= m(sa - \frac{1}{2}). \end{aligned} \tag{20}$$

Note that the term m is always larger than 0. Since $(sa)^2 - \frac{1}{2}^2 = -\frac{4a^2}{m}$, we know that

$$sa < \frac{1}{2}, \tag{21}$$

from which we have $f(p_{2x}, p_{2y}) < 0$. It is obvious that the inequality in (19) does not hold when $f(p_{2x}, p_{2y}) < 0$, which means the Hessian matrix at p_2 is not positive definite, and therefore the equilibrium p_2 is not stable.

2) For the equilibrium p_3, similarly one has

$$p_{3x}^2 + p_{3y}^2 = \frac{1}{2}ma^2(h - \frac{1}{2a})^2 = \frac{m}{4} - 2a^2 - \frac{s}{2}ma.$$

It is easy to see $p_{3x}^2 + p_{3y}^2 < p_{2x}^2 + p_{2y}^2 < \frac{1}{2}\left(m - 4a^2\right)$, which implies

$$f(p_{3x}, p_{3y}) = 2p_{3x}^2 + 2p_{3y}^2 + 6a^2 - {d_1^*}^2 - {d_2^*}^2 < 0.$$

Hence, the Hessian matrix at p_3 is not positive definite, and thus the equilibrium point p_3 is not stable.

In conclusion, when $K = 4$ and $d_1^* \neq d_2^*$, with almost all initial conditions, p_k converges to the correct equilibrium p_1.

(b). The case with $d_1^* = d_2^*$. We denote by $h := \sqrt{{d_1^*}^2 - a^2}$ the desired height of the triangular formation. Hence, p_k is governed by

$$\dot{p_k} = \begin{bmatrix} -x(2x^2 + 2y^2 + 4a^2 - 2h^2) \\ -y(2x^2 + 2y^2 - 2h^2) + 4a(ah - ay) \end{bmatrix}. \tag{22}$$

First, when $x = 0$, the first entry of (22) equals 0 and by substituting it into the second entry of (22) we have

$$-y(2y^2 - 2h^2) + 4a(ah - ay) = 0,$$

which implies

$$(y - h)(y^2 + hy + 2a^2) = 0.$$

Therefore, one obtains an equilibrium $p_1 = [0, h]^T$, which is the correct equilibrium and is asymptotically stable (by the gradient nature of the system (9)).

When $h \geq \sqrt{8}a$, we have the other two equilibrium points described by

$$p_2 = \begin{bmatrix} 0 \\ \sqrt{\frac{h^2}{4} - 2a^2} - \frac{h}{2} \end{bmatrix}, p_3 = \begin{bmatrix} 0 \\ -\sqrt{\frac{h^2}{4} - 2a^2} - \frac{h}{2} \end{bmatrix}.$$

(When $h = \sqrt{8}a$, the equilibrium points p_2 and p_3 coincide as the same point.) We now analyze their stability properties.

1) For the equilibrium p_2, its Hessian matrix can be calculated as

$$H = \begin{bmatrix} -h^2 - 2h\sqrt{\frac{h^2}{4} - 2a^2} & 0 \\ 0 & h^2 - 6h\sqrt{\frac{h^2}{4} - 2a^2} - 8a^2 \end{bmatrix}.$$

As the first entry of H, $-h^2 - 2h\sqrt{\frac{h^2}{4} - 2a^2}$, is smaller than 0, the Hessian matrix is not positive definite. Hence, the equilibrium p_2 is unstable.

2) For the equilibrium p_2, its Hessian matrix can be calculated as

$$H = \begin{bmatrix} -h^2 + 2h\sqrt{\frac{h^2}{4} - 2a^2} & 0 \\ 0 & h^2 + 6h\sqrt{\frac{h^2}{4} - 2a^2} - 8a^2 \end{bmatrix}.$$

To ensure p_3 is stable, we must have $-h^2 + 2h\sqrt{\frac{h^2}{4} - 2a^2} > 0$, which implies

$$2\sqrt{\frac{h^2}{4} - 2a^2} > h, \Rightarrow 4\left(\frac{h^2}{4} - 2a^2\right) > h^2, \Rightarrow -8a^2 > 0.$$

But the above inequality can never hold which leads to a contradiction. Hence, the Hessian matrix at equilibrium p_3 is not positive definite and p_3 is an unstable equilibrium.

Then we consider $2x^2 + 2y^2 + 4a^2 - 2h^2 = 0$. In this case, the second entry of $\dot{p}_k$ is

$$4a^2 y + 4a(ah - ay) = 4a^2 h, \tag{23}$$

We have $4a^2 h \neq 0$, which implies there exists no equilibrium.

In conclusion, when the control gain is set as $K = 4$, the formation control system will almost globally converge to the correct equilibrium. ■

D. Stability analysis of the hierarchical formation system

Based on the stability results derived above, in this subsection we present the stability analysis for a hierarchical formation system consisting of triangular shapes. The proof will be based on the stability theory for cascade systems.

Theorem 2: Consider a hierarchical triangulated formation system comprising multiple triangle shapes and n agents under both distance and area constraints. Suppose that the potential function associated with each agent is designed based on the hierarchical control strategy detailed in Section III-A. Specifically, agent 1 is stationary, agent 2 follows the control law of (3) and all other agents follow the control law of (6) with the area control gain $K = 4$. Then the overall hierarchical formation system is almost globally asymptotically stable, and all agents (except the stationary agent 1) converge to equilibrium positions which satisfy the distance and area constraints from almost global initial positions.

Proof: According to the hierarchical structure of the formation control law, the overall formation system can be considered as a cascade system and we will use the stability theory of cascade systems (see e.g., [9, Chapter 9] and [10]) to prove the almost global convergence. As proved in Lemma 1 (also see [1, Theorem 1]), if agent 2's initial position is not collocated with agent 1, agent 2 *almost globally converges to an equilibrium point* with the desired distance exponentially fast, and the converged equilibrium point will be constrained in the line $p_2(0) - p_1$. Based on the gradient control law and the fact that agent 1 is stationary, the distance function $\|p_2(t) - p_1\|$ always decreases, which implies that agent 2's trajectory, which is confined in the line $p_2(0) - p_1$, is always bounded. Furthermore, since agent 1's position is fixed, the motion of agent 2 is fixed in the line of $p_2(0) - p_1$ which reduces the 2-D dynamics of $\dot{p}_2$ to a 1-D dynamical system in the same line. As shown in Lemma 1, given an initial position $p_2(0)$ not collocated with p_1, the asymptotic convergence of the reduced 1-D dynamical system in the line implies the almost globally asymptotic convergence of agent 2's position to an equilibrium. For agent 3 with the control law of (6) under $K = 4$, Theorem 1 shows that, if agent 2 is at a desired equilibrium point, agent 3's trajectory almost globally converges to a desired equilibrium point and its position is always bounded by the nature of the gradient control law (6). Furthermore, the system (6) satisfies the assumption A0-A2 of [10], which, together with the ultimate boundedness property, proves that the cascade formation system with agents 2 and 3 is almost globally asymptotically stable by invoking [9, Corollary 9.3] and [10, Proposition 2]. As analyzed in the proof of Theorem 1, since the Hessian matrix at a correct equilibrium is always positive definite, this implies exponential convergence to a correct equilibrium.

For all other agents, by following a mathematical induction of Theorem 1 and repeating the same procedure of the above analysis via [9, Corollary 9.4] and [10, Proposition 2], it proves that the overall hierarchical formation system is almost globally asymptotically stable and all agents (except the stationary agent 1) converge to desired equilibrium points that satisfy correct distance and area constraints. ∎

IV. Simulation Results

A. Trajectory of agent k in the three-agent case with an area control gain $K = 4$

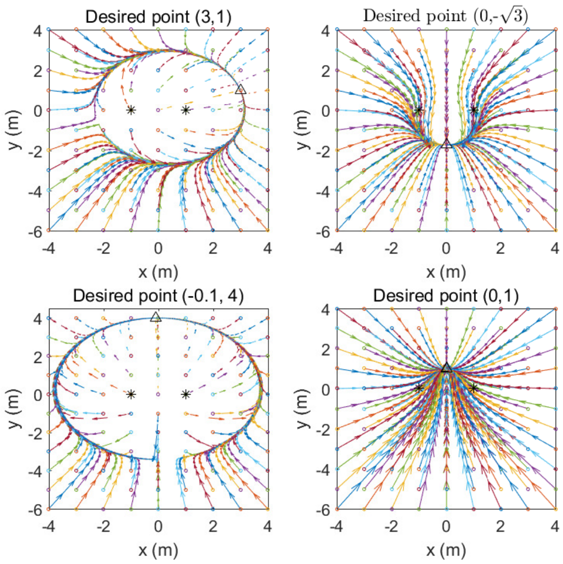

Fig. 3. Phase portrait of agent k when $K = 4$ with different initial positions and target shapes

We show several simulations on the equilibrium stability of Theorem 1. We consider four different triangular shapes and plot the trajectories of agent k with different initial positions. The star markers are the positions of agent i and j with fixed positions at $(-1, 0)$ and $(1, 0)$, respectively. Round markers indicate different initial locations of agent k, and the triangle markers are the final positions of agent k with different initial positions. It can be observed in Fig. 3 that in all cases p_k converges asymptotically with almost all initial positions to the desired equilibrium position which satisfies both the given distance and area constraints.

B. Stabilization of a hierarchical formation system

In this section, we apply the hierarchical control strategy to solve the formation stabilization problem with a desired shape shown in Fig. 1. According to the strategy in Section III-A, we fix the position of agent 1 with $(1, \sqrt{3})$ in the simulation. There are 5 layers in this formation control strategy (see Section III-A). The desired shapes in the hierarchical formation control strategy are specified as follows: cliques $(1, 2, 3)$ and $(6, 9, 10)$ are equilateral triangles, clique $(5, 8, 9)$ is an isosceles triangle, cliques $(4, 8, 7)$ and $(4, 5, 8)$ are right-angled triangles and the others are arbitrary triangles with desired shapes. The trajectories of all agents and their final converged positions are depicted in Fig. 4, which shows that in all cases of different triangle shapes, all agents can converge to desired equilibrium points that realize a correct target formation.

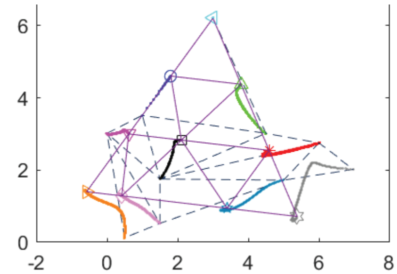

Fig. 4. The trajectories of all agents in a hierarchical formation, and their final positions with a converged shape

V. Conclusions

In this paper we have discussed formation shape control with both distance and area constraints, and proved an almost global convergence for formation shape stabilization by choosing a special control gain associated with the area control term. The results are applicable to general triangular formation shapes, which can avoid reflection ambiguity and thus ensure a unique shape up to translations. Based on the stability theory for cascade systems, we have also proved an almost global convergence for hierarchical formation systems comprised of multiple general triangular sub-formations.

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References

[1] T. Sugie, B. D. O. Anderson, Z. Sun, and H. Dong, "On a hierarchical control strategy for multi-agent formation without reflection," *Proc. of the 57th IEEE Conference on Decision and Control (CDC'18)*, pp. 2023–2028, 2018.

[2] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[3] Z. Sun, *Cooperative Coordination and Formation Control for Multi-agent Systems*. Springer, 2018.

[4] B. D. O. Anderson, Z. Sun, T. Sugie, S.-i. Azuma, and K. Sakurama, "Formation shape control with distance and area constraints," *IFAC Journal of Systems and Control*, vol. 1, pp. 2–12, 2017.

[5] E. D. Ferreira-Vazquez, E. Hernandez-Martinez, J.-J. Flores-Godoy, G. Fernández-Anaya, and P. Paniagua-Contro, "Distance-based formation control using angular information between robots," *Journal of Intelligent & Robotic Systems*, vol. 83, no. 3-4, pp. 543–560, 2016.

[6] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Systems*, vol. 28, no. 6, pp. 48–63, 2008.

[7] X. Chen, M.-A. Belabbas, and T. Başar, "Global stabilization of triangulated formations," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 172–199, 2017.

[8] Z. Sun and T. Sugie, "Identification of Hessian matrix in distributed gradient-based multi-agent coordination control systems," *Numerical Algebra, Control & Optimization, accepted and in press*, 2018.

[9] W. J. Terrell, *Stability and stabilization: an introduction*. Princeton University Press, 2009.

[10] D. Angeli and L. Praly, "Stability robustness in the presence of exponentially unstable isolated equilibria," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1582–1592, 2011.