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Modelling of Individual Behaviour in the DeGroot–Friedkin Self-Appraisal Dynamics on Social Networks

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***Abstract*— The DeGroot–Friedkin model describes how an individual's self-confidence in his or her own opinion evolves as that individual participates in a group discussing a sequence of topics; as the individual has more impact or less impact (termed social power) on a given topic discussion, his or her self-confidence increases or decreases due to the process of *reflected self-appraisal*. This paper proposes a broad generalisation of the DeGroot–Friedkin model by allowing each individual's self-appraisal process to be distorted by behavioural characteristics such as humility. We establish the generalised dynamical model for the evolution of individuals' social power (a measure of an individual's contribution to each topic discussion). For some types of individuals, whom we term "humble" and "unreactive", results are provided on the existence of equilibria, whether such equilibria are unique, and convergence to said equilibria. Simulations are used to illustrate that networks of "emotional" individuals, who at times act like humble individuals and at other times like arrogant individuals, can have at least two attractive equilibria.**

I. Introduction

In recent years, the systems and control community has become increasingly interested in models of opinion dynamics [1], in part due to its close relation with algorithms for coordination of multi-agent systems. Opinion dynamics models describe how an individual's opinion(s) evolve as he or she learns of, and processes, the opinion(s) of neighbouring individuals in the network by way of interactions.

The discrete-time French–DeGroot model (or simply DeGroot model) [2], [3] is the fundamental model of opinion dynamics. The DeGroot model has been expanded in many major directions, including the incorporation of confidence bounds in the Hegselmann–Krause model to capture homophily [4], [5], negative weights to capture antagonistic interactions in the Altafini model [6]–[8], and an individual's prejudice to his or her initial opinion in the Friedkin–Johnsen model [9]. We do not further detail such works, and instead refer the reader to survey papers such as [1].

This paper does deal with a significant and recent advancement of the DeGroot model, namely the DeGroot–Friedkin model [10]. In the theory of *reflected self-appraisal* (or simply self-appraisal) [11]–[13], an individual may update his or her "self-confidence" during group activities depending on how much impact he or she is having on said activity. Naturally, there is interest in capturing this process in opinion dynamics models. In the DeGroot model, a consensus of opinions (in which every individual has the same opinion) is eventually reached if the network satisfies some connectivity requirements, and it is possible to quantify the amount of relative contribution, or social power, each individual had in determining the final consensus value. The DeGroot–Friedkin model therefore proposed to capture an individual's self-appraisal dynamics to investigate how an individual's self-confidence in his or her own opinion value evolves along a sequence of topic discussions.

The dynamics of the DeGroot–Friedkin model was studied theoretically in [10] for strongly connected networks, weakly connected networks in [14]. These works show that for constant interaction topologies, each individual's self-confidence asymptotically converges to a steady state value, dependent only on the network structure, in the limit of the topic sequence. The works [15], [16] extended the model to incorporate dynamic (topic-varying) interaction topologies, and showed that each individual's initial self-confidence was forgotten exponentially fast. Noise in the self-appraisal process, and memory of past discussions, was introduced in [17]. Empirical tests were conducted on a modified self-appraisal model, in which the Friedkin–Johnsen model replaces the DeGroot model [18].

The DeGroot–Friedkin model proposes that after a given topic discussion has reached a consensus, an individual updates his or her self-confidence for the next topic to be *equal* to the relative contribution, termed *social power*, he or she had in that given topic. Thus, in the above discussed literature, an individual's "self-confidence" is synonymous with "social power". One can however postulate that some individuals may not update their self-confidence to be equal to their social power due to individual behaviour traits. For example, certain individuals are "arrogant" or "humble", and so might approach the next discussion with a higher or lower self-confidence than a "well-adjusted" individual given the same social power in the current issue.

A primary novel contribution of this paper to generalise the DeGroot–Friedkin model by proposing that each individual may have different behaviours, captured by a smooth mapping, so that an individual's self-confidence arising from the self-appraisal mechanism may be distorted from his or her true social power. This advance leads to a more

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realistic model, but which has significantly more complicated dynamics than the original model in [10]. We then establish the dynamical equations governing social power evolution for the generalised model and prove that the self-confidence trajectories retain desired properties of the original model, e.g. that consensus is reached for every issue. Four new classes of behaviour functions are proposed in addition to the original model function. Above, we already gave examples of an "arrogant" individual and a "humble" individual. Two other types of behaviour functions are also defined, which we term "emotional" and "unreactive". We then establish the existence of at least one non-trivial equilibrium for (possibly mixed) networks of humble, unreactive, and well-adjusted (original model function) individuals. For networks that have either humble and well-adjusted individuals, this equilibrium is unique, and convergence occurs for almost all initial conditions exponentially fast. Last, we present illustrative simulations to show that for networks of emotional individuals, multiple attractive equilibria can exist.

The rest of the paper is organised as follows. Section II provides mathematical notation and an introduction to graph theory. Section III introduces the original DeGroot–Friedkin model and our proposed generalisation to incorporate individual behaviour. Section IV provides several theoretical results which establish properties of the self-confidence trajectories. Conjectures arising from simulations are presented in Section V, and the paper is concluded in Section VI.

II. Notation and Preliminaries

To begin, we establish the mathematical notation to be used in this paper. The n-column vector of all ones and zeros is given by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively. The $n\times n$ identity matrix is given by $\boldsymbol{I}_n$. The i^{th} canonical base unit vector of $\mathbb{R}^n$ is denoted as $\mathbf{e}_i$. The 1-norm of a matrix is denoted by $\|\cdot\|_1$.

We say that a matrix $\boldsymbol{A}$ is nonnegative (respectively positive) if all of its entries a_{ij} are nonnegative (respectively positive), and is denoted by $\boldsymbol{A}\geq 0$ and $\boldsymbol{A}>0$ respectively. A square matrix $\boldsymbol{A}\geq 0$ is said to be row-stochastic if, for all $i=1,\ldots,n$, there holds $\sum_{j=1}^n a_{ij}=1$. The matrix $\text{diag}(x_i)$ is the diagonal matrix with the i^{th} diagonal entry being x_i. For a real square matrix $\boldsymbol{A}$, spectral radius is denoted by $\rho(\boldsymbol{A})$, while $\lambda_i(\boldsymbol{A})$ denotes the i^{th} eigenvalue of $\boldsymbol{A}$. The n-simplex is $\Delta_n=\{\boldsymbol{x}\in\mathbb{R}^n : 0\leq\boldsymbol{x}, \mathbf{1}_n^\top\boldsymbol{x}=1\}$, while the simplex excluding the corner points and the interior of the simplex are $\widetilde{\Delta}_n = \Delta_n\backslash\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ and $\text{int}(\Delta_n)=\{\boldsymbol{x}\in\mathbb{R}^n : 0<\boldsymbol{x}, \mathbf{1}_n^\top\boldsymbol{x}=1\}$, respectively. We further define the sets $\Xi_n=\{\boldsymbol{x}\in\mathbb{R}^n : 0\leq x_i\leq 1, i\in\{1,\ldots,n\}\}$, $\text{int}(\Xi_n)=\{\boldsymbol{x}\in\mathbb{R}^n : 0<x_i<1, i\in\{1,\ldots,n\}\}$. A square matrix $\boldsymbol{A}\geq 0$ is primitive if $\exists k\in\mathbb{N}$ such that $\boldsymbol{A}^k>0$ [19, Definition 1.12].

A. Graph Theory

For a given not necessarily symmetric matrix $\boldsymbol{A}\geq 0$, we associate with it a directed graph $\mathcal{G}[\boldsymbol{A}]=(\mathcal{V},\mathcal{E}[\boldsymbol{A}],\boldsymbol{A})$, where $\mathcal{V}=\{v_1,\ldots,v_n\}$ is the set of vertices of $\mathcal{G}[\boldsymbol{A}]$ and in the context of this paper, each vertex (node) represents an individual in a population of size n. An edge $e_{ij}=(v_i,v_j)$ is in the set of ordered edges $\mathcal{E}[\boldsymbol{A}]\subseteq\mathcal{V}\times\mathcal{V}$ if and only if $a_{ji}>0$. A self-loop for node v_i exists if $e_{ii}\in\mathcal{E}[\boldsymbol{A}]$. The edge e_{ij} is said to be incoming with respect to v_j and outgoing with respect to v_i, and connotes that v_j learns of some information (typically an opinion value) from v_i. A directed path is a sequence of edges of the form $(v_{p_1},v_{p_2}),(v_{p_2},v_{p_3}),\ldots$, where $v_{p_i}\in\mathcal{V}$ are distinct and $e_{p_ip_{i+1}}\in\mathcal{E}$. A graph $\mathcal{G}[\boldsymbol{A}]$ is *strongly connected* if and only if there is a path from every node to every other node, which is equivalent to $\boldsymbol{A}$ being *irreducible* [20]. A directed spanning tree is a directed graph formed by directed edges of the graph that connects all the nodes, and where every vertex apart from the root has exactly one incoming edge. A graph is said to contain a directed spanning tree if a subset of the edges forms a directed spanning tree. A directed cycle is a directed path that starts and ends at the same vertex, and contains no repeated vertex except for the initial (which is also the final) vertex. A graph is *aperiodic* if the largest integer k that divides the length of every cycle of the graph is $k=1$ [19]. Note that any graph with a self-loop is aperiodic. Results linking $\mathcal{G}[\boldsymbol{A}]$ to the primitivity of $\boldsymbol{A}$, and eigenvectors of $\boldsymbol{A}$, are now given.

Lemma 1 ([19, Proposition 1.35]). *The graph $\mathcal{G}[\boldsymbol{A}]$ is strongly connected and aperiodic if and only if $\boldsymbol{A}$ is primitive.*

Lemma 2 (Perron-Frobenius Theorem [20]). *For a row-stochastic irreducible $\boldsymbol{A}$, there are strictly positive left and right eigenvectors $\boldsymbol{u}^\top$ and $\mathbf{1}_n$ associated with the simple eigenvalue 1. With normalisation satisfying $\boldsymbol{u}^\top\mathbf{1}_n=1$, we call $\boldsymbol{u}^\top$ and $\mathbf{1}_n$ the* dominant *left and right eigenvectors of $\boldsymbol{A}$, respectively.*

III. Self-Appraisal Dynamics with Individual Behaviour

In this section, we introduce the original DeGroot–Friedkin model and then present a generalisation of the model. Consider a population of $n\geq 3$ individuals, labelled as $1,\ldots n$, with index $\mathcal{I}=\{1,\ldots,n\}$. The n individuals sequentially discuss topics, with the topic sequence being indexed by $\mathcal{S}=\{0,1,2\ldots,\}$. For topic $s\in\mathcal{S}$, let $z_i(s,t)\in\mathbb{R}$ denote individual i's opinion at time $t=0,1,2,\ldots$. For any $s\in\mathcal{S}$, individual i's opinion evolves as

$$z_i(s,t{+}1)=w_{ii}(s)z_i(s,t){+}(1{-}w_{ii}(s))\sum_{j=1}^n c_{ij}z_j(s,t) \quad (1)$$

where $w_{ii}(s)\in[0,1]$ is individual i's *self-confidence* or self-weight, and $w_{ij}(s)=(1-w_{ii}(s))c_{ij}$ is the weight assigned by individual i to individual j's opinion $z_j(s,t)$. Here, c_{ij} is the *relative*[1] *interpersonal weight* individual i accords the opinion $z_j(s,t)$ of individual j. We assume that $\sum_{j=1}^n c_{ij}=1$ and $c_{ii}=0$ for any $i\in\mathcal{I}$, which implies that $\sum_{j=1}^n w_{ij}(s)=1$ for all s and i. For simplicity, we denote $y_i(s)=w_{ii}(s)$, and assume that the initial conditions satisfy

[1]The term relative is used to stress that $c_{ii}=0$. In this model, individual i's self-confidence $w_{ii}(s)=y_i(s)$ evolves over $s\in\mathcal{S}$, and thus the weight individual i accords to individual $j\neq i$'s opinion is adjusted using $1-w_{ii}(s)$ and the relative weight c_{ij}.

Assumption 1. *The initial self-confidence $y_i(0)$ satisfies $0 \leq y_i(0) < 1, \forall\, i \in \mathcal{I}$ and $\exists j \in \mathcal{I} : y_j(0) > 0$.*

In compact form, one can then write

$$\boldsymbol{z}(s, t+1) = \boldsymbol{W}(s)\boldsymbol{z}(s,t), \quad (2)$$

where $\boldsymbol{z}(s,t) = [z_1(s,t), \ldots, z_n(s,t)]^\top$ is the vector of individuals' opinions and

$$\boldsymbol{W}(s) = \boldsymbol{Y}(s) - (\boldsymbol{I}_n - \boldsymbol{Y}(s))\boldsymbol{C} \quad (3)$$

is the row-stochastic influence matrix. Here, $\boldsymbol{Y}(s) = \text{diag}(y_i(s))$ is a diagonal matrix of the individuals' self-confidences and $\boldsymbol{C} = \{c_{ij}\}$ is the matrix of relative interpersonal weights among the network of n individuals. A strongly connected graph is a *star graph* if and only if there exists a unique "centre" node v_i such that every edge of the graph is incoming or outgoing with respect to v_i. We make the following assumption regarding the network structure.

Assumption 2. *The graph $\mathcal{G}[\boldsymbol{C}]$ is strongly connected, it is not a star graph*[2]*, and is constant for all $s \in \mathcal{S}$.*

Under Assumption 2, $\boldsymbol{C}$ is irreducible, and we choose a corresponding left positive eigenvector $\boldsymbol{\gamma}^\top = [\gamma_1, \ldots, \gamma_n]$ as the associated left eigenvector (see Lemma 2). Importantly, for non-star $\mathcal{G}[\boldsymbol{C}]$, we have that $\gamma_i < 1/2$ for all $i \in \mathcal{I}$ [10].

If $\exists i \in \mathcal{I}$ such that $y_i(s) = w_{ii}(s) > 0$ and $\nexists j \in \mathcal{I}$ such that $y_j(s) = 1$, then one can show that the graph $\mathcal{G}[\boldsymbol{W}(s)]$ is strongly connected and aperiodic, i.e. $\boldsymbol{W}(s)$ is primitive. If there exists a unique $i \in \mathcal{I}$ such that $y_i(s) = 1$ and $y_j(s) < 1$ for all $j \neq i$, then we can similarly show that $\mathcal{G}[\boldsymbol{W}(s)]$ has a directed spanning tree with v_i as the root node and v_i having no incoming edges. We will show in the sequel that for all $y_i(0)$ satisfying Assumption 1, $\mathcal{G}[\boldsymbol{W}(s)]$ is either strongly connected and aperiod, or has a directed spanning tree with the root node having no incoming edges, for all $s \in \mathcal{S}$. In both these cases, standard results on the DeGroot model [1] show that Eq. (2) converges as

$$\lim_{t\to\infty} \boldsymbol{z}(s,t) = \mathbf{1}_n \boldsymbol{x}^\top(s)\boldsymbol{z}(s,0) \quad (4)$$

where $\boldsymbol{x}^\top(s)$ is the left eigenvector of $\boldsymbol{W}(s)$ associated with the simple eigenvalue at 1. In particular, if $\mathcal{G}[\boldsymbol{W}(s)]$ is strongly connected and aperiodic, $\boldsymbol{x}^\top(s)$ is the dominant left eigenvector (see Lemma 2), and if $\mathcal{G}[\boldsymbol{W}(s)]$ has a directed spanning tree with root node v_i having no incoming edges, then $\boldsymbol{x}(s) = \mathbf{e}_i$. This implies that the opinions reach a consensus value of $\boldsymbol{x}(s)^\top \boldsymbol{z}(s,0)$. Thus, $x_i(s) \in [0,1]$ represents the *relative contribution* of individual i towards the final consensus value, and is termed the *social power* of individual i for issue s in [10], [16] to reflect this relative contribution.

The DeGroot–Friedkin model aims to capture the evolution of $y_i(s)$ by modelling *reflected self-appraisal*. More specifically, assuming convergence as in Eq. (4) is obtained, then for every $i \in \mathcal{I}$, individual i's self-confidence at the next issue $s+1$ becomes

$$y_i(s+1) = x_i(s). \quad (5)$$

In other words, at the end of discussion for topic s, individual i takes note of his or her relative contribution (social power) $x_i(s)$, and sets his or her self-confidence $y_i(s+1)$ for the next topic to be his or her social power $x_i(s)$. Then, discussion on topic $s+1$ begins, again using the DeGroot dynamics in Eq. (1) but with $s+1$ replacing s in that equation. It is worth noting that as y_i changes to $y_i(s+1)$, the constant relative weights c_{ij} are still used to scale the $w_{ij}, j \neq i$ to ensure that $\sum_{j=1}^n w_{ij} = 1$ holds for all $i \in \mathcal{I}$ as required in the DeGroot model. Let $\boldsymbol{y} = [y_1, \ldots, y_n]^\top$. We summarise the key results on $\boldsymbol{y}(s), s \in \mathcal{S}$ in the following lemma.

Lemma 3 ([10], [16])**.** *Suppose that Assumptions 1 and 2 hold and that every individual $i \in \mathcal{I}$ updates his or her self-confidence $y_i(s)$ according to Eq. (5) with opinion evolution captured by Eq. (1). Then, the following holds*

1) *The vector $\boldsymbol{y}(s)$ converges exponentially fast as $\lim_{s\to\infty} \boldsymbol{y}(s) = \boldsymbol{y}^*$, with the point $\boldsymbol{y}^* \in int(\Delta_n)$.*
2) *For any $i, j \in \mathcal{I}$, one has $y_i^* > y_j^*$ or $y_i^* = y_j^*$ if and only if $\gamma_i > \gamma_j$ or $\gamma_i = \gamma_j$, respectively, where γ_i is the i^{th} entry of the dominant left eigenvector $\boldsymbol{\gamma}^\top$ of $\boldsymbol{C}$.*

A. Generalisation of the DeGroot–Friedkin Model

We now propose a generalisation of the DeGroot–Friedkin model to capture an individual's behaviour during the *self-appraisal process.* In particular, we replace Eq. (5) with

$$y_i(s+1) = \phi_i(x_i(s)) \quad (6)$$

where $\phi_i(x_i(s))$ is a function capturing individual i's behaviour during the reflected self-appraisal process. The following assumption is placed on ϕ_i.

Assumption 3. *For every $i \in \mathcal{I}$, $\phi_i(x_i) : [0,1] \to [0,1]$ is a smooth monotonically increasing function satisfying $\phi_i = 0 \Leftrightarrow x_i = 0$ and $\phi_i = 1 \Leftrightarrow x_i = 1$.*

Note that the $x_i(s)$ for a fixed s necessarily sum to 1, because of how they are defined. However, no such restriction applies to the $y_i(s)$. Nevertheless, because of the restriction on the $x_i(s)$, it follows easily from Assumption 3 that if any $y_i(s)$ takes the value 1 for $s > 0$, the other $y_j(s), j \neq i$ all assume the value 0.

The map ϕ_i is a function modifying individual i's perception of social power. That is, individual i's social power on topic s is $x_i(s)$, but due to behaviour, e.g. arrogance or humility, observes a *distorted* value of social power $\phi_i(x_i(s))$ and thus updates his or her self-confidence for topic $s+1$, $y_i(s+1)$, to be equal to the observed (and distorted) social power $\phi_i(x_i(s))$. Obviously, the original DeGroot–Friedkin model is just Eq. (6) with ϕ_i as the identity map. In the sequel, we will propose and study four new classes of maps for ϕ_i to capture different types of behaviours. We now establish the dynamical equations for $x_i(s)$ (which is equivalent to establishing the dynamical equations for $y_i(s)$

[2] Star graphs result in convergence to $\lim_{s\to\infty} \boldsymbol{y}(s) = \mathbf{e}_i$ for general initial conditions $y_i(0), i \in \mathcal{I}$, where v_i is the centre node, which is a special case of the DeGroot–Friedkin dynamics.

given that ϕ_i is surjective for all i). Let us define a mapping $\boldsymbol{\Phi} : \boldsymbol{x}(s) \mapsto \boldsymbol{y}(s+1)$ by stacking together the functions ϕ_i in the obvious way. The following result establishes the mapping we shall use to study the dynamics of $\boldsymbol{x}(s), s \in \mathcal{S}$.

Theorem 1. *Suppose that Assumptions 1, 2 and 3 hold, and that for any topic $s \in \mathcal{S}$, the n individuals in the network discuss opinions according to Eq. (2) and update their self-confidence according to Eq. (6). Then,*

$$\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{y}(s+1)) = \boldsymbol{F}(\boldsymbol{\Phi}(\boldsymbol{x}(s))) \tag{7}$$

where

$$\boldsymbol{F}(\boldsymbol{y}) = \begin{cases} \mathbf{e}_i & \text{if } y_i = 1 \text{ for any } i \\ \alpha(\boldsymbol{y}) \begin{bmatrix} \frac{\gamma_1}{1-y_1} \\ \vdots \\ \frac{\gamma_n}{1-y_n} \end{bmatrix} & \text{otherwise} \end{cases} \tag{8}$$

with $\alpha(\boldsymbol{y}) = 1/\sum_{i=1}^{n} \frac{\gamma_i}{1-y_i}$. The map $\boldsymbol{F} \circ \boldsymbol{\Phi} : \Delta_n \mapsto \Delta_n$ is continuous and smooth on Δ_n, and $\boldsymbol{x}(s) \in \Delta_n$ and $\boldsymbol{y}(s) \in \Xi_n$ for all $s \in \mathcal{S}$.

Proof. First, let us consider the case where $y_i(s) = 1$ for some $s \in \mathbb{S}$, and without loss of generality further assume that $i = 1$. (As already noted, there can be at most one $y_i(s)$ assuming the value 1). Since $y_1(s) = 1$, one can verify using Eq. (3) that $\boldsymbol{W}(s)$ takes the form of

$$\boldsymbol{W}(s) = \begin{bmatrix} 1 & \mathbf{0}_{n-1}^\top \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} \tag{9}$$

where $\boldsymbol{W}_{21} \in \mathbb{R}_{\geq 0}^{n-1}$ and $\boldsymbol{W}_{22} \in \mathbb{R}_{\geq 0}^{(n-1)\times(n-1)}$. In particular, one can show that $\mathcal{G}[\boldsymbol{W}]$ has a directed spanning tree with root node v_1 having no incoming edges. Our discussions below Eq. (2) then allows us to conclude that $\boldsymbol{x}(s) = [1, 0, \ldots, 0]^\top$. Replacing s with $s+1$, it is then clear that $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{y}(s+1))$ as claimed.

We now consider the case where $\nexists i : y_i(s+1) = 1$. Using equation Eq. (3) for $\boldsymbol{W}(s+1)$ it is evident that

$$\begin{aligned} &\boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}\boldsymbol{W}(s+1) \\ &= \boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}\boldsymbol{Y}(s+1) + \boldsymbol{\gamma}^\top\boldsymbol{C} \\ &= \boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}\boldsymbol{Y}(s+1) \\ &\quad + \boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}(\boldsymbol{I}_n - \boldsymbol{Y}(s+1)) \\ &= \boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1} \end{aligned} \tag{10}$$

The positivity of the γ_i and the fact that $y_i(s+1) < 1$ guarantees that each entry of $\boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}$ is positive, and the vector is a left eigenvector of $\boldsymbol{W}(s+1)$ corresponding to the simple unity eigenvalue. Normalisation of $\boldsymbol{\gamma}^\top(\boldsymbol{I}_n - \boldsymbol{Y}(s+1))^{-1}$ then yields the map in Eq. (8).

That $\boldsymbol{F}$ is smooth and continuous was proved in [16, Corollary 2]. Assumption 3 yields that $\boldsymbol{\Phi} : \Xi_n \mapsto \Delta_n$ is continuous and smooth; the composition of two smooth and continuous maps is also smooth and continuous. Since $\sum_{i=1}^{n} F_i = 1$, we have $\boldsymbol{x}(s) \in \Delta_n$ for all $s \in \mathcal{S}$, and from Assumption 3 we conclude that $\boldsymbol{y}(s) \in \Xi_n$ for all $s \in \mathcal{S}$. ☐

We remark here that the slight abuse of notation $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{y}(s+1))$ arises because the DeGroot–Friedkin model is a two-stage model. For any given issue s, the vector of individuals' self-confidences $\boldsymbol{y}(s)$ helps to define the opinion dynamics matrix $\boldsymbol{W}(s)$, as in Eq. (3). The network first discusses opinions according to Eq. (2). Then, and at the end of discussion, i.e. when a consensus of opinions is reached, the social power vector $\boldsymbol{x}(s)$ is obtained and each individual i evaluates his or her self-confidence for the next issue $s+1$ as $y_i(s+1) = \phi_i(x_i(s))$, giving rise to Eq. (7).

If $y(s') = \mathbf{e}_i$ for some $i \in \mathcal{I}$ and $s' \in \mathcal{S}$, then clearly that individual i will continue to have maximal self-confidence and self-confidence, $y_i(s) = 1$ and $x_i(s) = 1$, while all other individual $j \neq i$ will have no self-confidence and social power for all $s > s'$. Thus, initial conditions satisfying $\exists i : y_i(0) = 1$ and $y_j < 1 \forall j \neq i$ leads to trivial and non-generic dynamics. The remainder of this paper therefore focuses on more general initial conditions satisfying Assumption 1 (which does not preclude the possibility that $\boldsymbol{y}(s') = \mathbf{e}_k$ for some $k \in \mathcal{I}$ and $s' > 0$).

B. Classes of Behaviour Functions

We now introduce four classes of behaviour functions, each of which is illustrated in Fig. 1a-1d.

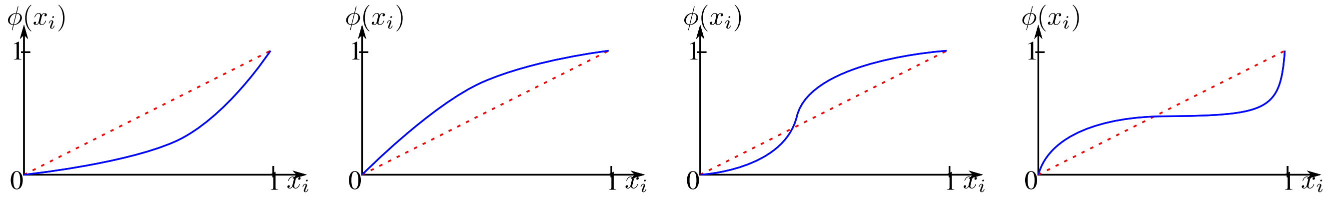
Humble Individuals: An individual i is said to be humble if $\phi_i(x_i) < x_i$ for all $x_i \in (0,1)$. With each individual i using Eq. (6) to update self-confidence $y_i(s)$, it follows that a humble individual almost always (except for $x_i = 0, 1$) *evaluates* his or her social power to be *lower* than the true social power x_i.

Arrogant Individuals: An individual i is said to be arrogant if $\phi_i(x_i) > x_i$ for all $x_i \in (0,1)$. As a consequence, an arrogant individual will almost always *perceive himself or herself to have a higher social power* than the true contribution he or she had to the discussion, x_i.

Emotional Individuals: Emotional individuals have a $\phi_i(x_i)$ function which, for some $x_i' \in (0,1)$, satisfies $\phi_i(x_i) < x_i$ for all $x_i \in (0, x_i')$, $\phi_i(x_i') = x_i'$ and $\phi_i(x_i) > x_i$ for all $x_i \in (x_i', 1)$. The "turning point" x_i' identifies the value of social power above which individual i overrates his or her contribution to the discussion, and below which he or she underrates his or her contribution to the discussion.

Unreactive Individuals: An individual i is said to be unreactive if, for some $x_i' \in (0,1)$, $\phi_i(x_i)$ satisfies $\phi_i(x_i) > x_i$ for all $x_i \in (0, x_i')$, $\phi_i(x_i') = x_i'$ and $\phi_i(x_i) < x_i$ for all $x_i \in (x_i', 1)$. In contrast to an emotional individual, the "turning point" x_i' identifies the value of social power above which individual i underrates his or her contribution to the discussion, and below which he or she overrates his or her contribution to the discussion.

Individuals for which $\phi(x) = x, \forall x$ are termed *well-adjusted* individuals. The proposed function classes satisfy Assumption 3, and may contain a wide range of different functions. Specific functional forms may be studied in future to obtain full convergence results, while this paper aims to establish an analysis framework and preliminary properties of the trajectory $\boldsymbol{y}(s), s \geq 0$ for the general functions.



(a) A humble individual. (b) An arrogant individual. (c) An emotional individual. (d) An unreactive individual.

Fig. 1: Illustrative examples of $\phi_i(x_i)$ (blue lines) which distort individual i's self-confidence y_i from true social power x_i. The dotted red line is a "well-adjusted" individual from the original DeGroot–Friedkin model, $\phi_i(x_i) = x_i$.

IV. Analysis of the System

Just as in the case where all ϕ_i are identity maps (which corresponds to the original problem examined in [10], [16]), it is of interest to consider whether there is a (unique) fixed point of the composite mapping $\boldsymbol{F} \circ \boldsymbol{\Phi}$, whether it is stable, under what conditions is convergence to a fixed point guaranteed, etc. First, we show that for networks containing only humble, unreactive, or well-adjusted individuals, $\boldsymbol{F} \circ \boldsymbol{\Phi}$ has a fixed point in $\text{int}(\Delta_n)$. Then, we analyse the Jacobian and establish that for networks of humble or well-adjusted individuals, there is a unique fixed point in $\text{int}(\Delta_n)$ that is convergent for all initial conditions satisfying Assumption 1.

Theorem 2. *Suppose that Assumptions 1, 2, and 3 hold, and that every individual is either humble, unreactive, or well-adjusted as defined in Section III-B. Define, for every $j \in \mathcal{I}$,*

$$r_j = \min_{j \in \{1,\dots,n\}} \left\{ \frac{1-2\gamma_j}{1-\gamma_j}, 1 - x'_j \right\} \tag{11}$$

where γ_j is the j^{th} entry of the dominant eigenvector $\boldsymbol{\gamma}^\top$ of $\boldsymbol{C}$, and x'_j is the inflection point of ϕ_j if individual j is an unreactive individual, and $x'_j = 1$ if j is a humble individual. Then, for any $r \in (0, r_j]$ there holds

$$x_j = 1 - r \Rightarrow G_j(\boldsymbol{x}) < 1 - r \tag{12}$$

where G_j is the j^{th} entry of the composite map $\boldsymbol{G} \triangleq \boldsymbol{F} \circ \boldsymbol{\Phi}$ in Eq. (7). Moreover, $\boldsymbol{G}$ has a fixed point $\boldsymbol{x}^* \in int(\Delta_n)$.

Proof. The positivity of γ_i and the fact that $y_i(0) < 1$ for all i guarantees that $x_i(1) > 0$ for all $i \in \mathcal{I}$. We now prove that Eq. (12) holds for $s = 1$, and by induction, one can immediately prove that Eq. (12) holds for all $s \geq 0$.

With $r \leq r_j$, and dropping the argument $s = 1$, there holds

$$\begin{aligned}
G_j(\boldsymbol{x}) &= \alpha(\boldsymbol{\Phi}(\boldsymbol{x})) \frac{\gamma_j}{1-\phi_j(x_j)} \\
&= \frac{1}{\frac{\gamma_j}{1-\phi_j(x_j)}\left(1 + \frac{\sum_{k\neq j}^n \gamma_k/(1-\phi_k(x_k))}{\gamma_j/(1-\phi_j(x_j))}\right)} \frac{\gamma_j}{1-\phi_j(x_j)} \\
&\leq \frac{1}{1 + \sum_{k\neq j}^n \frac{r}{\gamma_j} \frac{\gamma_k}{(1-\phi_k(x_k))}}
\end{aligned} \tag{13}$$

because $r = 1 - x_j \Rightarrow r \leq 1 - \phi_j(x_j)$. For well-adjusted or humble individuals, this is clear to see, since $\phi_j(x_j) \leq x_j$ for all $x_j \in [0,1]$. For unreactive individuals observe that $r \leq r_j$ and from Eq. (11) we have $r_j \leq 1 - x'_j$. This implies that $1 - x_j \leq 1 - x'_j \Rightarrow x_j \geq x'_j$. Since $\phi_j(x_j) \leq x_j$ for $x_j \geq x'_j$, we have $r \leq 1 - \phi_j(x_j)$. Because $1 - x_k < 1$, we obtain $\gamma_k/(1-x_k) > \gamma_k$, which implies that the right hand side of Eq. (13) obeys

$$\frac{1}{1 + \sum_{k\neq j}^n \frac{r}{\gamma_j} \frac{\gamma_k}{(1-\phi_k(x_k))}} < \frac{1}{1 + \sum_{k\neq j}^n \frac{\gamma_k r}{\gamma_j}} = \frac{\gamma_j}{\gamma_j + (1-\gamma_j)r} \tag{14}$$

because $\sum_{k\neq j}^n \gamma_k = 1 - \gamma_j$ from the definition of $\boldsymbol{\gamma}$. It follows from Eq. (13) and Eq. (14) that

$$\begin{aligned}
1 - r - G_j(\boldsymbol{x}) &> 1 - r - \frac{\gamma_j}{\gamma_j + (1-\gamma_j)r} \\
&= \frac{\gamma_j + (1-\gamma_j)r - r\gamma_j - (1-\gamma_j)r^2 - \gamma_j}{\gamma_j + (1-\gamma_j)r} \\
&= \frac{r(1-2\gamma_j) - r^2(1-\gamma_j)}{\gamma_j + (1-\gamma_j)r} \\
&= \frac{r(1-\gamma_j)\left[\frac{1-2\gamma_j}{1-\gamma_j} - r\right]}{\gamma_j + (1-\gamma_j)r}
\end{aligned} \tag{15}$$

From the definition of r_j in Eq. (11) and the fact that $\gamma_j < 1/2$ for non-star graphs, we conclude that $\frac{1-2\gamma_j}{1-\gamma_j} - r \geq 0$ for every $r \leq r_j$, which implies the right hand side of Eq. (15) is nonnegative. It follows that $1 - r > G_j(\boldsymbol{x})$.

Next, define the convex and compact set $\mathcal{A} = \{\boldsymbol{x} : 0 \leq x_i \leq 1 - \bar{r}, \forall i \in \mathcal{I}\}$, where $\bar{r} = \min_{j\in\mathcal{I}} \frac{1-2\gamma_j}{1-\gamma_j}$. Eq. (12) implies that $\boldsymbol{G}(\mathcal{A}) \subseteq \mathcal{A}$, and from Brouwer's Fixed Point Theorem [21], we conclude that there is at least one fixed point, call it $\boldsymbol{x}^*$, of $\boldsymbol{G}$ in $\mathcal{A}$. Since $x_i(s) > 0, \forall i \in \mathcal{I}$ for all $s > 0$, we further conclude that $\boldsymbol{x}^* \in \text{int}(\Delta_n)$. □

For networks containing arrogant or emotional individuals, the results in Theorem 2 do not necessarily hold. In fact, simulation counter-examples have been identified in which Eq. (12) is violated.

A. Incremental behaviour of the update map for the social power vector

It was discovered in [16] that the Jacobian of the mapping from $\boldsymbol{x}(s)$ to $\boldsymbol{x}(s+1)$ was useful for determining whether convergence to a fixed point occurred and whether this fix point was unique. We now explore the Jacobian, but it is necessarily more complicated since the corresponding mapping now involves the more general ϕ_i.

The function $\boldsymbol{F}$ is the mapping of $\boldsymbol{y}(s+1)$ to $\boldsymbol{x}(s+1)$. Notice that the mapping as we have defined it is actually s-independent. So in computing the Jacobian, denoted $\boldsymbol{J_F}$, and similar quantities, we shall often drop the dependence on s. We obtain that (see [16, Theorem 3] calculation details)

$$\begin{aligned} \frac{\partial F_i}{\partial y_i} &= \frac{x_i(1-x_i)}{1-y_i} \\ \frac{\partial F_i}{\partial y_j} &= -\frac{x_i x_j}{1-y_j} \;\; i \neq j \end{aligned} \tag{16}$$

We remark that this calculation does not rely on the x_i summing to 1. For future use below, we also introduce the matrix $\boldsymbol{H} = \text{diag}\left((1-x_i)^{-1}\right) \boldsymbol{J_F} \,\text{diag}\,(1-y_j)$ for $x_i \in (0,1)$. Observe that

$$h_{ii} = x_i \text{ and } h_{ij} = -\frac{x_i x_j}{1-x_i},\; j \neq i \tag{17}$$

A number of properties of the matrix $\boldsymbol{H}$ are established in [16]. For convenience, these are summarised as follows.

Lemma 4 ([16])**.** *Consider a set of $x_i, i = 1, \ldots, n, x_i \in (0,1)$ summing to 1, and a matrix $\boldsymbol{H}$ defined as in Eq. (17). Then $\boldsymbol{H}$ has one zero eigenvalue and all other eigenvalues are real, lie in $(0,1)$ and sum to 1. Moreover, $\|\boldsymbol{H}\|_1 < 1$.*

Given the mapping $\boldsymbol{\Phi} : \boldsymbol{x}(s) \to \boldsymbol{y}(s+1)$ as defined above Theorem 1, let $\boldsymbol{J_\Phi}$ denote the Jacobian, which is $\text{diag}(\phi_i'(x_i))$ with ϕ_i' denoting the derivative of ϕ_i with respect to x_i. Evidently the Jacobian of $\boldsymbol{G} = \boldsymbol{F} \circ \boldsymbol{\Phi}$ is $\boldsymbol{J_G} = \boldsymbol{J_F} \boldsymbol{J_\Phi}$. However, with the above definition of $\boldsymbol{H}$ we see that (with insertion of the topic indexing s)

$$\boldsymbol{J_G} = \text{diag}(1 - x_i(s+1)) \boldsymbol{H} \,\text{diag}\left(\frac{\phi_j'(x_j(s))}{1-\phi_j(x_j(s))}\right) \tag{18}$$

The eigenvalues of $\boldsymbol{J_G}$ are the same as those of the matrix

$$\boldsymbol{K} = \boldsymbol{H} \,\text{diag}\left(\frac{1-x_j(s+1)}{1-\phi_j(x_j(s))} \phi_j'(x_j(s))\right) \tag{19}$$

We remark that in [16], the corresponding expression was $\boldsymbol{K} = \boldsymbol{H} \,\text{diag}(\frac{1-x_j(s+1)}{1-x_j(s)})$ and this corresponds to taking $y_j(s+1) = \phi_j(x_j(s)) = x_j(s)$ in the expression for $\boldsymbol{K}$.

We are interested in fixed points of the mapping $\boldsymbol{G}$, and their stability. The stability of a fixed point $\boldsymbol{x}^*$ of $\boldsymbol{G}$ can be inferred from the eigenvalues of $\boldsymbol{K}$ at the fixed point [22], and we note then that, at such a point,

$$\boldsymbol{K} = \boldsymbol{H} \,\text{diag}\left(\frac{1-x_j^*}{1-\phi_j(x_j^*)} \phi_j'(x_j^*)\right) \tag{20}$$

Lemma 5. *Adopt the same hypothesis as Lemma 4 and let the functions $\phi_i, i \in \mathcal{I}$ satisfy Assumption 3. Then, if*

$$\frac{1-x_j}{1-\phi_j(x_j)} \phi_j'(x_j) \leq 1 \;\forall x_j \in (0,1), \forall j \tag{21}$$

the matrix $\boldsymbol{K}$ satisfies $\|\boldsymbol{K}\|_1 < 1$.

Proof. Denoting k_{ij} as the ij^{th} entry of $\boldsymbol{K}$, the condition Eq. (21) implies that $\|\boldsymbol{K}\|_1 \triangleq \max_{i \in \{1,\ldots,n\}} \sum_{j=1}^n k_{ji} \leq \max_{i \in \{1,\ldots,n\}} \sum_{j=1}^n h_{ji} \triangleq \|\boldsymbol{H}\|_1$. Thus, $\|\boldsymbol{K}\|_1 \leq \|\boldsymbol{H}\|_1 < 1$ (see Lemma 4). □

We remark that the condition on the $\phi_j(x_j)$ given in Lemma 5 is quite restrictive. In fact, an integration and use of the boundary condition that $\phi_j(0) = 0$ for all j yields $\phi_j(x_j) \leq x_j$ for all $x_j \in [0,1]$. In other words, Lemma 5 is applicable for networks with individuals that are either humble or well-adjusted. Nonetheless, the advantages of studying the incremental behaviour, viz. the Jacobian of $\boldsymbol{G}$, are evident and further study may allow us to expand our results to include other individual types. Nonetheless, an immediate consequence of Lemma 5 is the following result.

Theorem 3. *Suppose that Assumptions 1, 2 and 3 hold, and that for any topic $s \in \mathcal{S}$, the network of n individuals discuss opinions according to Eq. (2) and update their self-confidence according to Eq. (6). Suppose further that each individual $i \in \mathcal{I}$ is either (i) humble, or (ii) well-adjusted, as defined in Section III-B. Then, $\lim_{s\to\infty} \boldsymbol{x}(s) = \boldsymbol{x}^*$ exponentially fast, where $\boldsymbol{x}^* \in int(\Delta_n)$ is the unique fixed point of the map $\boldsymbol{G}$.*

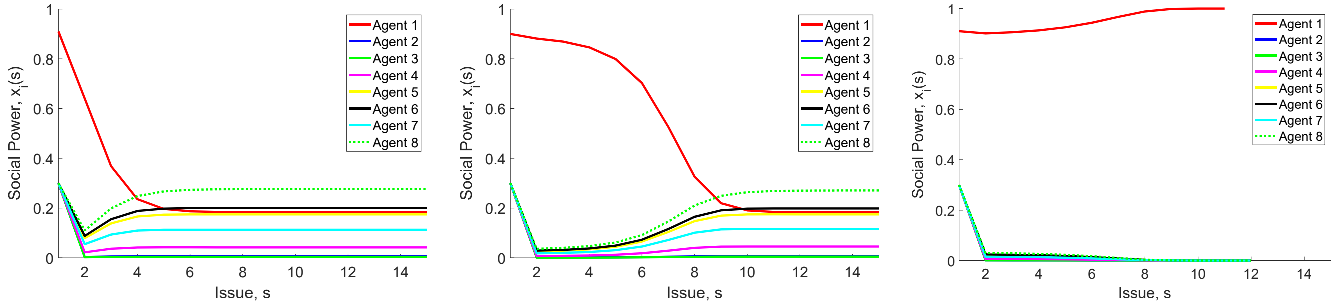
Proof. The proof is similar to the proof of convergence in [16, Theorem 3] for when $\boldsymbol{G} = \boldsymbol{F}$, i.e. $\boldsymbol{\Phi}$ is the identity mapping. The technical details, including application of nonlinear contraction analysis, are therefore omitted. □

V. Simulations

We now present a set of illustrative simulations to illustrate the existence of multiple attractive equilibria.

We consider a network of $n = 8$ individuals with an arbitrarily generated non-star graph $\mathcal{G}[\boldsymbol{C}]$ satisfying Assumption 2. We initialise the individuals with the self-confidence vector $\hat{\boldsymbol{y}}(0)$, which satisfies Assumption 1: $\hat{y}_1(0) = 0.9$, and $\hat{y}_j(0) = 0.3, \forall\, j \neq 1$. In Fig. 2a, every individual $i \in \mathcal{I}$ is "well-adjusted", i.e. $\phi_i(x_i) = x_i, \forall\, x_i \in [0,1]$. Consistent with Lemma 3, the social power vector $\boldsymbol{x}(s)$ converges exponentially fast to an equilibrium, $\boldsymbol{x}^* \in \text{int}(\Delta_n)$. In Fig. 2b, every individual $i \in \mathcal{I}$ is "emotional" with $\phi_i(x_i) = -0.5\cos(\pi x_i) + 0.5$ (such a ϕ satisfies Assumption 3). The initial conditions are again $\hat{\boldsymbol{y}}(0)$, and we see convergence of $\boldsymbol{x}(s)$ to a steady state $\boldsymbol{x}_1^* \in \text{int}(\Delta_n)$ (with convergence appearing to be exponentially fast). Last, consider Fig. 2c, in which every individual is "emotional", with the same type of ϕ_i function as in the simulation of Fig. 2b. We now initialise using $\tilde{\boldsymbol{y}}(0)$, where $\tilde{y}_1(0) = 0.91$, and $\tilde{y}_j(0) = \hat{y}_j(0) = 0.3, \forall\, j \neq 1$ which again satisfies Assumption 1, and differs from $\hat{\boldsymbol{y}}(0)$ only in that individual 1 has a slightly higher initial self-confidence. In this case, $\lim_{s\to\infty} \boldsymbol{x}(s) = \mathbf{e}_1$, which is different to $\boldsymbol{x}_1^*$, and corresponds to individual 1 eventually holding all of the social power in the network.

Several notable conclusion can be drawn from these simulations. First, consider the original DeGroot–Friedkin model. When initial conditions satisfy $0 \leq y_i(0) < 1, \forall\, i \in \mathcal{I}$ and $\exists j \in \mathcal{I} : y_j(0) > 0$, there holds $\lim_{s\to\infty} \boldsymbol{x}(s) = \mathbf{e}_i$ for some $i \in \mathcal{I}$ if and only if $\mathcal{G}[\boldsymbol{C}]$ is a star graph, with centre node v_i. From Fig. 2c, it is clear that emotional individuals can significantly alter the self-appraisal dynamics of the network to the extent that even for non-star networks, it is possible for a single individual to eventually hold all of the social power.



(a) well-adjusted individuals with $\hat{\boldsymbol{y}}(0)$. (b) Emotional individuals, with $\hat{\boldsymbol{y}}(0)$. (c) Emotional individuals, with $\tilde{\boldsymbol{y}}(0)$.

Fig. 2: Illustrative examples of the evolution of social power, $x_i(s)$, for a network of 8 individuals. Well-adjusted individuals have $\phi_i(x_i) = x_i$, while emotional individuals have $\phi_i(x_i) = -0.5\cos(\pi x_i) + 0.5$. Note that $\hat{\boldsymbol{y}}(0) \neq \tilde{\boldsymbol{y}}(0)$.

Second, in comparing the simulations for Fig. 2b and 2c, it is obvious that networks of emotional individuals may have multiple attractive equilibria, one of which is in $\text{int}(\Delta_n)$. In the original DeGroot–Friedkin model, there is a unique $\boldsymbol{x}^* \in \text{int}(\Delta_n)$ which admits almost global convergence, while all other equilbria are unstable [16, Corollary 2]. Last, and although not obvious due to the figure size, we observed that $\boldsymbol{x}^* \neq \boldsymbol{x}_1^*$; the emotional function changes the steady state social power of each individual. These observations are helpful indicators for future work directions.

VI. Conclusions

In this paper, we proposed a generalisation of the DeGroot–Friedkin model by allowing each individual's self–appraisal behaviour to be captured by a function. We established the dynamical equations that govern the evolution of social power for the generalised model, and proposed four new function classes to describe "humble", "arrogant", "emotional", and "unreactive" individuals in addition to the original "well-adjusted" individuals. We showed that networks having a mixture of humble, unreactive and well-adjusted individuals had at least one equilibrium inside the unit simplex.By studying the Jacobian of a relevant mapping, we then established that for networks containing a mixture of humble and well-adjusted individuals, and for almost all initial conditions, convergence to a unique equilibrium was guaranteed, with an exponential rate. Last, we used simulations to establish insight into networks of emotional individuals, including the existence of multiple attractive equilibria. Future work will focus on comprehensive convergence analysis for networks of arrogant, emotional, or unreactive individuals, and further analysis of the equilibria.

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