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Beyond consensus and polarisation: complex social phenomena in social networks

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Abstract: A fundamental aspect of society is the exchange and discussion of opinions between individuals, occurring in situations as varied as company boardrooms, elementary school classrooms and online social media. After a very brief introduction to the established results of opinion dynamics models, which seek to mathematically capture observed social phenomena, a brief discussion follows on several recent themes pursued by the authors building on the fundamental ideas.

In particular, a novel discrete-time model of opinion dynamics is used to establish how discrepancies between an individual's expressed and private opinions can arise due to stubbornness and a pressure to conform to a social norm. It is also shown that a few extremists can create "pluralistic ignorance", where *people believe* there is majority support for a position but in fact the position is *privately rejected* by the majority. We also analyze the way an individual's self-confidence can develop through contributing to discussions on a sequence of topics, reaching a consensus in each case, where the consensus value to some degree reflects the contribution of that individual to the conclusion. Last, we consider a group of individuals discussing a collection of logically related topics. In particular, we identify that for topics whose logical interdependencies take on a cascade structure, disagreement in opinions can occur if individuals have competing and/or heterogeneous views on how the topics are related, i.e. the logical interdependence structure varies between individuals.

Key Words: social networks, opinion dynamics, consensus, networked systems

1 Introduction

In 1956, John French Jr. introduced an agent-based model of opinion dynamics [1] to study how individuals exerted social power on each other during interactions in a network. The model has become known as the French–DeGroot (or simply DeGroot [2]) model and is the fundamental agent-based model of opinion dynamics which many subsequent works, including ours, build upon. The model assumes that each individual has an opinion on a given topic, and each individual interacts to learn of the opinions of that individual's neighbours. In doing so, the opinions evolve over time as each individual integrates learned opinion values of his/her neighbours with the individual's own opinion using a weighted averaging process. Eventually, a consensus is reached on the opinion value, i.e. there is agreement across the opinions of all individuals, if the network satisfies some connectedness conditions. This is illustrated in Fig. 1. Experimental validations of the DeGroot model are reported in e.g. [3].

While in this paper, we shall almost entirely avoid mathematical details, we record here the relevant equation in discrete-time:

$$x(k+1) = Ax(k) \tag{1}$$

With n interacting individuals, $x \in \mathbb{R}^n$, the entries of x are most commonly assumed to lie in the interval $[-1, 1]$ or sometimes $[0, 1]$; the quantity $x_i(k)$ is the opinion held at time k by individual i concerning the topic of interest, with a value of -1 (or, sometimes, 0) corresponding to the individual being wholly negative towards the topic, and a value of 1 corresponding to being completely positive. The matrix A is a *stochastic matrix*, and captures the way the individuals influence one another. Thus it is a matrix of nonnegative entries, and the row sums of A are 1. This means that a new value of $x_i(k+1)$ is a convex weighted linear combination $\sum_{i=1}^{n} a_{ij}x_j(k)$, and if all $x_j(k)$ assume the same value, then $x_i(k+1)$ takes this value also. Put another way, consensus values for the state vector $x(k)$, i.e. values for which all entries are equal, are equilibrium states of the equation (1). Further, they are the only equilibrium states given appropriate connectivity of the underlying network through which the individuals are interacting. The equilibrium state is reached exponentially fast, and its value depends on the initial condition $x(0)$. This occurs because the stochastic matrix property implies that A always has an eigenvalue at 1, and under suitable connectivity conditions, it is simple and all other eigenvalues have magnitude less than 1. For an introduction to stochastic matrices and their relevance in social networks, see [4].

A continuous-time formulation is also possible, and one has $\dot{x} = -Lx$ where L is a *Laplacian* matrix, i.e. its row sums are zero, and off-diagonal entries are nonpositive. Under a suitable connectivity condition, all eigenvalues of $-L$ are in the left half plane, except for one at the origin, and then $x(t)$ tends to a steady state in which all entries are the same [4, 5].

Beyond consensus, variations of the DeGroot model have been proposed to investigate how different social phenomena may arise. Examples include models which describe how opinions in a network can separate into multiple clusters, and how opinions can become polarised into opposing clusters due to negative influence [6, 7] or bias assimilation of information sources [8]

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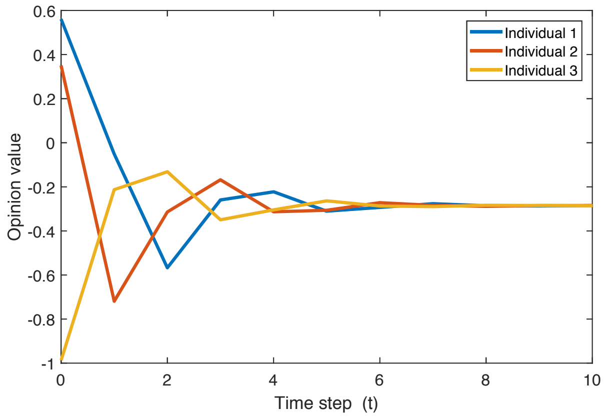

Fig. 1: The temporal evolution of opinions for three individuals in a social network according to the DeGroot model. "Social influence" acts to bring individuals' opinions closer together over time. Eventually, all individuals reach a consensus with the same opinion value.

The Hegselmann-Krause model captured homophily [9], where an individual interacts only with those others who have similar opinions. Over time, individuals can become separated into clusters, where the final opinions are the same within each cluster, but different between the clusters. The Altafini model introduced the concept of negative influence to capture antagonistic interactions among individuals who may, for any number of reasons, dislike or mistrust each other [6]. If the network is "structurally balanced", the opinions can become polarised into two opposing clusters. We refer the reader to the survey papers [4, 10] for a more comprehensive introduction to other works in opinion dynamics.

In the sections below, we introduce several distinct recent developments which seek to capture well-studied phenomena, of which most people have at least intuitive knowledge. The new developments proceed by introducing mathematical models which often allow the drawing of general conclusions regarding phenomena beyond a simple consensus of opinions, but are not extremely complicated to analyze. The areas of the three developments are as follows:

1) Exploring discrepancies between an individual's private and expressed (or public) opinion.
2) Understanding how an individual's self-confidence evolves through discussion in a group of a sequence of topics. The self-confidence reflects the individual's influence on the final (consensus) opinion.
3) Exploring what happens when a group of individuals are discussing a collection of issues which are somehow logically related.

2 Discrepancies In An Individual's Private and Expressed Opinion

Inspired by Solomon Asch's seminal experiments on conformity under pressure [11], one can formulate a novel opinion dynamics model to study how differences between an individual's expressed and private opinions can arise [12]. In particular, we assume that an individual expresses an opinion which is the individual's private opinion *altered due to a pressure to conform to the social network's average opinion*. In other words, the individual has some "resilience" to the pressure, but is not unaffected by it. We also assume each individual remains somewhat attached to the individual's initial opinion, which we call "stubbornness".

The mathematical form of the model is a modest two-stage adjustment of the DeGroot model. The DeGroot model assumes, as already introduced in equation (1), that

$$x_i(k+1) = a_{ii}x_i(k) + \sum_{j\neq i}^{n} a_{ij}x_j(k) \qquad (2)$$

To capture the idea of attachment to an initial opinion, or stubbornness in an individual, it can be replaced by the model known as the Friedkin-Johnsen model [13]:

$$x_i(k+1) = \lambda_i[a_{ii}x_i(k) + \sum_{j\neq i}^{n} a_{ij}x_j(k)] + (1-\lambda_i)x_i(0) \quad (3)$$

with $\lambda_i \in (0,1)$ known as the *susceptibility to influence* parameter (making $1-\lambda_i$ the individual's stubbornness). The second and further extension requires ascribing to the i-th agent a second scalar parameter termed resilience, in the following way. The quantity $x_i(t)$ constitutes the agent's *private* opinion, and there is now a second quantity associated with the agent, namely, a *public* opinion $\hat{x}_i(t)$ which other agents learn of, and is derived by combining the private opinion and the effect of group pressure (effectively, the social network's average opinion). This means that the update equations are as follows:

$$\begin{aligned} x_i(k+1) &= \lambda_i[a_{ii}x_i(k) + \sum_{j\neq i}^{n} a_{ij}\hat{x}_j(k)] \\ &\quad +(1-\lambda_i)x_i(0) \\ \hat{x}_i(k) &= \phi_i x_i(k) + (1-\phi_i)\hat{x}_{avg}(k-1) \end{aligned} \qquad (4)$$

Here $x_{ave}(k) = \sum_{i=1}^{n} x_i(k)/n$ and $\phi_i \in (0,1)$ is termed the *resilience*, i.e. the ability for individual i to withstand group pressure.

Study of the model [12] has led to several conclusions, which are illustrated in an example simulation in Fig. 2. We point out some of the most interesting results.

- The combination of (i) pressure to conform to the social norm x_{ave}, (ii) stubborn attachment to the individual's initial opinion, and (iii) the strong connectedness of the network, means that at equilibrium, for any given individual, that individual's private and expressed opinions are unequal.
- There is greater disagreement in the private opinions than in the expressed opinions at equilibrium. This is due to the effects of a pressure to conform to a social norm: people are more willing to voice agreement in a social network, but less willing to shift their private opinions. The smallest interval containing the private opinions actually "encloses" the smallest interval containing the expressed opinions at equilibrium.
- It is possible to estimate a lower bound on the level of disagreement in the final private opinions given the level of disagreement in the expressed opinions, and an estimate of how resilient the individuals are to the pressure to conform.

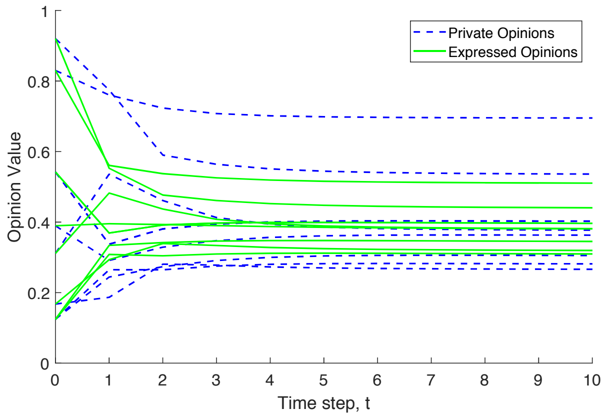

Fig. 2: Evolution of opinions for a network of 8 individuals. The opinions converge to a steady-state under mild assumptions on the network connectivity.

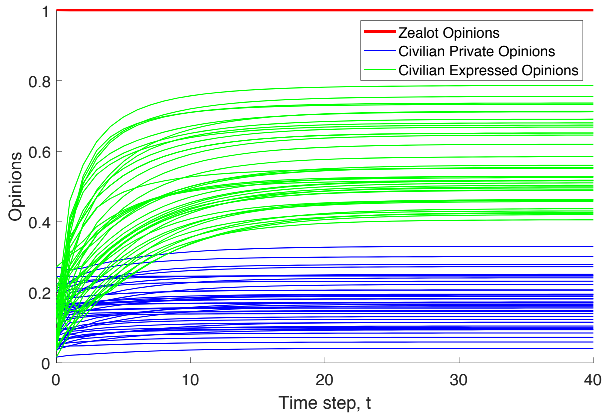

Fig. 3: Temporal evolution of opinions for a network of 200 individuals. There are 5 zealots, and for clarity, only 40 civilian (non-zealot) individuals are shown. Stubbornness of the civilians leads to only minor changes in their private opinions, but the zealots spread misinformation, causing the civilians' expressed opinions to shift significantly.

- The connectedness of the network leads to an individual's resilience to pressure to conform having a "propagating effect". Changing an individual's level of resilience to this pressure leads to a specific pattern of changes to every other individual's expressed opinion at equilibrium.

The model has also been used to accurately predict and explain Asch's conformity experiments. In particular, we are able to capture how an individual reacts when he or she is faced with a unanimous group of other individuals who openly question his or her belief in an indisputable fact. Some individuals would bend to the pressure of the unanimous group while yet others would resist; we show that each reaction recorded by Asch can be accurately predicted by different values for the parameter pair λ_i, ϕ_i, i.e. susceptibility to influence and resilience to the group pressure.

Pluralistic ignorance is a social phenomenon where the majority of a population privately reject an opinion position, but people believe there is majority support for that position, e.g. in the 1960s, white Americans misidentified how much support there was for racial segregation [14]. In Fig. 3, our model shows that stubborn extremists (zealots) placed at well connected nodes in the network can create massive pluralistic ignorance in the general population. Not only does this create confusion and misinformation about the true desires of the population, it is known that large differences in expressed and private opinions that are sustained for a long time can foster discontent among individuals, leading to unexpected and drastic actions [15]. This may be of particular interest in studying how misinformation spreads through high profile media figures, or hostile bot accounts on social media (such as the Twitter bots of a foreign country commenting on political matters). We also expect this to be closely linked with a social phenomenon called *the spiral of silence* [16], which roughly states that a person is less likely to voice an opinion if he sees that everyone else is moving away from that opinion. We hope that study and development of these models (including incorporation of a spiral of silence mechanism) will reveal to us deeper insight about the role extremists play in creating a divergence in the private and expressed opinions of the general population, and establish effective countermeasures. For example, it is not immediately clear whether it is better to (i) introduce a new set of extremists that are on the opposite side of the opinion spectrum to the original zealots (which might risk polarising the network), or (ii) spread more moderate opinions (which might risk being lost in the presence of the original zealots).

3 Evolution of Individual Social Power

Suppose an individual is participating in a group discussion which covers a number of different issues, $1, 2, \ldots, s, \ldots$. Each issue is discussed through to consensus, then the next issue is discussed, and so on. Suppose that the individual perceives during this process that they have less and less impact on the outcome of each discussion. Consequently, they become less and less confident of their own opinion. (The converse situation of having more and more impact and rising confidence can also occur of course). The term *social power* has been used to describe this self-confidence. (There is no allowance for overly confident or overly humble individuals; thus self-confidence is meant to accurately reflect influence on the discussion outcome).

How does a person evaluate their influence on a discussion, and how does the updating of self-confidence affect discussion on the next topic? We start with the first of these questions. The discussion of any one issue proceeds according to the DeGroot equation $x(k+1, s) = A(s)x(k, s)$ where we allow for A to vary with issues, hence the dependence on the issue index s in a manner more precisely indicated below. We refer to a_{ii}, the ii^{th} entry of A, as individual i's self-confidence, or social power. Consensus is reached for topic s and if $\zeta^\top(s)$ is a left eigenvector of $A(s)$ corresponding to the simple eigenvalue 1, and normalized to have all positive entries adding to unity, as may usually be done, then there holds

$$\lim_{k\to\infty} x(k, s) = \zeta^\top(s)x(0, s)\mathbf{1} = \sum_{i=1}^{n} \zeta_i(s)x_i(0, s)\mathbf{1} \quad (5)$$

where $\mathbf{1}$ denotes a vector all of whose entries are 1. Evidently, the weighting applied to the contribution of agent i to the consensus value for issue s is simply $\zeta_i(s)$. The properties of $\zeta(s)$ ensure this is a normalized value, i.e. $\sum_i \zeta_i(s) = 1$.. Now for the next issue, $A(s+1)$ is adjusted to reflect a change in each individual's self-confidence $a_{ii}(s+1)$ arising from the process of reflected self-appraisal. In particular, the $a_{ii}(s+1)$ of $A(s+1)$ is taken as $\zeta_i(s)$, and the remaining entries are are scaled by $1-\zeta_i(s)$ to ensure that the i-th row of $A(s+1)$ adds up to 1. This replacement indicates that for issue $s+1$, agent i weights its own opinion relative to the opinions of others by the same weight as its contribution to the consensus value in issue s. It also means that the nature of the interactions between agents, discounting any self-weighting, is constant with s. If agent 1 finds agent 2 twice as reliable as agent 3 for issue 1, that proportionality relationship will hold for all issues. However, what does change is the overall weight agent 1 gives to all opinions other than his own, since in adjusting the weighting given to his own opinion to be $a_{ii}(s+1) = \zeta_i(s)$, a compensating adjustment for weighting placed on others' opinions, $a_{ij}(s+1) = (1-\zeta_i(s))a_{ij}(s)$ is necessary to ensure that the sum of the weights $a_{ij}(s+1)$ remains at 1.

This model is known as the *DeGroot–Friedkin* model. It was studied in [17] and there it was established that the sequence of vectors of self-confidences asymptotically approaches a unique limit as the number of issues tends to infinity, except for certain pathological exceptional cases.

Our own work has taken this farther in several respects, [18]. Suppose that the group in question is a cabinet of ministers. Each week they might meet and regularly discuss issues relating to defence, social security and the economy. Because of the different expertise of the different ministers, it would be logical for minister 1 to vary the relative weight he or she puts on the opinions of ministers 2 and 3 in discussing issues of a different character. Further, the composition of such a group can change over time. Accommodating such time-variation, including the specialization of periodic time-variation, is difficult, but not impossible. In fact, one can show an exponential convergence result. Even with changing issues, the vector of self-confidences converges to a nonconstant trajectory that is independent of the initially assumed vector of self-confidences. In the event that the issues are governed by a constant set of weights, the limiting trajectory is of course constant–as identified by [17]. In the event that issues are revisited on a periodic basis, such as in a cabinet, the trajectory is periodic.

Figure 4 depicts the evolution of social power for three out of six individuals, engaged in discussing topics with periodically varying relative weights of neighbors. Notice that different initial conditions for each individual, $\hat{x}_i(0) \neq \tilde{x}_i(0)$ for each $i = 1, 3, 6$, give rise to trajectories which converge; the limiting trajectory of each individual's social power $x_i(s)$ is determined only by the periodically varying topology structure.

In summary, our key conclusion is that sequential opinion discussion removes perceived or initial social power/self-confidence exponentially fast. True social power/self-confidences evolving via reflected self-appraisal, obtained in the limit of the sequence of topic discussions, is dependent only on the sequence of topology structures, i.e. the distinct agent-to-agent interactions.

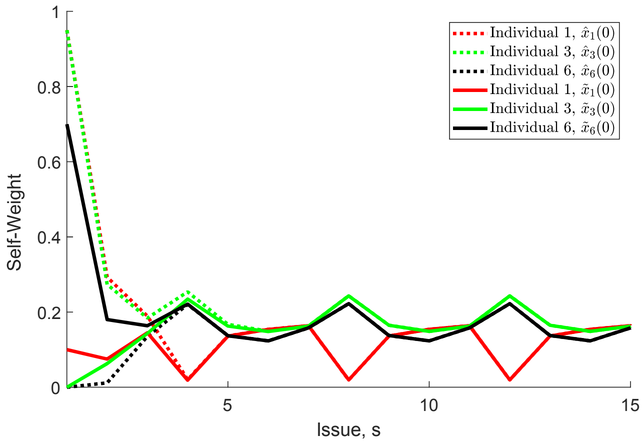

Fig. 4: Evolution of individuals' social powers over a sequence of issue discussions for a network of 6 individuals. For each $i = 1, 3, 6$, $\hat{x}_i(0) \neq \tilde{x}_i(0)$.

4 Consensus Across Logically Related Issues

What do we mean by the term "logically related issues"? We provide an example. Full acceptance of the truth of a topic will correspond to a state value of 1, and full rejection to a state value of -1. Consider two topics being simultaneously discussed; 1) mentally challenging tasks are just as exhausting as physically challenging tasks and 2) that Go and chess should be considered sports in the Olympics. Clearly a person who believes topic 1 is true is more likely to believe topic 2 is true. Let individual i's opinion vector be $\mathbf{x_i} = [x_i^1, x_i^2]^\top$. (We use the bold face to distinguish this situation from the earlier problems which restrict consideration at any one time to a single topic). For topic 1, if x_i^1 is positive (respectively negative), then individual i believes (to some degree) mentally challenging tasks are just as exhausting (respectively not as exhausting) as physically challenging tasks. For topic 2, if x_i^2 is positive (respectively negative) then individual i believes (to some degree) Go and chess should be considered (respectively not considered) an Olympic sports. The notion of logical interdependence is captured by the concept of a *logic matrix*, which for individual i is denoted as C_i. To capture in a model how individuals interact when interdependent topics are involved, [19, 20] proposed to combine the idea of the DeGroot model with the notion of logic matrices. With $A = (a_{ij})$ a stochastic matrix capturing the network effect of interaction among individuals, one has

$$\mathbf{x}_i(k+1) = \sum_{j=1}^{n} a_{ij} C_i \mathbf{x}_j(k) \tag{6}$$

To clarify the purpose of the logic matrix, consider an individual i with no neighbors, so that (6) becomes

$$\mathbf{x}_i(k+1) = C_i \mathbf{x}_i(k) \tag{7}$$

and consider the above mentioned example regarding Go and chess as Olympic sports. An individual i may hold the belief that an event can only be an Olympic sport if it is exhausting.

One possible logic matrix is given by

$$C_i = \begin{bmatrix} 1 & 0 \\ 0.7 & 0.3 \end{bmatrix} \tag{8}$$

Suppose that $\mathbf{x}_i(0) = [1, -0.7]^\top$, i.e. initially agent i believes mentally challenging tasks are as exhausting as physically challenging tasks, but does not believe Go and chess should be Olympic sports. According to (7), $x_i^1(1) = x_i^1(0)$ and $x_i^2(1) = 0.7x_i^1(0) + 0.3x_i^2(0) = 0.49$. That is agent i's evaluation on the logical interdependence of topic 2 on topic 1 (because agent i holds the view that an event can only be an Olympic sport if it is exhausting) causes $x_i^2(k)$ to shift from -0.7 to 0.49; thus, agent i's opinion on topic 2 in one step has been altered due to i's opinion on topic 1. Eventually, $\lim_{k\to\infty} \mathbf{x}_i(k) = [1, 1]^\top$, and agent i comes to believe (after some internal reflection) that Go and chess should be Olympic sports. On the other hand, if agent i does not initially believe mentally challenging tasks are as exhausting as physically challenging tasks, e.g. $x_i^1(0) = -0.8$, then $\lim_{k\to\infty} \mathbf{x}_i(k) = [-0.8, -0.8]^\top$ and agent i believes chess should not be an Olympic sport.

Another individual j might take the view that Go and chess being Olympic sports *implies that mentally challenging tasks are to some degree as exhausting as physically challenging tasks* and so may have

$$C_j = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{bmatrix} \tag{9}$$

The nonzero 12 entry indicates that agent j views topic 1 as being logically dependent on topic 2.

While the above C_i and C_j are row-stochastic, we do not in general require C_i to be row-stochastic (though other constraints will apply, depending on whether we are working with continuous- or discrete-time models). For convenience, we will restrict attention to one possibility, namely discrete-time models. What properties then should a matrix C_i have? It is crucial that in the absence of inputs from other individuals, an individual's own belief structure should be consistent, in the sense of giving rise to a steady-state opinion vector, which should generically be nonzero. Thus the equation (7) for arbitrary initial conditions with each entry confined to $[-1, 1]$ should have a transient solution, including a limiting solution, such that the transient and limit also have entries confined to this interval. The requisite conditions are set out in detail in [19]. A parallel restriction applies in continuous time of course [21].

Unsurprisingly, if all C_i are the same, from an algebraic point of view, equation (6) for the entire network is rather easy to analyze, with the aid of Kronecker products. In deed, [19] considers networks of individuals all having the same C_i (along with other aspects including stubbornness as described by the Friedkin–Johnsen model, see (3)). The work [20] does consider heterogeneous C_i, but only row-stochastic C_i (which is restrictive, as it does not allow for negative couplings between topics). Moreover, the stability result in [20] requires at least one individual to have some stubbornness. Our work [22] has sought to focus on what happens *to the final opinion distribution* when the C_i are distinct, and thus makes the assumption that the individuals are not stubborn; we now record several key outcomes which are different from those in [19, 20]

Notice that the matrix C_i above is lower triangular. This is a common situation, reflecting the fact that individual i's belief system flows from one or more axioms, or truths which they consider to be indisputable. Our work begins by focusing on such lower triangular C_i.

First, we note the possibly nonobvious fact that all models of this type are stable, given connectivity conditions standard in opinion dynamics models and assumptions imposed on the logic matrix C_i, arising from the need for (7) to describe a consistent belief structure, as discussed above.

Next, we consider the issue of whether consensus on individual topics can be achieved. Suppose to begin that there are two topics under discussion, and that topic 2 depends on topic 1 but topic 1 does not depend on topic 2, for every individual. Thus

$$C_i = \begin{bmatrix} 1 & 0 \\ c_{21,i} & c_{22,i} \end{bmatrix} \tag{10}$$

If all individuals have the same sign for the 21 entry of their C matrix, i.e. there are no *competing logical interdependence structures*, consensus is reached in steady state on both topics. On the other hand, if there are two individuals p, q for which $c_{12,p}$ and $c_{12,q}$ have opposite signs, then consensus will not be achieved on topic 2. Evidently, major but not complete differences in belief structures will not destroy consensus. This is illustrated in Fig. 5 and 6, where there are 6 individuals discussing 2 topics, with C_i taking the form of (10). In Fig. 5, individuals 1,2,3 have the same $C_i = \hat{C}$ matrix, while individuals 4,5,6 have the same $C_j = \tilde{C}$ matrix. The two logic matrices satisfy $\hat{C} \neq \tilde{C}$ but have the same sign pattern; $c_{21,i}$ is negative and $c_{22,i}$ is positive. Fig. 5 shows that the individuals reach a consensus for each of topics 1 and 2, with the consensus values for the two topics being different. Fig. 6 shows a simulation where five individuals $i = 1, 2, \ldots, 5$ have the same C_i, and whose opinions on topic 2 are represented by the solid blue lines. The sixth individual j, whose opinion on topic 2 is indicated by the dashed blue line, has C_j with $c_{21,j}$ having opposite sign to $c_{21,i}$. As can be seen, opinions converge to a persistent disagreement, even for the five individuals with the same logic matrix C_i.

The situation with three or more topics is more subtle. Suppose that all C_i have the form

$$C_i = \begin{bmatrix} 1 & 0 & 0 \\ c_{21,i} & c_{22,i} & 0 \\ c_{31,i} & c_{32,i} & c_{33,i} \end{bmatrix} \tag{11}$$

Then discrepancies in signs in the second row between corresponding elements will in general prevent consensus from being reached on topic 3 as well as on topic 2. Also *minor* differences of value (without sign difference) in any of the entries of the third row can lead to lack of consensus in topic 3. Thus there is a significant distinction to be drawn between the two topic case and the three (or more) topic case.

The failure of consensus is not just minor; there can arise disagreement of opinions from minor differences in the values of the entries. Providing a comprehensive but heuristic explanation of why this occurs is elusive. Models such

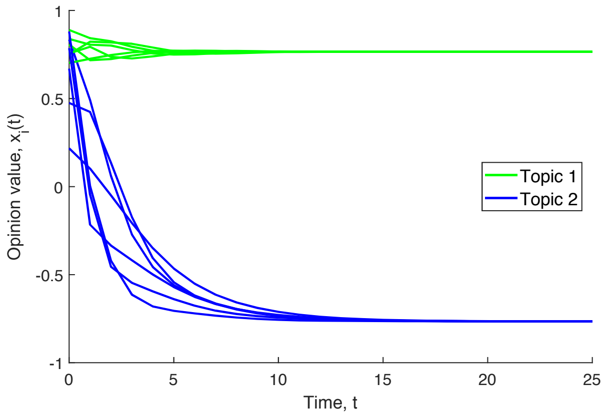

Fig. 5: Evolution of 6 individuals' opinions for two logically interdependent topics, with topic 1 independent, and topic 2 dependent on topic 1. The individuals have heterogeneous C_i, but all C_i matrices have the same sign pattern, i.e. there are no competing logical interdependencies.

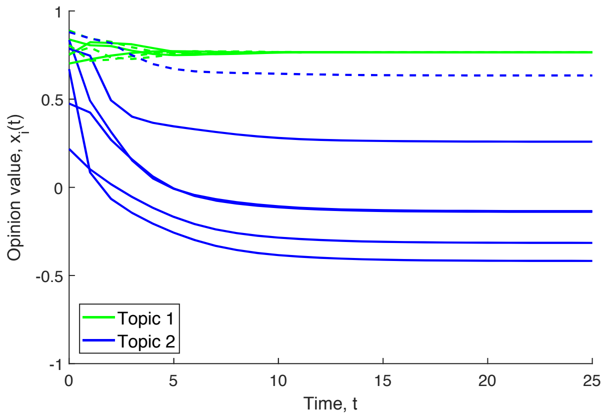

Fig. 6: Evolution of 6 individuals' opinions for two logically interdependent topics, with topic 1 independent, and topic 2 dependent on topic 1. The solid blue lines represent individuals with the same C_i matrix, and the dotted blue line represents an individual j whose 21 entry of C_j has opposite sign to the 21 entry of the other C_i.

as the Hegselmann–Krause and Altafini models lead to final opinion distributions that are said to show "weak diversity of opinions" [7, 23], where there may be clusters of different opinion values but the opinion values within a cluster are identical. As a result of stubbornness, the Friedkin–Johnsen model and the model introduced in Section 2 can lead to final opinion distributions that show "strong diversity of opinions", where there is a range of opinion values, and if there is a cluster of opinions, then the opinions within the cluster are of similar but not equal value. Strong diversity is observed in many scenarios in real-world social networks [7, 23]. Our reported findings are of particular interest as we have observed that disagreement, and in particular *strong diversity in the final opinions*, e.g. in Fig. 6, can occur *even though* each individual uses (6) in an attempt to bring their opinions closer together, and there is no presence of stubbornness [13], bounded-confidence [9], negative influence [6], or the other usual processes to which disagreement may be attributed.

As a general rule, much the same set of conclusions arise in social network modelling whether one uses a continuous- or discrete-time model. In the case of multiple interdependent topics however, we need to note one clear difference [21]. When one moves to continuous time, one can consider a time scale associated with the process of any one individual arriving at a set of opinions consistent with their belief structure (i.e. the continuous time version of (7)), from an arbitrary initial set of values, and one can consider a time scale associated with the individual's interactions with other individuals, thereby causing some modification, and maybe even achieving consensus between individuals. In the event that the second time scale is comparable with the first, instability may result. Put another way, one must not rush the process of arriving at consensus between individuals, when there are multiple related topics to consider. People can only adjust to the opinions of others so fast: go faster, and fracture and instability may result.

5 Future Directions

A number of exciting future directions exist for the themes discussed in this paper. Aside from the usual extensions to consider time-varying interactions, gossip-based algorithms and incorporation of other accepted features such as bounded-confidence or stubbornness, we briefly cover specific extensions to interesting social phenomena in three separate points, each related to one of the above three main sections.

A number of social phenomena have been recorded in the social science literature that relates to differences in an individual's private and expressed opinions, which deserve to be examined in detail using agent-based opinion dynamics models. Chief among these is the "spiral of silence", which is related to our tendency as humans to predict behavior. In this case, the "spiral of silence" posits that a person is less likely to express an opinion if that person sees the general public is moving away from that opinion position. It would also be of great interest to study such models for online social networks, where the 1-9-90 rule is prevalent in many discussion forums. Roughly speaking, 1% of users create content, 9% of users participate by posting opinions, and 90% of users do not participate at all [24]; how do the private opinions of the 90% evolve as a result of the 10% expressing opinions, and what happens if these ratios change?

With regards to the evolution of social power, a natural extension is to model individual behavior into the reflected self-appraisal process. Currently, it is assumed that each individual perfectly self-appraises the amount of impact that individual had in the previous topic discussion. However, one can consider an individual who is humble (or arrogant) and thus self-appraises himself or herself to have less impact than is true (or more impact than is true). Other opinion dynamics models, such as the Friedkin-Johnsen or Altafini models, may also be incorporated.

In the study of multiple related topics, a comprehensive account must be provided on the logical interdependence structures and their impacts on the final opinion distribution. For example, irreducible logic matrices appear to not have the same tendency to generate disagreement even when there is heterogeneity. This may provide insight into which belief structures are more robust in generating consensus, and which belief structures are more likely to lead to disagreement.

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