Improved Doppler Positioning Techniques for Stand-Off Scenarios

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In “stand-off” scenarios, an estimate of the bearing to an emitter can be easily calculated from measurements of the received frequency, position, and velocity of receivers. This technique is only effective if the measured frequencies can be reliably associated with a single emitter that is far away. We show how to determine if the receivers’ measurements correspond to a single emitter in a stand-off scenario, eliminate outliers, and estimate the unknown transmit frequency.

I. INTRODUCTION

In many civilian and defense situations, it is necessary to determine the location of radio frequency (RF) emitters such as mobile phones, radio transmitters, or emergency beacons. There are a number of methods for locating RF emitters such as time difference of arrival, direction finding, and Doppler positioning. A summary of a variety of localization techniques are presented in Poisel’s book [1].

In this article, we concentrate on Doppler positioning, and in particular, its use to achieve direction finding of a stationary emitter. Doppler positioning is a passive localization technique, meaning that the localization depends only on receiving signals, as opposed to transmitting them (a process that can announce the tracker’s intent and location). Passive techniques have the advantage of being more stealthy and requiring less power for operation or carriage of the associated payload such as a radar transmitter, which means that they are well suited for use in unmanned aerial vehicles (UAVs) doing continuous surveillance work, reducing the frequency required for refueling or minimizing the broadcasting of UAV location information.

Doppler positioning has a long history. It was used to locate Russia’s first satellite SPUTNIK [2] and for surveillance by the SR-71 aircraft during the cold war [3]. Prior to the deployment of the global positioning system, the TRANSIT system was used for the global satellite navigation [4], and it used a Doppler positioning technique.

To determine the location of a stationary emitter using Doppler positioning, it is in principle sufficient to measure the (Doppler-shifted) carrier frequency of the transmitter signal and to know the location and velocity of the sensor at a number of different locations. Small low cost UAVs equipped with a GPS receiver and a software-defined radio can easily obtain these measurements [5].

Although Doppler positioning is a well-established technique, there are still some advances to be made [6]–[9]. When UAVs are used to locate an emitter it is often advantageous for the UAVs to be far enough away from the emitter to avoid being detected or subject to active interference. In this article, we describe a method for verifying, using Doppler-shifted frequency measurements, that an emitter is remote (as far as the UAV positions are concerned) so that the direction vector can be assumed to be constant at all the points at which the emitter Doppler shift is measured. The emitter is assumed to be stationary so that all Doppler shifting is attributable to the UAV’s velocity only.

Based on this knowledge, we explain how to determine the direction of the emitter when the emitter is sufficiently far away from the points at which the Doppler-shifted emitter frequency is received. An initial estimate of direction can be obtained by solving a least-squares problem; however, we show how to obtain an improved estimate of the emitter’s direction by minimizing a constrained Lagrangian. The emitter direction is determined using the Gauss–Newton
method for solving the constrained Lagrangian, initializing the iterations with the least-squares solution. Simulations show that this refinement significantly improves the bearing estimate when compared with the simple least-squares estimator. We also consider how additional measurements can be treated so as to decide whether or not they are outliers.

This article differs from previous work in the following three key areas.

1) It includes a measure \( \chi \) of how valid the “stand-off” approximation is, i.e., that the detected signals come from a single source far enough away so that the bearing of the emitter is roughly the same for all receiver locations (and any estimation of the range is essentially unreliable). This measure is also related to the possible direction of the arrival error in a way that has not previously been recognized.

2) It exploits the fact that the bearing vector is a unit vector to set up the problem of estimating this unit vector in terms optimizing a constrained Lagrangian, with this approach yielding significant improvements in the variance of the bearing estimates.

3) A new measure \( \xi \) is proposed for determining whether a new measurement should be considered an outlier (and therefore removed from bearing estimate calculations). To the best of our knowledge, this has not been done previously.

The outline of this article is as follows. In the next section, we present the fundamental Doppler positioning equations and show how they can be transformed into a linear equation with the aid of “nuisance” variables that are the quadratic terms of the emitter’s location in Cartesian coordinates. We present the solutions to the linearized equations and show that if the noise on the measured frequency is sufficiently small, the estimated location is in good agreement with the true location. In Section III, we examine “stand-off” scenarios in which the maximum distance between received signal locations is small compared with the distance from the mean receiver location to the emitter. In this section, we derive a stand-off indicator, which is used to determine if a set of measurements is consistent with the stand-off scenario, and separately, we consider how to decide whether a new frequency measurement is from an emitter sufficiently far away for the “stand-off” approximation to continue to be valid. We demonstrate the validity of this indicator with some examples. In Section III-C, we show how to obtain an improved estimate of the bearing to the emitter in a stand-off scenario by using the constrained cost function with a Lagrange multiplier. We show that this estimate is a significant improvement on the least-squares estimator. In the last section, before the conclusions, we show how to accommodate the case when the carrier frequency of the transmitter is not known precisely.

II. LINEARIZATION VIA THE INTRODUCTION OF NUISIBLE VARIABLES

In this section, we will explain how the basic localization problem can be reformulated to a linear equation. This will subsequently help us in studying stand-off scenarios in which the target is remote from the UAV sensing it.

Throughout this paper, we allow the possibility of a 2-D or 3-D environment. Until Section IV, we will assume that the unshifted emitter frequency is precisely known, as opposed to a nominal value being known. One could only assume that it is precisely known if, for example, there was a separate stationary receiver providing the information, or the emitter was actually reflecting signals due to its being illuminated by a radar, with the position, velocity, and precise frequency of the radar transmitter known to the receiver.

Assume that the receiver makes \( N \) measurements of the frequency of a signal associated with an emitter. In the absence of noise, the \( i \)th frequency measurement by the receiver, call it \( f_i \), is related to the position \( r_i \) and velocity \( \dot{r}_i \) of the receiver, the position of the emitter \( p \) and the unshifted emitter frequency \( f_0 \) by the Doppler effect equation:

\[
f_i = f_0 \left( 1 + \frac{\hat{r}_i^\top (p-r_i)}{c \| p-r_i \|} \right), \quad i = 1, 2, \ldots, N. \tag{1}
\]

(Here, the over-dot indicates the derivative with respect to time \( \frac{\text{d}}{\text{d}t} \), the notation \( x^\top \) and \( \| x \| \) denote the transpose and Euclidean norm of a vector \( x \), respectively, and \( c \) is the speed of light). It is convenient to define a Doppler parameter

\[
\gamma_i \equiv c \left( \frac{f_i}{f_0} - 1 \right)
\]

so that (1) can be written as

\[
\gamma_i^2 \left( p^\top - r_i^\top \right) (p-r_i) = (r_i^\top (p-r_i))^2. \tag{2}
\]

To avoid unnecessary duplication in recording separately 2-D and 3-D cases, we shall write many of the relevant equations just for the 3-D case, especially where it is obvious that the corresponding 2-D case can be obtained by just setting certain variables to zero. The coordinates of the vectors \( p^\top \) and \( r_i^\top \) will be taken as \([x, y, z]\) and \([x_i, y_i, z_i]\), respectively.

For \( N \) independent measurements, we can write (2) in the linear algebra form as

\[
A w = b \tag{3}
\]

where the matrix \( A \) and vector \( b \) are known from the measurements, and the vector \( w \) contains entries depending on

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1This equation assumes that the speed of the receiver is much less than the speed of light so that relativistic effects can be ignored, and that the refractive index of air is constant and unity.
the emitter position, all being given by

\[
A = \begin{bmatrix}
\alpha_i^x & \alpha_i^y & \alpha_i^z & -2x_i\dot{y}_i & -2x_i\dot{z}_i \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_N^x & \alpha_N^y & \alpha_N^z & -2x_N\dot{y}_N & -2x_N\dot{z}_N \\
-2\dot{y}_1\dot{z}_1 & 2\dot{z}_1^2 & 2\dot{z}_1^2 & 2\dot{z}_1^2 & \\
\vdots & \vdots & \vdots & \vdots & \\
-2\dot{y}_N\dot{z}_N & 2\dot{z}_N^2 & 2\dot{z}_N^2 & 2\dot{z}_N^2 \\
\end{bmatrix}
\]

\[
w^\top = \begin{bmatrix} x^2 & y^2 & z^2 & xy & xz & yz & x & y & z \end{bmatrix}
\]

\[
b = \begin{bmatrix}
(r_1^\top \dot{r}_1)^2 - y_1^2 \dot{r}_1^\top \dot{r}_1 \\
\vdots \\
(r_N^\top \dot{r}_N)^2 - y_N^2 \dot{r}_N^\top \dot{r}_N \\
\end{bmatrix}
\]

with

\[
\alpha_i^x \equiv y_i^2 - x_i^2, \quad \alpha_i^y \equiv y_i^2 - y_i^2, \quad \alpha_i^z \equiv y_i^2 - z_i^2
\]

and

\[
\zeta_i^x = r_i^\top \dot{r}_i x_i - y_i^2 x_i \\
\zeta_i^y = r_i^\top \dot{r}_i y_i - y_i^2 y_i \\
\zeta_i^z = r_i^\top \dot{r}_i z_i - y_i^2 z_i.
\]

In the 2-D case, columns 3, 5, 6, and 9 of \(A\) drop out, as do the corresponding entries of the vector \(w\). Also, in the remaining entries of \(A\), the quantities \(z_i\) are all zero for all \(i\).

If there are nine or more measurements (five or more in the 2-D case), the matrix \(A\) may have full column rank, and indeed simulations reveal that for generic UAV trajectories and points on those trajectories, this property holds.\(^2\)

Suppose that \(A\) does have a full column rank. If we assume that all measurements are noiseless, we can find \(w\) and the target’s location is then given by the last three (two in the 2-D case) entries of \(w\). Note that if there are more than nine (five in the 2-D case) measurements one can express \(w\) as

\[
w = (A^\top A)^{-1} A^\top b. \tag{4}
\]

If measurements are contaminated by noise and \(N > 9\) (or \(N > 5\) in the 2-D case), then (3) will not have an exact solution. However, in general (4) yields a least-squares solution of (3), and again, the last three (or two) entries of the solution can be taken as the emitter position estimate. Of course if there are nine (five) measurements exactly, then with or without noise, this equation reduces to

\[
w = A^{-1} b. \tag{5}
\]

When the emitter is remote from the UAV positions, one cannot expect, especially in the noisy situation, this equation to yield accurate values. However, in the nonremote case, and as illustrated by simulations later, the solution given from the last three (two) entries of (4) can be satisfactory.

There is even an improvement to the solution algorithm that can be pursued but, given that our main interest in this paper is the treatment of stand-off scenarios, we will do this in detail elsewhere. This improvement arises from the fact that the only solutions, or approximations to solutions, are those of the form

\[
w = [x^2 y^2 z^2 xy xz yz x y z]^\top \tag{6}
\]
i.e., those for which the first six entries are obtainable as quadratic expressions involving the last three. Put another way, we are interested not in the usual least-squares solution of (3) but rather a constrained least-squares solution, obtained using six independent quadratic constraints. (A corresponding conclusion of course holds in the 2-D case, there being three constraints). As the techniques for achieving this are quite different to those applying when the emitter is remote, we avoid further discussion here.

A. Simulated Geolocation Results

Unless otherwise stated, the following parameters were used in all the simulations: both the emitter and receiver locations lie in the same 2-D plane, the receiver moves in a perfect circle centered at zero with a radius of 10 km and a speed of 200 m/s, the receiver position and velocity measurements are noiseless, the measured frequency is calculated at five evenly spaced intervals about a circle of radius 10 km, the transmitted frequency \(f_0\) is 300 MHz, and the measured frequency noise is zero mean Gaussian with a standard deviation of 10 Hz.

Fig. 1 depicts the situation in which the emitter is 30-km away.

The \(A\) matrix in (3) is a square matrix for this example, and no attempt was made to enforce the quadratic constraints on the equation solution mentioned earlier so the emitter location was obtained from (5).
these scenarios, the estimate of the range to the emitter is either unreliable or unnecessary. In the next subsection, we address the question of deciding on the basis of a collection of measurements whether or not a stand-off scenario exists. Assuming then that such a scenario has actually been identified, it becomes of interest to determine the bearing of the emitter and indeed Section III-C describes how to estimate the bearing to the emitter when the emitter is far from the receiver locations. The final subsection considers the question of deciding whether a new measurement is consistent with the stand-off scenario already identified.

In a number of occasions in what follows, we will work with singular values of various matrices. We include in Appendix A some relevant background on singular value decomposition.

A. Testing for a Stand-Off Scenario

Initially, we discuss the 3-D case. Suppose that the true position \([x, y, z]\) of the emitter is far away from each of the receiver positions \([x_i, y_i, z_i]\) at which a Doppler-shifted signal is received, and the origin of the coordinate system is located close to the receiver positions, thus implying \(\sqrt{x^2 + y^2 + z^2}\) assumes a large value. It is clear intuitively that the greater the stand-off distance, the harder it will be to determine the range of the emitter, though it might well be straightforward to still obtain its direction (elevation and azimuth). Put another way, if \([x, y, z]\) is one solution to the relevant equations, then for any constant \(s\) with \(s > 1\) (and maybe for some values less than 1, but this does not concern us) \([sx, sy, sz]\) should also be an approximate solution. In terms of the measurement matrix \(A\) and measurement vector \(b\), none of whose entries are expected to be large even when one at least of \(x, y,\) and \(z\) is large, we have both

\[
\begin{bmatrix}
    x^2 \\
    y^2 \\
    z^2 \\
    xy \\
    xz \\
    yz \\
    x \\
    y \\
    z
\end{bmatrix} \approx b \quad \text{and} \quad
\begin{bmatrix}
    s^2x^2 \\
    s^2y^2 \\
    s^2z^2 \\
    s^2xy \\
    s^2xz \\
    s^2yz \\
    sx \\
    sy \\
    sz
\end{bmatrix} \approx b
\]

from which we conclude (by letting \(s\) become large) that with \(\bar{A}\) denoting the first six columns of \(A\), there must hold

\[
\begin{bmatrix}
    x^2 \\
    y^2 \\
    z^2 \\
    xy \\
    xz \\
    yz \\
    x \\
    y \\
    z
\end{bmatrix} \approx \theta
\]

(7)

where

\[
\bar{A} =
\begin{bmatrix}
\alpha_1^x & \alpha_1^y & \alpha_1^z & -2x_1y_1 & -2x_1z_1 & -2y_1z_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_N^x & \alpha_N^y & \alpha_N^z & -2x_Ny_N & -2x_Nz_N & -2y_Nz_N
\end{bmatrix}.
\]

In turn, this means that the first six columns of \(A\) are approximately linearly dependent, or that \(\bar{A}\) has a singular value very close to 0.

The changes for the 2-D case are trivial, where \(\bar{A}\) has three columns, corresponding to the first three columns of the five column matrix \(A\).

The stand-off emitter indicator is defined as

\[
\chi \equiv 100 - 100 \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}
\]

(8)

where \(\sigma_{\text{max}}\) and \(\sigma_{\text{min}}\) denote the largest and smallest singular values of \(\bar{A}\), respectively. The stand-off indicator is zero if \(\sigma_{\text{max}}\) and \(\sigma_{\text{min}}\) are the same, and it is 100 if \(\sigma_{\text{min}}\) is zero.

Fig. 2 shows a 2-D case where the emitter is 15 km from the center of the receiver locations. In this case, the stand-off indicator is \(\chi = 36\). By contrast when the emitter is 150 km from the center of the receiver locations, as shown in Fig. 3, the stand-off indicator is \(\chi = 95\).

B. Direction of Arrival Variation and Singular Values

In this subsection, our aim is to indicate a relation between the angular spread of directions between different...
positions of the receiver and a stationary target on the one hand, and the stand-off parameter $\chi$ on the other hand.

To proceed, we first note that the squared Doppler equation (2) can be written as

$$y_t^2 u_i^t u_i = (r_1^t u_i)^2$$  \hspace{1cm} (9)

if we write the displacement vector $p - r$ in terms of range and a direction vector

$$p - r_1 = \rho_i u_i \quad \text{where} \quad u_i = \begin{bmatrix} \cos \theta_i \cos \phi_i \\ \cos \theta_i \sin \phi_i \\ \sin \theta_i \end{bmatrix}.$$  \hspace{1cm} (10)

Equation (9) can be reformulated in terms of $\bar{A}$ as

$$\bar{a}_i^t \mu_i = 0 \quad \text{where} \quad \bar{a}_i$$ is the $i$th row of $\bar{A}$ and

$$\mu_i = \begin{bmatrix} \cos^2 \theta_i \cos^2 \phi_i \\ \cos^2 \theta_i \sin^2 \phi_i \\ \sin^2 \theta_i \end{bmatrix}.$$  \hspace{1cm} (12)

In the noiseless case and if the emitter were infinitely far away, then the directions from the receivers to the emitter would all be the same, $\mu_i = \mu_i$ and

$$\bar{a}_i^t \mu_i = 0, \, \forall \, i.$$  

If the emitter is far away but not infinitely far away, then for some $\mu$ and small $\delta \mu_i$

$$\mu_i = \mu + \delta \mu_i$$

and the squared Doppler equation becomes (to first order in the angular variation in $\theta$ and $\phi$) away from the values associated with $\mu$

$$\bar{a}_i^t \mu = -\bar{a}_i^t \delta \mu_i = -\bar{a}_i^t M \delta \phi_i \quad \text{where} \quad M = \frac{\partial \mu}{\partial \varphi} = \begin{bmatrix} -\cos \phi \sin 2\theta & -\cos \phi \sin 2\theta \\ -\sin \phi \cos 2\theta & \cos \phi \cos 2\theta \\ \sin 2\theta & -\frac{1}{2} \sin \phi \sin 2\theta \\ \frac{1}{2} \cos \phi \cos 2\theta & \frac{1}{2} \cos \phi \cos 2\theta \end{bmatrix}.$$  \hspace{1cm} (13)

To assist in the following, we note that by expressing $\cos^4 \phi + \sin^4 \phi$ as $1 - 2 \cos^2 \phi \sin^2 \phi$, using the Pythagorean identity and the double angle formulas, a simple calculation yields the following bound on the Euclidean norm of $m_\phi$:

$$\|m_\phi\|^2 = \sin^2 2\theta \left(1 - (1/4) \sin^2 2\phi\right) + 1 \leq 2$$  \hspace{1cm} (15)

with the upper bound attained at $\phi = 0, \theta = \pi/4$.

Likewise

$$\|m_\theta\|^2 = \cos^2 \theta \left(\cos^2 \theta \left(\sin 2\phi\right)^2 + 1\right) \leq 2$$  \hspace{1cm} (16)

with the upper bound achieved at $2\phi = \pi/2, \theta = 0$.

We have discussed the ways in which the variation in $\delta \varphi$ reflected in the singular values of $\bar{A}$.

In Appendix A, we show that

$$\sigma_{\min}(\bar{A}) \|\mu\| \leq \|\bar{A} \mu\| \leq \sigma_{\max}(\bar{A}) \|\mu\|.$$  \hspace{1cm} (17)

Also, it is shown in Appendix B that the bounds on $\mu$ (where $\mu$ is defined by (12) with the subscripts removed) are $\sqrt{2/3} \|\mu\| \leq 1$. Accordingly, there holds

$$\sqrt{2/3} \sigma_{\min}(\bar{A}) \leq \|\bar{A} \mu\| \leq \sigma_{\max}(\bar{A}).$$  \hspace{1cm} (18)

Next, let $\Delta \theta, \Delta \phi$ denote the maximum magnitudes of $\delta \theta_i, \delta \phi_i$, respectively. Then, it is easily seen that (using the triangle inequality for the first inequality step)

$$\|\bar{A} \mu\| \leq \left|\bar{a}_1^t \mu_1 \delta \phi_1 \right| \leq \left|\bar{a}_i^t \mu_i \delta \phi_i \right| \leq \sigma_{\max}(\bar{A}) \|m_\phi\| \Delta \phi$$  \hspace{1cm} (19)

As the lower bound on $\|\bar{A} \mu\|$ is $\sqrt{2/3} \sigma_{\min}(\bar{A})$ and the upper bound on both $\|m_\phi\|$ and $\|m_\theta\|$ is $\sqrt{2}$, it follows that $\sqrt{2/3} \sigma_{\min}(\bar{A}) \leq \sqrt{2} \sigma_{\max}(\bar{A}) \max\{\Delta \theta, \Delta \phi\}$, and hence

$$\max\{\Delta \theta, \Delta \phi\} \geq \frac{1}{2 \sqrt{3}} \frac{\sigma_{\min}(\bar{A})}{\sigma_{\max}(\bar{A})}.$$  \hspace{1cm} (19)

For example if the stand-off flag is $\chi = 95$ as in Fig. 3, the angle error is greater than $1^\circ$, however if on $\chi = 36$ as in Fig. 2, it is greater than $11^\circ$.

C. Bearing Estimation With Unitary Constraint

Suppose that it has been determined (using the singular value test, or any other means) that a stand-off scenario exists. In this case, it is appropriate to switch from seeking to localize the target to seeking to determine its direction from the array of points at which the receiver measures Doppler data.

In a stand-off scenario

$$f_i \approx f_0 \left(1 + \frac{r_i^t u}{c}\right).$$  \hspace{1cm} (20)

The least-squares solution to (20) is

$$u = (B^t B)^{-1} B^t g.$$  \hspace{1cm} (21)
where

\[
B = \begin{bmatrix}
    f_1^T \\
    \vdots \\
    f_N^T
\end{bmatrix},
\quad
g = c \begin{bmatrix}
    \frac{\beta}{\rho} - 1 \\
    \vdots \\
    \frac{\beta}{\rho} - 1
\end{bmatrix} = \begin{bmatrix}
    \gamma_1 \\
    \vdots \\
    \gamma_N
\end{bmatrix}.
\]

In a noisy situation, there can be no guarantee that \( u \) derived by (21) is a unit vector, and it would seem prudent to seek to use this piece of information to reduce the impact of noise on the quality of the direction estimate. A straightforward way of incorporating the constraint is to reflect the constraint that \( u \) is a unit vector through use of a Lagrange multiplier. Finding a least-squares solution of a linear equation subject to a single quadratic constraint is a standard problem for which there is a straightforward algorithm. The Lagrangian is

\[
L = (u^T B^T - g^T) (B u - g) - \lambda (u^T u - 1).
\]  

This takes an extremum when

\[
\frac{\partial L}{\partial u} = 0 \Rightarrow (B^T B - \lambda I) u = B^T g \quad (23a)
\]

and

\[
\frac{\partial L}{\partial \lambda} = 0 \Rightarrow u^T u = 1. \quad (23b)
\]

At once we see that, with

\[
W(\lambda) = B^T B - \lambda I,
\]

\[
u = W^{-1}(\lambda)B^T g
\]

and

\[
1 = g^T B (W^{-1}(\lambda))^T W^{-1}(\lambda)B^T g.
\]  

One way to solve the equations resulting from the Lagrange multiplier introduction is to start with an estimate of \( u \) obtained from the standard least-squares solution (thus not necessarily of unit length) and an arbitrary estimate of \( \lambda \), say zero. Then, we calculate an improved estimate. Our new estimates of \( u \) and \( \lambda \) are \( u \rightarrow u + d u \) and \( \lambda \rightarrow \lambda + d \lambda \), respectively. Using (23) and keeping only terms of first order in \( d \lambda \) and \( d u \), we obtain

\[
\begin{bmatrix}
0 & 2u^T \\
-u & B^T B - \lambda I
\end{bmatrix}
\begin{bmatrix}
d\lambda \\
du
\end{bmatrix} = \begin{bmatrix}
1 - u^T u \\
B^T g - B^T B u + \lambda u
\end{bmatrix}.
\]  

(24)

The iterative scheme for determining the direction vector \( u \) is thus as follows.

1) The initial estimate of \( \lambda \) is \( \lambda_0 = 0 \), and we use (23a) to determine the initial estimate of \( u_0 = (B^T B)^{-1}B^T g \).
2) Improved estimates of \( \lambda \) and \( u \) are \( \lambda \rightarrow \lambda + d \lambda \) and \( u \rightarrow u + d u \), where \( d \lambda \) and \( d u \) are determined by solving (25).
3) Determine how close the length of the direction vector is to unity

\[
t = |u^T u - 1|.
\]

4) If \( t > 0.01 \), repeat steps 2 and 3 with \( u \rightarrow u + d u \) and \( \lambda \rightarrow \lambda + d \lambda \); if \( t < 0.01 \), the algorithm has converged to sufficient accuracy.

A comparison between the accuracy of linear estimates of the bearing using (21) and the scheme detailed above is given in Fig. 4. To generate this plot, 1000 simulations were run. The configuration parameters in these simulations were the same as previous simulations except the speed of the UAVs was normally distributed with mean zero and standard deviation 200 m/s, and their headings uniformly distributed. In this case, the error of the bearing estimate has improved by approximately 20%.

D. Outlier Detection

Suppose that \( u \) is determined from \( N \) frequency measurements by solving (23), and then, another measurement is made. How do we determine if this new measurement belongs to the same emitter? Again, we will use ideas of singular value decomposition to deal with this issue.

Suppose that, based on \( N \) measurements, we have concluded a target is in a stand-off situation. Suppose a further measurement is taken. Can we decide whether or not this measurement is consistent with the stand-off notion without solving the problem from the start using all \( N + 1 \) measurements, i.e., can we exploit what we have learnt with \( N \) measurements to simplify the task?

In linear algebra terms, we are asking under what circumstances the addition of a row to a matrix which is known to have a small singular value will cause the enlarged matrix to still have a small singular value.

Suppose that the initially given matrix \( \bar{A} \) is \( N \times 6 \) and has singular value decomposition, as given by (33) in Appendix A, where now the diagonal matrix \( \Sigma \) and orthogonal matrix \( Q \) are \( 6 \times 6 \). Suppose also that the minimum singular value \( \sigma_6 \) is small relative to the other singular values. (In the 2-D case, \( \bar{A} \) is \( N \times 3 \) and the smallest singular value is \( \sigma_3 \). We shall confine our discussion to the 3-D case in this section; changes for the 2-D case are trivial). By the lemma in Appendix A, \( \| \bar{A} \|_Q \) is small. Suppose an \( (N + 1) \)th measurement is defined by a six-dimensional row vector

\[
\bar{a}_{N+1}^T = \begin{bmatrix}
\bar{A}_{N+1,1} & \bar{A}_{N+1,2} & \ldots & \bar{A}_{N+1,6}
\end{bmatrix}.
\]
If $a_{N+1}^\top$ is an inlier, we expect it to be confined to the space spanned by $\{q_1^\top, \ldots, q_i^\top\}$, or equivalently, we expect $a_{N+1}^\top$ to be orthogonal to $q_6$, i.e., $a_{N+1}^\top q_6 = 0$. In practice because the measurements are noisy and the emitter is not infinitely far away, we expect the component of $a_{N+1}^\top$ parallel to $q_6$ to be small, but nonzero, i.e.

$$\frac{|a_{N+1}^\top q_6|}{||a_{N+1}||} \ll 1.$$  

A good measure of acceptance of a new measurement is the percentage of $a_{N+1}$ perpendicular to $q_6$ as

$$\xi = 100 - 100 \times \frac{|a_{N+1}^\top q_6|}{||a_{N+1}||}. \tag{26}$$

We now consider a 2-D example. When the extra measurement comes from an emitter at a bearing of approximately 103$^\circ$ different from the original emitter, the inlier score is $\xi = 8$, as illustrated in Fig. 5. By contrast if the bearing of the extra emitter is only 10$^\circ$ off that of the original emitter (as shown in Fig. 6), then the inlier score is $\xi = 91$.

**IV. NOMINAL RATHER THAN PRECISE EMITTER FREQUENCY IS KNOWN**

In this section, we shall assume that a nominal emitter frequency, $f_0$ is known to the receiver or receivers, while the true emitter frequency is $f_0(1 + \Delta)$, where $\Delta$ is a small fractional offset, and not known to the receivers.

In particular, we shall assume the validity of two approximations, namely, that $\Delta^2$ and $\Delta(f_0 - 1)$ are both negligible in comparison with $\Delta$. In broad terms, the localization task is one of finding the emitter coordinates and $\Delta$, or in a stand-off scenario, the emitter bearing and $\Delta$. Rather than give great detail, we shall simply summarize the key changes, which effectively make the equations more complicated.

The fundamental Doppler equation (1) is varied, to become

$$f_i = f_0 (1 + \Delta) \left(1 + \frac{\dot{r}_i^\top (p - r_i)}{c||p - r_i||} \right). \tag{27}$$

The equation replacing (2) is

$$c^2 \left(\frac{f_i}{f_0} - 1\right)^2 ||p - r_i||^2 = (\dot{r}_i^\top (p-r_i))^2. \tag{28}$$

Replacing $(1 + \Delta)^{-1}$ by $1 - \Delta$ yields

$$c^2 \left(\frac{f_i}{f_0} - 1\right)^2 ||p - r_i||^2 \approx (\dot{r}_i^\top (p-r_i))^2. \tag{29}$$

Next, using the second approximation referred to previous, replace $\frac{f_i}{f_0}$ by simply $\Delta$. There results

$$c^2 \left((\frac{f_i}{f_0} - 1) - \Delta\right)^2 ||p - r_i||^2 \approx (\dot{r}_i^\top (p-r_i))^2.$$  

or using the Doppler parameter definition $\gamma_i = c \left(\frac{f_i}{f_0} - 1\right)$, there results

$$c^2 (\gamma_i/c - \Delta)^2 ||p - r_i||^2 \approx (\dot{r}_i^\top (p-r_i))^2. \tag{29}$$

From a series of such equations, one can construct a variation on the earlier linear equation, but involving a larger vector of unknowns. Previously, we had $A w = b$ with

$$w = [x^2 \quad y^2 \quad z^2 \quad xy \quad xz \quad yz \quad x \quad y \quad z]^\top.$$

It is necessary that the adjusted definition for $w$ reflects the incorporation in (29) of $-2\gamma/c\Delta ||p - r_i||^2$ and of $c^2 \Delta^2 ||p - r_i||^2$, with $\Delta$ an unknown. The upshot is that an additional ten entries must be included in the vector $w$, viz. $\Delta x, \Delta y, \Delta z, \Delta^2 x, \Delta^2 y, \Delta^2 z, \Delta^2 z$. There are of course additional corresponding columns of $A$, but the measurement vector $b$ is unaltered.

The same rationale as previously for considering the occurrence of a stand-off scenario leads to the conclusion that a certain submatrix of $A$ will have an approximately zero singular value, viz. that associated with those entries of $w$ that depend quadratically on $x$, $y$, and $z$. These are the first six columns of the $A$ matrix as previously, and the additional two columns corresponding to the new entries
of $w$ given by $\Delta(\chi^2 + y^2 + z^2)$ and $\Delta^2(\chi^2 + y^2 + z^2)$, this being a total of eight columns in all.

When an emitter is in a stand-off scenario, but the exact emitter frequency is unknown, the estimation of direction is straightforwardly treated. We replace (20) by

$$f_i = f_0 (1 + \Delta) \left( 1 + \frac{\bar{r}_i^T u}{c} \right) \approx f_0 \left( 1 + \frac{\bar{r}_i^T u}{c} + \Delta \right). \quad (30)$$

(Here, we have neglected the second-order term $\Delta(\bar{r}_i^T u)$ in comparison with the first-order terms $\bar{r}_i^T u$ and $\Delta$.) We define a modification $B_\Delta$ of the matrix $B$ used earlier by

$$B_\Delta = \begin{bmatrix} \bar{r}_1^T c \\ \bar{r}_2^T c \\ \vdots \\ \bar{r}_N^T c \end{bmatrix}. \quad (31)$$

The vector $g$ remains as before, viz $[\gamma_1 \gamma_2 \ldots \gamma_\nu]^T$. The least-squares solution to the equation set (30) is

$$\begin{bmatrix} u \\ \Delta \end{bmatrix} = (B_\Delta^T B_\Delta)^{-1} B_\Delta^T g. \quad (32)$$

The results for a particular example are shown in Fig. 7.

V. CONCLUSION

This article considers methods for determining the direction of arrival of an emitter signal based only the measured frequency of the emitter signal at the sensors and the navigation data (positions and velocities of the sensors). The methods are applicable in a 2-D or 3-D ambient space. In solving the relevant equations, it is crucial to determine that two conditions are met: the emitter is at a “stand-off” distance, and that each measurement is associated with a unique emitter. To test these two conditions, we create a “stand-off” indicator and an “inlier” indicator. As the values of these indicators approach 100%, we can be confident that the conditions are met to produce a reliable estimate of the emitter’s direction based on the frequency and navigation data measurements.

In the initial sections of this article, we assumed that the emitter frequency was either known or could be measured precisely. In the last section, we relaxed this condition and showed that if the emitter’s transmitter frequency was nominally, but not precisely, known then the emitter’s direction could still be estimated to a good degree of accuracy.

APPENDIX

A. Singular Value Decomposition and Measurement Association

Let $\tilde{A} \in \mathbb{R}^{N \times K}$ be a matrix with more rows than columns. (In our application, $K$ will take the value 3 or 6 according as we treat an ambient 2-D or 3-D environment, while as before, $N$ is the number of measurements). The singular value decomposition of $\tilde{A}$ is of the form

$$\tilde{A} = \tilde{P} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q^\top$$

where $\tilde{P}$ and $Q$ are real orthogonal matrices of size $N \times N$ and $K \times K$, respectively, and $\Sigma$ is a square diagonal matrix $diag[\sigma_1, \sigma_2, \ldots, \sigma_K]$ with nonnegative diagonal entries decreasing down the diagonal. These are the singular values of $\tilde{A}$.

It is standard that for any vector $x$, there holds (with the norm sign as earlier denoting a Euclidean norm)

$$\|\tilde{A}x\| \leq \sigma_1(\tilde{A}) \|x\|.$$ \quad (34)

Slightly less well known is the following inequality involving the smallest singular value:

$$\|\tilde{A}x\| \geq \sigma_K(\tilde{A}) \|x\|.$$ \quad (35)

We shall also have occasion to use a refinement of the last inequality, as follows.

LEMMA 1 Let $q_K$ denote the last column of $Q$. Then $\|\tilde{A}q_K\| = \sigma_K$, i.e., $q_K$ is the solution of the problem of minimizing $\|\tilde{A}x\|$ over $\|x\| = 1$.

PROOF The definition of $q_K$ and the orthogonality of $Q$ ensures that $Q^\top q_K = [0, 0, 0]^\top$ and so

$$\tilde{A}q_K = \tilde{P} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} q_K = \tilde{P} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \tilde{P} \begin{bmatrix} 0 \\ \sigma_K \vdots \end{bmatrix}. \quad (35)$$

Since $\tilde{P}$ is orthogonal, it is immediate that $\|\tilde{A}q_K\| = \sigma_K$. However, since the singular values of $\tilde{A}$ are precisely the eigenvalues of $\tilde{A}^\top \tilde{A}$, the minimum of $x^\top \tilde{A}^\top \tilde{A} x$ over $\|x\| = 1$ is $\sigma_K^2$, and so (35) is verified.

B. Bounds on the Magnitude of $\mu$

The squared magnitude $\mu$ (given by (12) with the subscripts removed) is

$$\|\mu\|^2 = \cos^4 \theta (\cos^4 \phi + \sin^4 \phi) + \sin^4 \theta 
+ \cos^4 \theta \cos^2 \phi \sin^2 \theta + \cos^2 \theta \sin^2 \phi \cos^2 \phi 
+ \cos^2 \theta \sin^2 \phi \sin^2 \phi.$$
Noting that \( \cos^4 \phi + \sin^4 \phi = 1 - 2 \cos^2 \phi \sin^2 \phi \) and using the Pythagorean identity \( \cos^2 \phi + \sin^2 \phi = 1 \), we note that the squared magnitude reduces to
\[
\| \mathbf{\mu} \|^2 = \cos^4 \theta \left( 1 - \cos^2 \phi \sin^2 \phi \right) + \sin^2 \theta.
\]
This is clearly a maximum when \( \cos \phi \sin \phi = 0 \). When this occurs \( \| \mathbf{\mu} \|^2 \leq \cos^2 \theta + \sin^2 \theta \), which reduces to
\[
\| \mathbf{\mu} \|^2 \leq 1 - \cos^2 \theta \sin^2 \theta \leq 1
\]
where once again, we have made use of the Pythagorean identity.

To determine the minimum value of \( \| \mathbf{\mu} \|^2 \), we note that
\[
1 - \cos^2 \phi \sin^2 \phi = 1 - \frac{1}{4} (\sin 2\phi)^2 \geq \frac{3}{4}
\]
hence
\[
\| \mathbf{\mu} \|^2_{\text{min}} = \frac{3}{4} \cos^4 \theta + \sin^2 \theta.
\]
This is an extremum when the derivative is zero, i.e.
\[
\frac{d \| \mathbf{\mu} \|^2_{\text{min}}}{d \theta} = \sin \theta \cos \theta \left( 2 - 3 \cos^2 \theta \right) = 0.
\]
This equation has three solutions \{ \sin \theta = 0, \cos \theta = 0, \cos^2 \theta = 2/3 \} and the minimum occurs when \( \cos^2 \theta = 2/3 \) and the corresponding value of \( \| \mathbf{\mu} \|_{\text{min}} = \sqrt{2/3} \). Combining this result with the one given previously, we determine
\[
\sqrt{2/3} \leq \| \mathbf{\mu} \| \leq 1.
\]

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