Cooperative Localization of a GPS-Denied UAV Using Direction-of-Arrival Measurements

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A GPS-denied unmanned aerial vehicle (UAV) (Agent B) is localized through inertial navigation system alignment with the aid of a nearby GPS-equipped UAV (Agent A), which broadcasts its position at several time instants. Agent B measures the signals’ direction of arrival with respect to Agent B’s inertial navigation frame. Semidefinite programming and the orthogonal Procrustes algorithm are employed,

1. INTRODUCTION

Unmanned aerial vehicles (UAVs) play a central role in many defense reconnaissance and surveillance operations. Formations of UAVs can provide greater reliability and coverage when compared to a single UAV. To provide meaningful data in such operations, all UAVs in a formation must have a common reference frame (typically the global frame). Traditionally, UAVs have access to the global frame via GPS. However, GPS signals may be lost in urban environments and enemy-controlled airspace (jamming). Overcoming loss of GPS signal is a hot topic in research [1] and offers a range of different problems in literature [2], [3]. Without access to global coordinates, a UAV must rely on its inertial navigation system (INS). Stochastic error in on-board sensor measurements causes the INS frame to accumulate drift. At any given time, drift can be characterized by a rotation and translation with respect to the global frame and is assumed to be independent between UAVs in a formation. INS frame drift, therefore, cannot be modeled deterministically. Information from global and INS frames must be collected in order to determine the drift between frames and align the INS frame with the global frame. We describe this process as cooperative localization when multiple vehicles interact for this purpose.

Signals of opportunity (SOP), such as AM/FM radio, digital television, or cellular communication, can serve as references to assist in characterizing the misalignment between navigation frames of multiple agents. Recent contributions in this field include [4]–[6]. In contexts where SOP are either unavailable or unreliable, various measurement types such as distance between agents and direction of arrival of a signal (we henceforth call DOA)1 can be used for this process. In the context of UAVs, additional sensors add weight and consume power. As a result, one generally aims to minimize the number of measurement types required for localization. This paper studies a cooperative approach to localization using DOA measurements.

When two or more GPS-enabled UAVs can simultaneously measure directions with respect to the global frame toward the GPS-denied UAV, the location of the GPS-denied UAV is given by the point minimizing distances to the half-line loci derived from the directional measurements [7]–[9]. Operational requirements may limit the number of nearby GPS-enabled UAVs to one single agent. We, therefore, seek a solution that does not require simultaneous measurements to a single point.

When the GPS-denied agent is able to simultaneously measure directions with respect to its local INS frame toward multiple landmarks with known global coordinates, and accuracy is improved through maximum likelihood estimation. The method is validated using flight data and simulations. A three-agent extension is explored.

1A bearing generally describes a scalar measurement between two points in a plane, whereas a direction of arrival is a vector measurement between two points in a 3-D ambient space (as considered in this paper).
triangulation-based measurements can be used to achieve localization. This problem is studied in a three-dimensional (3-D) space in [10] and in a two-dimensional (2-D) space in [11] and [12]. If only one landmark bearing can be measured at any given time, a bearing-only simultaneous localization and mapping (SLAM) algorithm may be used to progressively construct a map of the environment on the condition each landmark is seen at least twice. Alignment of a GPS-denied agent’s INS frame could then be achieved by determining the rotation and translation between the map’s coordinate frame and the global coordinate frame. In practice, landmark locations may be unknown, or there may be no guarantee they are stationary or permanent, and hence, we require a localization algorithm, which is independent of landmarks in the environment. Iterative filtering methods such as the extended Kalman filter (EKF) are often required when drift is significant between updates. In [13], an EKF is used to estimate drift in the context of marine localization. In our problem context, the drift is sufficiently slow to be modeled as stationary over short periods. We are motivated to formulate a localization algorithm, which does not involve an iterative filtering technique.

Without reliance on landmarks, the only directional measurements available are between the GPS-denied and GPS-enabled UAVs. Given the small size of their airframes with respect to their separation distance, these UAVs are modeled as point agents, and therefore, one single-directional measurement is available at any given time. A stationary target is localized by an agent using bearing-only measurements in a 2-D space [14], [15] and in a 3-D space [16]. A similar problem is considered in [17], in which a mobile source is localized using measurements received at a stationary receiver using an iterative filtering technique. However, for operational reasons, the agent requiring localization may be unable to broadcast signals, or agents involved may not be allowed to remain stationary. In such instances, the approaches in [14]–[17] are not suitable. Commonly used computer vision techniques such as structure from motion [18] require directional measurements toward multiple stationary points or toward a stationary point from multiple known positions. This is not possible in our problem context. The measurement and motion requirements we are imposing, therefore, represent a significant technical challenge. One algorithm satisfying all the requirements above was proposed in [19], in which two agents perform sinusoidal motion in 2-D ambient space. Directional measurements are used to obtain the distance between Agents A and B, but localization of B in the global frame is not achieved.

Motivated by interest from Australia’s Defence Science and Technology Group, this paper seeks to address the problem of localizing a GPS-denied UAV with the assistance of a GPS-enabled UAV, which we will call Agents B and A, respectively. Both agents move arbitrarily in a 3-D space. Agent B navigates using an INS frame. Agent A broadcasts its position in the global coordinate frame at discrete instants in time. For each broadcast, Agent B is able to take a DOA measurement toward Agent A.

The problem setup and the solution we propose are both novel. In particular, while the literature discussed above considers certain aspects from the following list, none consider all of the following aspects simultaneously.

1) The network consists of only two mobile agents (and is therefore different to the sensor network localization problems in the literature).
2) There is no a priori knowledge or sensing of a stationary reference point in the global frame.
3) Both UAVs are free to execute arbitrary motion in a 3-D space, with the exception of a small number of geometrically unsolvable trajectory pairs.
4) Cooperation occurs through unidirectional signal transmission. Agent A broadcasts its global position (acquired using GPS) to Agent B (which is GPS-denied).

When performing nonroutine operations in unfamiliar environments, any combination of these four aspects may be required with short notice. As a result, we are motivated to determine a reliable general solution to the cooperative localization problem.

In [20], this problem is studied in a 2-D space using bearing measurements, but the added (third) dimension in our paper means two scalar quantities, not one, are obtained per measurement. This significantly complicates the problem, thus requiring new techniques to be introduced.

In our proposed solution, we localize Agent B by identifying the relationship between the global frame ( navigated by Agent A) and the inertial navigation frame of Agent B. The relationship is identified by solving a system of linear equations for a set of unknown variables. The nature of the problem means that quadratic constraints exist on some of the variables. To improve robustness against noisy measurements, we exploit the quadratic constraints and use semidefinite programming (SDP), and the orthogonal Procrustes algorithm to obtain an initial solution for maximum likelihood estimation (MLE); this combined approach is a key novel contribution of this paper.

We evaluate the performance of the algorithm by using a real set of trajectories and Monte Carlo simulations. Sets of unsuitable trajectories are identified, in which our proposed method cannot feasibly obtain a unique solution. Finally, we explore an extension of the algorithm to a three-agent network, in which two agents are GPS-denied.

The rest of this paper is structured as follows. In Section II, the problem is formalized. In Section III, a localization method using a linear equation formulation is proposed. Section IV extends this method to SDP to produce a more robust localization algorithm. In Section V, an MLE method is presented to refine results further. Section VI presents simulation results to evaluate the performance of\footnote{No constraints exist on the trajectories other than the physical limitations of the aircraft. See Subsection IV-F for further details on unsuitable trajectories.}
The rotation and translation of Agent B’s local INS frame (B₂) with respect to the global frame (A₁) evolves via drift. Although this drift is significant over long periods, frame B₂ can be modeled as stationary with respect to frame A₁ over short intervals. During these short intervals, the following measurement process occurs multiple times. At each time instant k, the following four activities occur simultaneously:

1. Agent B records its own position in the INS frame \( p_B^B(k) \).
2. Agent A records and broadcasts its position in the global frame \( p_A^A(k) \).
3. Agent B receives the broadcast of \( p_A^A(k) \) from Agent A and measures this signal’s DOA using instruments fixed to the UAV’s fuselage. This directional measurement is, therefore, naturally referenced to the body-fixed frame \( B_3 \).
4. Agent B’s attitude, i.e., orientation with respect to the INS frames \( B_2 \) and \( B_3 \), is known. An expression for the DOA measurement referenced to the axes INS frames \( B_3 \) can, therefore, be easily calculated.

A DOA measurement, referenced to a frame with axes denoted \( x, y, z \), is expressed as follows.

1. Azimuth \((\theta)\): angle formed between the positive \( x \)-axis and the projection of the free vector from Agent B toward Agent A onto the \( xy \) plane.
2. Elevation \((\phi)\): angle formed between the free vector from Agent B toward Agent A and \( xy \) plane. The angle is positive if the \( z \) component of the unit vector toward Agent A is positive.

The problem addressed in this paper, namely the localization of Agent B, is achieved if we can determine the relationship between the global frame \( A_1 \) and the local INS frame \( B_2 \). This information can be used to determine global coordinates of Agent B at each time instant:

\[
p_B^A(k) = [u_B(k), v_B(k), w_B(k)]^\top.
\]

(Passing between the global frame \( A_1 \) and the local INS frame of Agent B \( B_2 \) is achieved by a rotation of frame axes (defined by a rotation matrix, call it \( R^B_{A_1} \)) and translation \( t^B_{A_1} \) of frame. For instance, the coordinate vector of the position of Agent A referenced to the INS frame of Agent B is

\[
p_A^B(k) = R^B_{A_1} p_A^A(k) + t^B_{A_1}.
\]

We, therefore, have

\[
p_A^B(k) = R^B_{A_1} p_B^B(k) = R^B_{A_1} p_A^A(k) - t^B_{A_1} p_A^B(k).
\]

Early sections in this paper (covering up to and including the employment of the orthogonal Procrustes algorithm) appeared in less detail in the conference paper [21]. Additions have been made to these sections—the literature review is now more extensive, and the role of different coordinate frames is much more explicitly set out; the performance of the algorithm is now validated on real flight trajectories. Analysis of unsuitable trajectories, ML refinement, and the three-agent extension are further extensions beyond [21].

\[\text{II. PROBLEM DEFINITION}\]

Two agents, which we call Agents A and B, travel along arbitrary trajectories in a 3-D space. Agent A has GPS and, therefore, navigates with respect to the global frame. Because Agent B cannot access GPS, it has no ability to self-localize in the global frame, but can self-localize and navigate in a local inertial frame by integrating gyroscope and accelerometer measurements.

This two-agent localization problem involves four frames, as shown in Fig. 1. The importance of each frame, and its use in obtaining the localization, will be made clear in the following. Frames are labeled as follows.

1) The global frame is \( A_1 \) (available only to Agent A).
2) The local INS frame of Agent B is denoted by \( B_2 \).
3) The body-centered INS frame of Agent B (axes of frames \( B_2 \) and \( B_3 \) are parallel by definition) is denoted \( B_3 \). \( B_3 \) is now validated on real flight trajectories. Analysis of unsuitable trajectories, ML refinement, and the three-agent extension are further extensions beyond [21].
where \( R_{B_A}^{B_B} = R_{B_A}^{A_1} \) and \(-R_{A_1}^{B_B}t_{A_1}^{B_B} = t_{A_2}^{A_2} \). The localization problem can be reduced to solving for \( R_{B_A}^{B_B} \in SO(3) \) with entries \( r_{ij} \) and \( t_{A_1}^{B_B} \in \mathbb{R}^3 \) with entries \( t_i \).

The matrix \( R_{B_A}^{B_B} \) is a rotation matrix if and only if \( R_{A_1}^{B_B}R_{B_A}^{B_B} = I_3 \) and \( \det(R_{B_A}^{B_B}) = 1 \). As will be seen in the following, these constraints are equivalent to a set of quadratic constraints on the entries of \( R_{B_A}^{B_B} \). In total, there are 12 entries of \( R_{B_A}^{B_B} \) and \( t_{A_1}^{B_B} \) to be found as we work directly with \( r_{ij} \).

III. LINEAR SYSTEM METHOD

This section presents a linear system (LS) method to solving the localization problem. Given enough measurements, the LS approach can achieve exact localization when using noiseless DOA measurements, so long as Agents A and B avoid a set of unsuitable trajectories (which are detailed in Subsection IV-F), in which rank deficiency is encountered. Building on this, Section IV introduces nonlinear constraints to the linear problem defined in this section to improve accuracy in the presence of noise.

A. Forming a System of Linear Equations

The following analysis holds for all \( k \) instants in time; hence, we drop the argument \( k \). The DOA measurement can be represented by a unit vector pointing from Agent B to Agent A. This vector is defined by azimuth and elevation angles \( \theta \) and \( \phi \) referenced to the local INS frame \( B_2 \), and its coordinates in the frame \( B_2 \) are given by

\[
\hat{q}(\theta, \phi) = [\hat{q}_1, \hat{q}_2, \hat{q}_3] = [\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi]^T.
\]

Define \( \hat{q} = \|p_A^{B_B} - p_B^{B_B}\| \) as the Euclidean distance between Agents A and B (which is not available to either agent).

Scaling to obtain the unit vector \( \hat{q} \) gives

\[
\hat{q}(\theta, \phi) = \frac{1}{\hat{q}} \begin{bmatrix} x_A - x_B, y_A - y_B, z_A - z_B \end{bmatrix}^T.
\]

Applying (4) yields

\[
\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{bmatrix} = \frac{1}{\hat{q}} \begin{bmatrix} r_{11}u_A + r_{12}v_A + r_{13}w_A + t_1 - x_B \\ r_{21}u_A + r_{22}v_A + r_{23}w_A + t_2 - y_B \\ r_{31}u_A + r_{32}v_A + r_{33}w_A + t_3 - z_B \end{bmatrix}.
\]

The left-hand vector is calculated directly from DOA measurements. Cross-multiplying entries 1 and 3 of both vectors eliminates the unknown \( \hat{q}_3 \), and rearranging yields

\[
(u_A\hat{q}_3)r_{11} + (v_A\hat{q}_3)r_{12} + (w_A\hat{q}_3)r_{13} - (u_A\hat{q}_1)r_{31} - (v_A\hat{q}_1)r_{32} - (w_A\hat{q}_1)r_{33} + (\hat{q}_3)t_1 - (\hat{q}_1)t_3 = (\hat{q}_2)x_B - (\hat{q}_1)z_B.
\]

Similarly, from the second and third entries in (8), we have

\[
(u_A\hat{q}_3)r_{21} + (v_A\hat{q}_3)r_{22} + (w_A\hat{q}_3)r_{23} - (u_A\hat{q}_2)r_{31} - (v_A\hat{q}_2)r_{32} - (w_A\hat{q}_2)r_{33} + (\hat{q}_3)t_2 - (\hat{q}_2)t_3 = (\hat{q}_3)y_B - (\hat{q}_2)z_B.
\]

Notice that both (9) and (10) are linear in the unknown \( r_{ij} \) and \( t_i \) terms. Given a series of \( K \) DOA measurements

\[
\begin{align*}
A\Psi &= b, \quad A \in \mathbb{R}^{2K \times 12} \quad (11) \\
\text{where } A \text{ and } b \text{ are completely known, containing } &\theta(k), \phi(k), \quad \rho_A^{B_A}, \quad \text{and } \rho_B^{B_B}. \quad \text{The 12-vector of unknowns } \Psi \text{ is defined as}
\end{align*}
\]

\[
\Psi = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{31} & r_{32} & r_{33} & t_1 & t_2 & t_3 \end{bmatrix}^T.
\]

Entrywise definitions of \( A \) and \( b \) are provided in an extended version of this paper [22]. These entries of \( \Psi \) can be used to reconstruct the trajectory of Agent B in the global frame using (5), and therefore, solving (11) for \( \Psi \) constitutes a solution to the localization problem. In the noiseless case, if \( K \geq 6 \) and \( A \) is of full column rank, (11) will be solvable.

B. Example of the LS Method in a Noiseless Case

We demonstrate the LS method using trajectories performed by aircraft operated by the Australian Defence Science and Technology Group. Positions of both Agents A and B within the global frame and Agent B within the INS frame were measured by on-board instruments, whereas we generated a set of calculated DOA values using the above recorded real measurements.

These trajectories are plotted in Fig. 2. We will make additional use of this trajectory pair in the noisy measurement case presented in Section IV and in the MLE refinement of the noisy case localization result in Section V. Extensive Monte Carlo simulations demonstrating localization for a large number of realistic\(^7\) flight trajectories are left to the noisy measurement case.

The quantities \( R_{B_A}^{B_B} \) and \( t_{A_1}^{B_B} \) and the DOA measurements are tabulated in the extended version of this paper [22]. Using (11), \( R_{B_A}^{B_B} \) and \( t_{A_1}^{B_B} \) were obtained exactly for the given flight trajectories; the solution is the green line in Fig. 2.

\(^7\)By realistic, we mean that the distance separation between successive measurements is consistent with UAV flight speeds and ensures that the UAV does not exceed an upper bound on the turn/climb rate. Further detail is provided in the extended version of this paper [22].
IV. SDP METHOD

This section presents an SDP method for localization, extending from the LS approach presented in Section III. This method reduces the minimum required number of DOA measurements to obtain a unique solution and is more robust than LS in terms of DOA measurement noise, and unsuitable trajectories are reduced. Results from this section will serve as an initialization of our localization method, which will be optimized using MLE in Section V.

Rank-relaxed SDP is used to incorporate the quadratic constraints on certain entries of $\Psi$ arising from the properties of rotation matrices. The inclusion of rotation matrix constraints in SDP problems has been used previously to jointly estimate the attitude and spin rate of a satellite [23], and in camera pose estimation using structure from motion (SFM) techniques when directional measurements are made to multiple points simultaneously [24]. We now apply rank-relaxed SDP in a novel context to achieve INS alignment of Agent B, sufficient for its localization. Finally, the orthogonal Procrustes algorithm (O) is used to compensate for the rank relaxation of the SDP. This section concludes by discussing trajectories which cause DOA localization techniques to become unsuitable under certain conditions.

A. Quadratic Constraints on Entries of $\Psi$

Rank-relaxed SDP (in the presence of inexact or noise contaminated data) benefits from the inclusion of quadratic constraint equations. We now identify 21 quadratic and linearly independent constraint equations on entries of $R_{B_k}^A$, which all appear in $\Psi$ in (12). Recall the orthogonality property of rotation matrices; by computing each entry of $R_{B_k}^A R_{A_i}^{B_k}$ and setting these equal to entries of $I_3$, and denoting the $i$th entry of $\Psi$ as $\psi_i$, we define constraints

$$C_i = \psi_{3i-2}^2 + \psi_{3i-1}^2 + \psi_{3i}^2 - 1 = 0, \quad i = 1, 2, 3$$

$$C_4 = \psi_1 \psi_4 + \psi_2 \psi_5 + \psi_3 \psi_6 = 0$$

$$C_5 = \psi_1 \psi_4 + \psi_2 \psi_5 + \psi_3 \psi_6 = 0$$

$$C_6 = \psi_4 \psi_7 + \psi_5 \psi_8 + \psi_6 \psi_9 = 0.$$  (13c)

To simplify notation, we call $C_{j:k}$ the set of constraints $C_i$ for $i = j, \ldots, k$. Similarly, by computing each entry of $R_{B_k}^A R_{A_j}^{B_k}$ and setting these equal to $I_3$, we define constraints $C_{7:12}$, which are omitted due to space limitations, and notice that the sets $C_{1:6}$ and $C_{7:12}$ are clearly equivalent. Further constraints are required to ensure $\det(R_{B_k}^A) = 1$.

Cramer’s formula states that $R_{B_k}^{-1} = \text{adj}(R_{B_k}^A)/\det(R_{B_k}^A)$, where $\text{adj}(R_{B_k}^A)$ denotes the adjugate matrix of $R_{B_k}^A$. Orthogonality of $R_{B_k}^A$ implies $R_{B_k}^A = \text{adj}(R_{B_k}^A)^\top$. By computing entries of the first column of $Z = R_{B_k}^A - \text{adj}(R_{B_k}^A)^\top$ and setting these equal to 0, we define constraints $C_{13:15}$ as follows:

$$C_{13} = \psi_1 - (\psi_5 \psi_9 - \psi_6 \psi_8) = 0$$

$$C_{14} = \psi_4 - (\psi_3 \psi_8 - \psi_2 \psi_9) = 0$$

$$C_{15} = \psi_7 - (\psi_2 \psi_6 - \psi_3 \psi_5) = 0.$$  (14c)

Similarly, by computing the entries of the second and third columns of $Z$ and setting these equal to 0, we define constraints $C_{16:18}$ and $C_{19:21}$, respectively. Due to space limitations, we omit presenting them. The complete set $C_{1:21} = C_\psi$ constrains $R_{B_k}^A$ to be a rotation matrix. The set of constraints is not an independent set, e.g., $C_{16}$ is equivalent to $C_{2:12}$. The benefits of the inclusion of dependent constraints are discussed further in Section IV-C.

 Due to these additional relations, localization requires azimuth and elevation measurements at a minimum of four instants only ($K = 4$), as opposed to six instants required in Section III.

B. Formulation of the Semidefinite Program

The goal of the semidefinite program is to obtain

$$\arg\min_{\Psi} ||A\Psi - b||_2$$  \quad (15)

subject to $C_\psi$. Equivalently, we seek $\arg\min_{\Psi} ||A\Psi - b||^2_2$ subject to $C_\psi$. We define the inner product of two matrices $U$ and $V$ as $\langle U, V \rangle = \text{trace}(UV^\top)$. One obtains

$$||A\Psi - b||^2_2 = \langle P, X \rangle$$  \quad (16)

where $P = [A \ b]^\top [A \ b]$ and $X = [\Psi^\top - I][\Psi^\top - I]^{\frac{1}{2}}$ is a rank 1 positive-semidefinite matrix. The constraints $C_\psi$ can also be expressed in inner product form. For $i = 1, \ldots, 21, C_i = 0$ is equivalent to $(Q_i, X) = 0$ for some easily determined $Q_i$. Solving for $\Psi$ in (15) is, therefore, equivalent to solving for

$$\arg\min_{\Psi} \langle P, X \rangle$$  \quad (17)

$$X \geq 0$$

$$\text{rank}(X) = 1$$

$$X_{13,13} = 1$$

$$(Q_i, X) = 0, \quad i = 1, \ldots, 21.$$  (21)

C. Rank Relaxation of a Semidefinite Program

This semidefinite program is a reformulation of a quadratically constrained quadratic program (QCQP). Computationally speaking, QCQP problems are generally NP-hard. A close approximation to the true solution can be obtained in polynomial time if the rank-1 constraint on $X$, i.e., (22), is relaxed. A full explanation of semidefinite relaxation and discussion on its applicability can be found in [25]. This relaxation significantly increases the dimension of the SDP solver’s codomain. A notable is that dependent constraints that are linearly independent over $\mathbb{R}$ within $C_\psi$, such as sets $C_{16}$ and $C_{7:12}$, cease to be redundant when expressed as in (21). Hypothesis testing using extensive simulations confirmed with confidence above 95% that inclusion of quadratically dependent constraints improves the localization accuracy.

$^6$All matrices $M$, which can be expressed in the form of $M = v^\top v$, where $v$ is a row vector, are positive-semidefinite matrices.
The solution $X$ obtained through rank-relaxed SDP is typically not a rank-1 matrix when DOA measurements are noisy. However, the largest singular value of $X$ is generally multiple orders of magnitude greater than the second largest singular value. A rank-1 approximation to $X$, which we call $\hat{X}$, is obtained by evaluating the singular value decomposition of $X$, and then setting all singular values except the largest equal to zero. From $\hat{X}$, one can then use the definition of $X$ to obtain the approximation of $\Psi$, which we will call $\hat{\Psi}$. Entries $\hat{\psi}_i$ for $i = 10, 11, 12$ can be used immediately to construct an estimate for $t^{B_3}_{A_1}$, which we will call $\hat{t}$. Entries $\hat{\psi}_i$ for $i = 1, \ldots, 9$ will be used to construct an intermediate approximation of $R^{B_3}_{A_1}$, which we call $\hat{R}$, and which we will refine further.

D. Orthogonal Procrustes Algorithm

Due to the relaxation of the rank constraint (19) on $X$, it is no longer guaranteed that entries of $\hat{\Psi}$ strictly satisfy the set of constraints $C_\Psi$. Specifically, $\hat{R}$ may not be a rotation matrix. The orthogonal Procrustes algorithm is a commonly used tool to determine the closest orthogonal matrix (denoted $\overline{R}$) to a given matrix, $\hat{R}$. This is given by

$$\overline{R} = \arg\min_\Psi \|\Omega - \hat{R}\|_F,$$

subject to $\Omega\Omega^T = I$, where $\|\cdot\|_F$ is the Frobenius norm.

When noise is high, the above method occasionally returns $\hat{R}$ such that $\det(\hat{R}) = -1$. In this case, we employ a special case of the orthogonal Procrustes algorithm [26] to ensure that we obtain rotation matrices and avoid reflections by flipping the last column in one of the unitary matrix factors of the singular value decomposition.

The matrix $\overline{R}$ and vector $\overline{t}$ are the final estimates of $R^{B_3}_{A_1}$ and $t^{B_3}_{A_1}$ using SDP and the orthogonal Procrustes algorithm. The estimate of Agent B’s position in the global frame is

$$\overline{P}^{B_3}_{A_1} = \overline{R}(p^{B_3}_{A_1} - \overline{t}).$$

For convenience, we use SDP+O to refer to the process of solving a rank-relaxed semidefinite program and then applying the orthogonal Procrustes algorithm to the result.

E. Example of SDP+O Method With Noisy DOA Measurements

In this subsection, we apply the SDP+O method to perform localization in a noisy DOA measurement case using the real trajectory example from Section III. A popular practice for performing DOA measurements from Agent B toward Agent A is to use fixed RF antennas and/or optical sensors on board Agent B’s airframe. The horizontal RF antenna typically has a larger aperture (generally around four times, owing to the physical layout of a fixed-wing UAV) than the vertical RF antenna. As a result, errors in azimuth and elevation measurements, referenced to the body-fixed frame $B_4$, are modeled by independent zero-mean Gaussian distributed variables with different standard deviations, denoted $\sigma_\phi$ and $\sigma_\chi$.

Strictly speaking, physical sensors return azimuth and elevation measurements in the interval $[0^\circ, 360^\circ]$, which means that each noise is expected to follow a von Mises distribution, which generalizes a Gaussian distribution to a circle [27]. In our case, we approximate the von Mises distribution by a Gaussian distribution because noise is sufficiently small. In this example, we assume that body-fixed frame azimuth and elevation measurement errors have standard deviations of $\sigma_\phi = 0.5^\circ$ and $\sigma_\chi = 2^\circ$.

Samples of Gaussian error with these standard deviations were added to body-fixed frame ($B_3$) elevation and azimuth measurements calculated as described in Section III. These were converted to DOA measurements referenced to the INS frame $B_3$. The SDP+O algorithm was used to obtain $\overline{R}$ and $\overline{t}$ using the agents’ position coordinates in their respective navigation frames and the noisy DOA values. The reconstructed trajectory $\overline{P}^{B_3}_{A_1}$ is plotted in Fig. 2 with the dotted black line. Position data of the reconstructed trajectory $\overline{P}^{B_3}_{A_1}$ are tabulated in [22].

REMARK 1 The accuracy of the SDP+O solution in the noiseless case was observed to deteriorate when entries in the true translation vector ($t_i$ for $i = 1, 2, 3$) are large. This is due to a form of inherent regularization in the SDP solver Yalmip [28]. When the approximate magnitude of the norm $\|t^{B_3}_{A_1}\|$ is known, one approach is to introduce a scaling coefficient before entries $t_i$ for $i = 1, 2, 3$ in (9) and (10) equal to the approximate norm of $\|t^{B_3}_{A_1}\|$.

In [22], we discuss a controlled shifting algorithm, which may be applied if an approximation of $t^{B_3}_{A_1}$ is known a priori.

F. Unsuitable Trajectories for DOA Localization

In this subsection, we are motivated to identify trajectories of Agents A and B which may lead to multiple solutions for $\overline{R}$ and $\overline{t}$ in the noiseless case and consequently unreliable solutions in the noisy case. We discuss three scenarios.

1) Agent A’s motion is planar.
2) The agents’ trajectories produce equal DOA measurements with respect to Agent B’s INS frame.
3) Agent A’s trajectory is a point or straight line.

The first scenario is an example of conditional unsuitability. When Agent A’s motion is planar, matrix $A$ in (11) is rank deficient, and a unique solution cannot be obtained by solving (11). In contrast, by introducing quadratic constraints through SDP, the correct solution is obtained.

The second and third scenarios are examples where the inability to discern a unique solution is not because of an algorithmic deficiency, but rather because there is geometric ambiguity arising from unsuitable trajectories. These scenarios are ill-posed using DOA measurements alone.

In the second scenario, DOA measurements expressed with respect to the local INS frame $B_3$ are equal at each time instant. This may occur in the near-field case, as illustrated in Fig. 3, or in the far-field case, where the distance between Agents A and B is sufficiently large that DOA measurements become approximately equal despite each agent’s trajectory remaining arbitrary. In the near-field case, range sensing technology (if available) can be used to replace or supplement directional measurements, in which

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Authorized licensed use limited to: CURTIN UNIVERSITY. Downloaded on June 10,2020 at 00:57:35 UTC from IEEE Xplore. Restrictions apply.
A. Likelihood Function Derivation

In this section, DOA values are always expressed with respect to the body-fixed frame of Agent B ($B_k$) to exploit the independence of azimuth and elevation measurement errors. This is a change from Sections III and IV, in which DOA measurements were generally expressed with respect to the local INS frame $B_2$. The transformation between coordinate frames $B_2$ and $B_4$ is known to Agent B.

Suppose that body-fixed frame measurements of azimuth and elevation $\Theta(k)$ and $\Phi(k)$ are contaminated by zero mean Gaussian noise as follows:

1) $\widetilde{\Theta}(k) = \Theta(k) + \xi_\Theta$, $\xi_\Theta \sim N(0, \sigma_\Theta^2)$.
2) $\widetilde{\Phi}(k) = \Phi(k) + \xi_\Phi$, $\xi_\Phi \sim N(0, \sigma_\Phi^2)$.

To calculate noiseless azimuth and elevation measurements, an expression must be derived for the position of Agent A in Agent B’s body-fixed frame $B_k$. Observe that

$$p_{B_k}^A(k) = R_{B_2B_k}(k) (R_{A_1A}(k) + t_{A_1}) + t_{B_k}(k). \quad (22)$$

To help distinguish coordinate reconstructions based on estimates of $\vec{R}$ and $\vec{t}$ from true coordinates, reconstructed positions will be explicitly expressed as functions of $\vec{R}$ and $\vec{t}$:

$$p_{B_k}^A(k, \vec{R}, \vec{t}) = R_{B_2B_k}(k) (R_{A_1A}(k) + \vec{t}) + t_{B_k}(k). \quad (23)$$

By definition of azimuth and elevation in Section II, we have

$$\theta_{B_k}(k, \vec{R}, \vec{t}) = \arcsin \left( \frac{p_{B_k}^A(k, \vec{R}, \vec{t})_z}{||p_{B_k}^A(k, \vec{R}, \vec{t})||} \right) \quad (24)$$

$$\phi_{B_k}(k, \vec{R}, \vec{t}) = \arctan \left( \frac{p_{B_k}^A(k, \vec{R}, \vec{t})_y}{p_{B_k}^A(k, \vec{R}, \vec{t})_x} \right) \quad (25)$$

where $p_{B_k}^A = [p_{B_k}^A]_x, p_{B_k}^A]_y, p_{B_k}^A]_z^\top$. The likelihood function for the set of DOA measurements is defined as follows:

$$L(p_{B_k}^A, p_{B_k}^B | \vec{R}, \vec{t}) = \prod_{k=1}^K \exp \left[ -\frac{(\theta_{B_k}(k) - \theta_{B_k}(k, \vec{R}, \vec{t}))^2}{2\sigma_\theta^2} \right] \times \prod_{k=1}^K \exp \left[ -\frac{(\phi_{B_k}(k) - \phi_{B_k}(k, \vec{R}, \vec{t}))^2}{2\sigma_\phi^2} \right]. \quad (26)$$

It can be shown that maximizing $L(p_{B_k}^A, p_{B_k}^B | \vec{R}, \vec{t})$ is equivalent to minimizing

$$\sum_{k=1}^K \left[ \frac{(\theta_{B_k}(k) - \theta_{B_k}(k, \vec{R}, \vec{t}))^2}{2\sigma_\theta^2} + \frac{(\phi_{B_k}(k) - \phi_{B_k}(k, \vec{R}, \vec{t}))^2}{2\sigma_\phi^2} \right]. \quad (27)$$

B. Optimization Using Gradient Descent

Possible parameterizations for the rotation matrix $\vec{R}$ include Euler angles, quaternion representation, and Rodrigues rotation formula. In this paper, we parameterize
by a 3-vector of Euler angles, and is presented in Fig. 2 as the SO\textsubscript{R}t\textsuperscript{8} is a 3-vector. This \textit{R} and \textit{R}7 and \((\textit{R}_B \sim \text{U}(t = \sigma))\), and \(\sigma\) is known, the error of rotation was generated by sampling \([0.1\alpha, \beta, \gamma^2\) − \(\textit{R}\)]. The position error is defined as the average Euclidean distance between true global coordinates of Agent B and estimated global coordinates over the \(K\) measurements taken, divided (to secure normalization) by the average distance between aircraft.

\[
\text{error}(p_{B1}^{k}) = \frac{\sum_k ||p_{B1}^{k} - p_{A1}^{k}(k)||}{Kd}
\]

where \(p_{B1}^{k} = \text{R}^\top(p_{B1}^{k} - \bar{t})\), and \(d\) represents the average distance between aircraft.

B. Monte Carlo Simulations Using SDP+O and ML

In this subsection, we summarize the results of Monte Carlo simulations to evaluate the expected performance of the SDP+O method and the SDP+O+ML method.

Pairs of realistic trajectories for Agents A and B are generated in accordance with a series of assumptions related to real flight dynamics listed in the extended version of this paper [22]. To represent the drift in the INS of Agent B, rotations \(\text{B}_A\) were generated by independently sampling three Euler angles \(\alpha, \beta, \gamma\), where \(\alpha, \beta, \gamma \sim U(-\pi, \pi)\), and translations \(t_{B1} = [t_1, t_2, t_3]^\top\) were generated by sampling entries \(t_1, t_2, t_3 \sim U(-600, 600)\).

As discussed in Subsection IV-E, we assume that the standard deviations of measurement error in the body-fixed frame \(B_4\) satisfy \(\sigma_0 = 4\sigma_\phi\). We vary the DOA error by \(\sigma_0 \in [0.1^\circ, 1^\circ, 2^\circ]\). Errors in the order of \(\sigma_\phi = 0.1^\circ\) are representative of an optical sensor, whereas the larger errors are representative of antenna-based (RF) measurements.

VI. SIMULATION RESULTS

In this section, we use extensive simulations of realistic flight trajectories to evaluate the effects of errors in body-fixed frame azimuth and elevation measurements.

In the preliminary conference paper [21], the LS+O method was found to collapse when small amounts of noise were introduced to DOA measurements, whereas the rotation error increased linearly with respect to DOA measurement noise when using the SDP+O method. The SDP+O method is the superior method, and there is no reason to employ LS+O.

A. Metrics for Error in \(\overrightarrow{R}\) and \(\bar{t}\)

This paper uses the geodesic metric for rotation [30]. All sequences of rotations in three dimensions can be expressed as one rotation about a single axis [31]. The geodesic metric on \(SO(3)\) defined by

\[
d(R_1, R_2) = \arccos\left(\frac{\text{tr}(R_1^\top R_2) - 1}{2}\right)
\]

is the magnitude of angle of rotation about this axis [32]. Where \(R_{A1}^k\) is known, the error of rotation \(\overrightarrow{R}\) is defined as \(d(\overrightarrow{R}, R_{A1}^k)\). The position error is defined as the average Euclidean distance between true global coordinates of Agent B and estimated global coordinates over the \(K\) measurements taken, divided (to secure normalization) by the average distance between aircraft.

\[
\text{error}(p_{B1}^{k}) = \frac{\sum_k ||p_{B1}^{k} - p_{A1}^{k}(k)||}{Kd}
\]

where \(p_{B1}^{k} = \overrightarrow{R}^\top(p_{B1}^{k} - \bar{t})\), and \(d\) represents the average distance between aircraft.

C. Example of Maximum Likelihood Refinement of SDP+O Solution

In this subsection, we demonstrate the benefits of MLE. Maximum likelihood (ML) was performed using the real flight trajectory data presented in Section III. The resulting reconstructed trajectory \(p_{A1}^k\) is presented in Fig. 2 as the solid black line, and its coordinates are tabulated in [22].

Additionally, in this subsection, we present the decrease and convergence in the value of frame rotation error and reconstructed position error\(^7\) over successive iterations of the gradient descent algorithm in Figs. 5(a) and (b).

The error in INS frame rotation is reduced by over 60%, and the reconstructed position error of Agent B is reduced by over 70% by iterating the gradient descent algorithm. This represents a significant gain with respect to the SDP+O estimate, which served as the initialization point of the gradient descent. Monte Carlo simulations covering a large set of trajectories are presented in Section VI.
In this section, we relax the condition preventing GPS-denied agents from broadcasting signals - a trilateral approach may be more resilient to DOA measurement error and/or unsuitable trajectories and may perhaps require fewer DOA measurements from each aircraft than simply repeating the two-agent localization algorithm with each GPS-denied agent. We introduce a GPS-denied Agent C, whose local INS frame has rotation and translation parameters \( R_{B2} \) and \( I_{A1} \) with respect to the global frame. We conclude this section by discussing the challenges involved in generalizing our findings to arbitrary \( n \)-agent networks.

### A. Measurement Process in the Three-Agent Network

To describe measurements within a network of more than two agents, one minor notation change is required: DOA measurements made by Agent I toward Agent J will henceforth be expressed in the INS coordinate frame of Agent I as \((\theta^I_J, \phi^I_J)\). At each time instant \( k \) in the discrete-time process, we have the following.

1. Agents A and B interact as per the two-agent case.
2. Agent C receives the broadcast of Agent A’s global coordinates and measures this signal’s DOA with respect to frame \(C_2\), which we denote \((\theta^C_A, \phi^C_A)\).
3. Agent C broadcasts its position with respect to its INS frame \( p^C_2 \), as well as the measurement \((\theta^C_A, \phi^C_A)\) to Agent B, who also takes a DOA measurement toward Agent C. This measurement is denoted \((\theta^B_C, \phi^B_C)\).

All DOA and position measurements are relayed to Agent B, who performs the localization algorithm discussed in the following.

### B. Forming System of Linear Equations in the Three-Agent Network

In Section III, the LS \( A\Psi = b \) was formed using relations stemming from the collinearity of the vector \((p^A_B - p^B_B)\) and the vector in the direction of DOA measurement \((\theta^B_A, \phi^B_A)\). We refer to this system of equations as \( S_{AB} \), where the subscript references the agents involved. A similar system \( S_{AC} \) can be constructed independently using Agent C’s DOA measurements toward Agent A and \( p^C_2 \).

In the three-agent network, Agent B also measures the DOA toward Agent C’s broadcast, with respect to Agent B’s local INS frame \(B_2\). To exploit the collinearity of the vectorial representation of the DOA measurement \((\theta^C_B, \phi^C_B)\) and \((p^B_2 - p^B_B)\), an expression for the position coordi-
nate vector $p_C^B$ is required. As achieved in (7) and (8) in Section III, this position may be expressed in terms of entries of $R_{BC}^{B_i}$ and $t_{BC}^{B_i}$, and the LS $S_{BC}$ may be defined similarly to $S_{AB}$ in Section III. Systems $S_{AB}, S_{AC}$, and $S_{BC}$ can be assembled, forming a large system of linear equations $S_{ABC}$ with 36 scalar measurements and three translation vector entries per agent pair.

At each time instant $k$ for $k = 1, \ldots, K$, two linear equations are obtained from each DOA measurement of $(\theta_{BC}^A, \phi_{BC}^A), (\theta_{CA}^A, \phi_{CA}^A), (\theta_{AB}^A, \phi_{AB}^A)$. As a result, six linear equations are obtained at each time instant. Performing the measurement process six times ($K = 6$) produces 36 linear equations. Generically, in the noiseless case, a unique solution exists for $K = 6$ time instants. When using only the LS method, the three-agent localization problem requires the same minimum number of time instants as solving two independent two-agent localization problems concurrently, and yet requires more DOA measurements than the sum of the number of measurements required in two separate two-agent localization problems. However, quadratic relationships among $R_{BC}^{B_i}, t_{BC}^{B_i}, R_{CA}^{C_i}, t_{CA}^{C_i}, R_{AB}^{A_i},$ and $t_{AB}^{A_i}$ significantly reduce the required number of time instants ($K$) at which measurements occur.

C. Quadratic Constraints in Three-Agent Network and Example

It is possible, using the rotational and translational relationships between the three frames, to obtain a total of 99 linearly independent quadratic constraints for a system of 36 unknown variables. Exact details are given in [22] for the interested reader, but omitted here for spatial considerations.

Rank-relaxed SDP can be used to obtain solutions for each INS frame’s rotation and translation with respect to the global frame, and the orthogonal Procrustes algorithm can be applied to each individual resulting rotation matrix. This defines the three-agent SDP+O method.

To illustrate successful localization in the three-agent case, realistic trajectories were defined for Agents A, B, and C for $K = 3$ time instants. These are presented in Fig. 8. Only Agents B and C were assigned random INS frame rotations and translations, as prescribed in Section VI, and the three-agent SDP+O method was used to obtain estimates of $R_{BC}^{B_i}, t_{BC}^{B_i}, R_{CA}^{C_i}, t_{CA}^{C_i},$ and $t_{AB}^{A_i}$. Each directional measurement consists of two scalar measurements, and hence, a total of $3 \times 2 \times K = 18$ scalar measurements were obtained. Localization was successful, which demonstrates that only three time instants ($K = 3$) are required for the three-agent SDP+O algorithm to obtain the exact solution in the noiseless case. Earlier, it was established that a minimum of six time instants were required to achieve a unique solution in the three-agent case using LS+O, and a minimum of four time instants were required to achieve a unique solution in the two-agent case using SDP+O. We have, therefore, demonstrated that a trilateral algorithm can achieve localization of two GPS-denied agents in fewer measurement time instants than applying the bilateral algorithm twice independently. We note that this extension to three agents is not applicable if the measurement graph is a tree because measurements are required between each pair of agents within the three-agent network.

D. Challenges in Extension to n-Agent Networks

Advancing to arbitrary $n$-agent networks requires results on bearing rigidity of a graph. Though results exist when all agents share the same reference frame [35]–[37], there is no such result when, as in our problem, agents have different reference frames. We note that algebraic conditions for 3-D bearing localizability based on the rank of generalized versions of the rigidity matrix have recently been identified in [25] and [23]. There is also the risk of an explosion in computational complexity due to a potentially exponential increase in the number of variables (entries of rotation matrices and translation vectors) that need to be determined. Further discussion can be found in [22].

VIII. CONCLUSION

This paper studied a cooperative localization problem between a GPS-denied and a GPS-enabled UAV. A localization algorithm was developed in two stages. We showed that an LS of equations built from six or more measurements yielded the localization solution for generic trajectories. The second stage considered the inclusion of quadratic constraints due to rotation matrix constraints. Rank-relaxed SDP was used, and the solution adjusted using the orthogonal Procrustes algorithm. This gave the algorithm greater resilience to noisy measurements and unsuitable trajectories. MLE was then used to improve the results of the algorithm. Simulations were presented to illustrate the performance of the algorithm. Finally, an approach was outlined to extend the two-agent solution to a three-agent network, in which only one agent has global localization capacity. Future work may include implementation on aircraft to perform localization in real time and validate our Monte Carlo analysis.
on measurement noise. Agent localization using distance measurements will be investigated in future work to compare the advantages and disadvantages of different sensing technologies for the same problem (see also [38]). We also hope to extend our trilateral algorithm to larger networks by establishing further theory on bearing rigidity when agents do not share a common reference frame.

REFERENCES


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