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A Distributed Algorithm with Scalar States for Solving Linear Equations

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Abstract— **Based on a combination of consensus and conservation, the paper develops a distributed update for solving linear equations by multi-agent networks, in which each agent only knows just a small part of the overall equation and can only communicate with its nearby neighbors. In the proposed distributed update, each agent knows only two scalar entries of the defining matrix of the overall equation and controls just two scalar states. Given the underlying networks to be connected and undirected, the proposed distributed update enables agents to collaboratively achieve a solution to the overall equation. Analytical proof is provided for the exponential convergence of the proposed update, which is also validated by numerical simulations.**

I. Introduction

The idea of consensus, which enables states of all agents in the network to reach an agreement regarding a certain quantity of interest through local coordination [1]–[4], has played a significant role in many distributed algorithms for computations and optimizations [5]–[11]. Along this direction, consensus-based distributed algorithms for solving linear equations have recently been developed [12]–[16], in which each agent updates its state to be a solution to part of the overall linear equation while reaching consensus with all its nearby neighbors. Nice as these distributed linear equation solvers are, they all inherently require each agent to know at least one complete row of the overall equation and control a state of size equal to the number of unknowns. (i.e. for an equation $Ax = b$, agent i knowns the i-th row of A, the i-th row of b and controls a state of dimension equal to that of x.) This means there is limited application of these consensus-based distributed equation solvers in solving large-scale equation with millions of unknowns which often arise from light scattering calculations [17], electromagnetism computations [18], computational fluid dynamics [19], boundary control of PDEs [20], large-scale linear regression [21], and so on. Recognition of this difficulty has motivated us to develop distributed algorithms for addressing the conflict between agents' limited capability of storage/computing and the practical needs for solving large-scale equations.

The aim of this paper is to develop a distributed update for solving linear equations in multi-agent networks, in which each agent only needs to know one **scalar entry** of the overall matrix associated with the linear equations, one **scalar entry** of the known vector part of the equation and controls two **scalar states**. The proposed update allows agents to cooperatively solve the overall linear equation by a combination of consensus and conservation, in which states of those agents corresponding to the same column of the equation, reach a consensus while agents corresponding to the same row of the equation obeys a conservation requirement. Given the underlying networks to be undirected and connected, exponential convergence of the proposed algorithm is guaranteed, which is also validated by simulations.

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Notation: Throughout this paper, we let $\mathbf{1}_r$ denote a vector in $\mathbb{R}^r$ with all its components equal to 1; let I_r denote the $r \times r$ identity matrix. Let M', $\ker M$ and $\operatorname{image} M$ denote the conjugate transpose, the kernel and the image of a matrix M, respectively. For $i = 1, 2, \cdots, r$, let $\operatorname{col}\{A_1, A_2, \cdots, A_r\}$ denote a column stack of matrices A_i, which is $\begin{bmatrix} A_1' & A_2' & \cdots & A_r' \end{bmatrix}'$, let $\operatorname{diag}\{A_1, A_2, \cdots, A_r\}$ denote the block diagonal matrix with A_i the ith diagonal block entry. Let $\otimes$ denote the Kronecker product. Let $\mathbf{n} = \{1, 2, ..., n\}$ and $\mathbf{m} = \{1, 2, ..., m\}$.

II. Problem formulation

In a network of mn agents, each agent ij, $i \in \mathbf{m}$ and $j \in \mathbf{n}$, is able to communicate with certain other agents called its *neighbors*. The neighbor relation here is assumed to be symmetric. A neighbor of agent ij is called its *row neighbor* if it is labeled as $i\bar{j}$ with $\bar{j} \in \mathbf{n}$, and is called its *column neighbor* if it is labeled as $\bar{i}j$ with $\bar{i} \in \mathbf{m}$. Let $\mathcal{N}_{ij}^R$ and $\mathcal{N}_{ij}^C$ denote the set of agent ij's row neighbors and column neighbors, respectively. Let $\mathbb{G}_i^R$, $i \in \mathbf{m}$ denote an n-node undirected graph consisting of nodes $i1, i2, ..., in$, in which there is an undirected edge connecting ij and $i\bar{j}$ if and only if ij and $i\bar{j}$ are row neighbors. Let $\mathbb{G}_j^C$, $j \in \mathbf{n}$ denote an m-node undirected graph consisting of nodes $1j, 2j, ..., mj$, in which there is an undirected edge connecting ij and $\bar{i}j$ if and only if ij and $\bar{i}j$ are column neighbors. One example of such an multi-agent network with $m = 3$ and $n = 4$ is shown in Fig. 1.

In this paper, we aim to employ the multi-agent network described above to cooperatively solve a linear equation

$$A\boldsymbol{x} = b, \qquad A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m \tag{1}$$

which is known to have a priori solution. Suppose each agent ij, $i \in \mathbf{m}$ and $j \in \mathbf{n}$, knows $a_{ij} \in \mathbb{R}$, which is the ijth entry

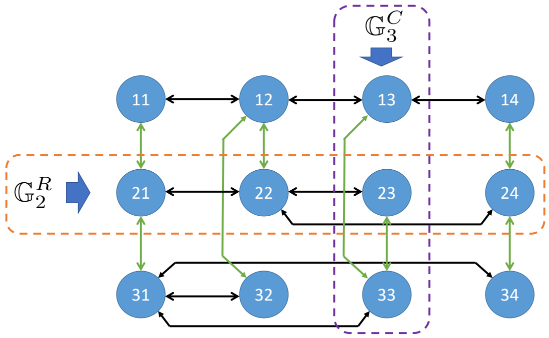

Fig. 1. An Example of a Multi-Agent Network with $m = 3$ and $n = 4$

of A, and $b_{ij} \in \mathbb{R}$, where

$$\sum_{j=1}^{n} b_{ij} = b_i \tag{2}$$

with b_i the ith entry of b, $i \in \mathbf{m}$. Note that b_i are given, but the b_{ij} are not, they can be any vectors that satisfy (2) (a particular case is that one b_{ij} equals b_i and the others are zero). Let each agent ij control a scalar state $x_{ij}(t) \in \mathbb{R}$. The **problem** of interest is to develop a distributed update to drive each $x_{ij}(t)$ to converge exponentially fast to constant scalars x_{ij}^* such that

$$\textbf{Row Conservation: } \sum_{j=1}^{n}(a_{ij}x_{ij}^* - b_{ij}) = 0, \;\; i \in \mathbf{m} \tag{3}$$

and

$$\textbf{Column Consensus: } x_{1j}^* = x_{2j}^* = \cdots = x_{mj}^* = y_j, \;\; j \in \mathbf{n}. \tag{4}$$

From (2), (3) and (4), one has $\sum_{j=1}^{n} a_{ij}y_j = b_i, \;\; i \in \mathbf{m}$. It follows that $Ay = b$, where $y = \text{col}\,\{y_1, y_2, ..., y_n\}$. Thus the constants x_{ij}^* are said to *collaboratively form a solution* to $Ax = b$ if and only if they satisfy (3) and (4).

III. A Distributed Update

In order to develop a distributed update for all x_{ij} to reach both row conservation in (3) and the column consensus in (4), we introduce one additional scalar state $z_{ij} \in \mathbb{R}$ for each agent ij. The update proposed for each agent ij, $i \in \mathbf{m}$ and $j \in \mathbf{n}$ is as follows:

$$\begin{aligned} \dot{x}_{ij} =& - a_{ij}\left(a_{ij}x_{ij} - b_{ij} - \sum_{ik\in\mathcal{N}_{ij}^R}(z_{ij} - z_{ik})\right) \\ &- \sum_{kj\in\mathcal{N}_{ij}^C}(x_{ij} - x_{kj}) \end{aligned} \tag{5}$$

$$\dot{z}_{ij} = a_{ij}x_{ij} - b_{ij} - \sum_{ik\in\mathcal{N}_{ij}^R}(z_{ij} - z_{ik}) \tag{6}$$

Here, the first line of update (5) and (6) form a flow that achieves the row-conservation (3) while the second line of update (5) is for achieving the column consensus in (4). It is worthy noting that the updates (5)-(6) are **distributed** in the sense that each agent ij only uses information from its row and column neighbors. Different from existing consensus-based distributed linear equation solvers, the states controlled by each agent in the updates (5)-(6) are **two scalar states**.

IV. Analysis and Main Result

Before proceeding on, we let

$$\boldsymbol{x}_i(t) = \text{col}\,\{x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\} \in \mathbb{R}^n \tag{7}$$

$$\boldsymbol{z}_i(t) = \text{col}\,\{z_{i1}(t), z_{i2}(t), \cdots, z_{in}(t)\} \in \mathbb{R}^n \tag{8}$$

$$\bar{A}_i = \text{diag}\,\{a_{i1}, \cdots, a_{in}\}, \quad \bar{b}_i = \text{col}\,\{b_{i1}, \cdots, b_{in}\} \tag{9}$$

and $L_{\mathbb{G}_i^R} \in \mathbb{R}^{n\times n}$ denoting the Laplacian matrix of $\mathbb{G}_i^R$. Then equations (5) and (6) for $i \in \mathbf{m}$ can be written in the following compact form:

$$\dot{\boldsymbol{x}}_i = - \bar{A}_i'\left(\bar{A}_i\boldsymbol{x}_i - \bar{b}_i - L_{\mathbb{G}_i^R}\boldsymbol{z}_i\right) - \boldsymbol{\psi}_i \tag{10}$$

$$\dot{\boldsymbol{z}}_i = \bar{A}_i\boldsymbol{x}_i - \bar{b}_i - L_{\mathbb{G}_i^R}\boldsymbol{z}_i \tag{11}$$

where $\boldsymbol{\psi}_i = \text{col}\,\{\psi_{i1}, \psi_{i2}, ..., \psi_{in}\}$ with

$$\psi_{ij}(t) = \sum_{kj\in\mathcal{N}_{ij}^C}(x_{ij} - x_{kj}), \;\; j \in \mathbf{n}, i \in \mathbf{m}. \tag{12}$$

From (12), one has

$$\begin{bmatrix}\psi_{1j}\\ \psi_{2j}\\ \vdots \\ \psi_{mj}\end{bmatrix} = L_{\mathbb{G}_j^C}\begin{bmatrix}x_{1j}\\ x_{2j}\\ \vdots \\ x_{mj}\end{bmatrix}, \quad j \in \mathbf{n}. \tag{13}$$

where $L_{\mathbb{G}_i^C} \in \mathbb{R}^{m\times m}$ is the Laplacian matrix of $\mathbb{G}_i^C$. Let

$$\boldsymbol{x}(t) = \text{col}\,\{\boldsymbol{x}_1(t), \cdots, \boldsymbol{x}_m(t)\} \in \mathbb{R}^{mn} \tag{14}$$

$$\boldsymbol{\psi}(t) = \text{col}\,\{\boldsymbol{\psi}_1(t), \cdots, \boldsymbol{\psi}_m(t)\} \in \mathbb{R}^{mn} \tag{15}$$

Let $P \in \mathbb{R}^{mn\times mn}$ denote a row permutation matrix with $P'P = I_{mn}$ such that

$$P\boldsymbol{x} = \text{col}\,\{x_{11}, x_{21}, ..., x_{m1}, ..., x_{1n}, x_{2n}, ..., x_{mn}\}. \tag{16}$$

Then by (13), one has

$$P\boldsymbol{\psi} = \hat{L}_C(P\boldsymbol{x}) \tag{17}$$

where

$$\hat{L}_C = \text{diag}\,\{L_{\mathbb{G}_1^C}, \cdots, L_{\mathbb{G}_n^C}\}. \tag{18}$$

Thus one has

$$\boldsymbol{\psi} = P'\hat{L}_C P\boldsymbol{x} \tag{19}$$

We further let

$$\boldsymbol{z}(t) = \text{col}\,\{\boldsymbol{z}_1(t), \cdots, \boldsymbol{z}_m(t)\} \in \mathbb{R}^{mn} \tag{20}$$

and

$$\hat{A} = \text{diag}\,\{\bar{A}_1, \cdots, \bar{A}_m\}, \quad \hat{b} = \text{col}\,\{\bar{b}_1, \cdots, \bar{b}_m\}, \tag{21}$$

$$\hat{L}_R = \text{diag}\,\{L_{\mathbb{G}_1^R}, \cdots, L_{\mathbb{G}_m^R}\} \tag{22}$$

Then (10)-(11) can be written as

$$\dot{\boldsymbol{x}} = - \hat{A}'\left(\hat{A}\boldsymbol{x} - \hat{b} - \hat{L}_R\boldsymbol{z}\right) - P'\hat{L}_C Px \tag{23}$$

$$\dot{\boldsymbol{z}} = \hat{A}\boldsymbol{x} - \hat{b} - \hat{L}_R\boldsymbol{z}. \tag{24}$$

which is equivalent to

$$\dot{\boldsymbol{\xi}} = Q\boldsymbol{\xi} + h \tag{25}$$

with

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}, \quad Q = \begin{bmatrix} -\hat{A}'\hat{A} - P'\hat{L}_C P & \hat{A}'\hat{L}_R \\ \hat{A} & -\hat{L}_R \end{bmatrix}, \quad h = \begin{bmatrix} \hat{A}'\hat{b} \\ -\hat{b} \end{bmatrix}. \tag{26}$$

To analyze the convergence of (25), we propose a lemma to characterize eigenvalues of Q.

Lemma 1: Let

$$M = \begin{bmatrix} -M_1'M_1 - M_2 & M_1'M_3 \\ \\ M_1 & -M_3 \end{bmatrix}$$

where the M_i are real, $i = 1, 2, 3$, and M_2 and M_3 are positive semi-definite. Then all eigenvalues of M are real negative or 0. Moreover, if 0 is an eigenvalue of M, it must be non-defective[1].

The proof of Lemma 1 will be given in the Appendix. By this lemma and by establishing the convergence of the linear time-invariant system (25) to a constant steady state, one has the following main result.

Theorem 1: Suppose $Ax = b$ has at least one solution and all $\mathbb{G}_i^R$, $i \in \mathbf{m}$ and $\mathbb{G}_j^C$, $j \in \mathbf{n}$ are connected and undirected. Then under the distributed updates (5)-(6), all $x_{ij}(t)$ converge exponentially fast to constant vectors x_{ij}^*, which collaboratively form a solution to $Ax = b$ by satisfying(3)-(4).

Proof of Theorem 1: We first prove that there exists a constant vector $\hat{\boldsymbol{\xi}} = \text{col}\,\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}\}$ which is an equilibrium of (25). Recall that there exists a constant vector $y \in \mathbb{R}^n$ such that $Ay = b$. Then

$$\sum_{j=1}^{n} (a_{ij}y_j - b_{ij}) = 0, \quad i \in \mathbf{m} \tag{27}$$

with y_j the jth entry of y. This along with the definition of $\bar{A}_i$, $\bar{b}_i$ in (9) yields

$$\mathbf{1}_n'(\bar{A}_i y - \bar{b}_i) = 0, \quad i \in \mathbf{m}. \tag{28}$$

By the property of Laplacian matrices $L_{\mathbb{G}_i^R}$ of connected graphs, one has

$$\text{image } L_{\mathbb{G}_i^R} = \ker \mathbf{1}_n' \tag{29}$$

It follows that

$$(\bar{A}_i y - \bar{b}_i) \in \text{image } L_{\mathbb{G}_i^R}, \quad i \in \mathbf{m} \tag{30}$$

and therefore, there exists a constant vector $\hat{z}_i \in \mathbb{R}^n$ such that

$$\bar{A}_i y - \bar{b}_i - L_{\mathbb{G}_i^R}\hat{z}_i = 0, \quad i \in \mathbf{m} \tag{31}$$

[1]An eigenvalue is non-defective if and only if its algebraic multiplicity equals its geometric multiplicity. In other words, the Jordan block corresponding to a non-defective eigenvalue is diagonal.

from which, the definitions of $\hat{A}$, $\hat{b}$ in (21), one has

$$\hat{A}\hat{\boldsymbol{x}} - \hat{b} - \hat{L}_R\hat{\boldsymbol{z}} = 0 \tag{32}$$

with

$$\hat{\boldsymbol{x}} = \mathbf{1}_m \otimes y, \quad \hat{\boldsymbol{z}} = \{\hat{z}_1, \cdots, \hat{z}_m\}.$$

Recall that $P \in \mathbb{R}^{mn \times mn}$ is a row permutation matrix defined in (16). Then

$$P\hat{x} = \text{col}\,\{y_1\mathbf{1}_m, y_2\mathbf{1}_m, ..., y_n\mathbf{1}_m\}$$

This equation and the definition of $\hat{L}_C$ in (18) imply

$$\hat{L}_C P\hat{\boldsymbol{x}} = \text{diag}\,\{y_1 L_{\mathbb{G}_1^C}\mathbf{1}_m, ..., y_n L_{\mathbb{G}_n^C}\mathbf{1}_m\} = 0 \tag{33}$$

From (32) and (33), one has $\hat{\boldsymbol{\xi}} = \text{col}\,\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}\}$ is an equilibrium of (25), that is

$$Q\hat{\boldsymbol{\xi}} + h = 0 \tag{34}$$

Second, we analyze the convergence of the error

$$\boldsymbol{e}(t) = \boldsymbol{\xi}(t) - \hat{\boldsymbol{\xi}}. \tag{35}$$

From (25) and (34), one has

$$\dot{\boldsymbol{e}} = Q\boldsymbol{e} \tag{36}$$

By noting the structure of Q in (26) and the fact that the Laplacian matrices $\hat{L}_C$ and $\hat{L}_R$ are symmetric and positive semi-definite, one by Lemma 1 concludes that all eigenvalues of Q are real negative or 0. Moreover, if 0 is an eigenvalue of Q, it must be non-defective. Thus there exists a constant vector $v \in \ker Q$ such that $\boldsymbol{e}(t)$ of the linear time-invariant system (36) converges to v exponentially fast [22]. Thus $\hat{\boldsymbol{\xi}} = \text{col}\,\{\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{z}}(t)\}$ converges exponentially fast to a constant vector $\hat{\boldsymbol{\xi}}^* = \text{col}\,\{\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{z}}^*\}$, where

$$\begin{bmatrix} \hat{\boldsymbol{x}}^* \\ \hat{\boldsymbol{z}}^* \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{z}} \end{bmatrix} + v, \quad v \in \ker Q. \tag{37}$$

Partition the constant vector $\hat{\boldsymbol{x}}^*$ such that

$$\hat{\boldsymbol{x}}^* = \text{col}\,\{\boldsymbol{x}_1^*, \cdots, \boldsymbol{x}_m^*\} \tag{38}$$

where

$$\boldsymbol{x}_i^* = \text{col}\,\{x_{i1}^*, x_{i2}^*, ..., x_{in}^*\} \in \mathbb{R}^n \tag{39}$$

with $x_{ij}^* \in \mathbb{R}$. Evidently, we have that $x_{ij}(t)$ converges to x_{ij}^* exponentially fast. Thus in the following one only needs to show that all these x_{ij}^* satisfy equations (3)-(4).

From (34) and (37), one has $\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{z}}^*$ is an equilibrium of the system (23)-(24). Then

$$\hat{A}\hat{\boldsymbol{x}}^* - \hat{b} - \hat{L}_R\hat{\boldsymbol{z}}^* = 0 \tag{40}$$

$$P'\hat{L}_C P\hat{\boldsymbol{x}}^* = 0 \tag{41}$$

Let $\hat{\boldsymbol{z}}^* = \text{col}\,\{\boldsymbol{z}_1^*, \boldsymbol{z}_2^*, ..., \boldsymbol{z}_m^*\}$ with $\boldsymbol{z}_i^* \in \mathbb{R}^n$. From (40) and the definitions of $\hat{A}, \hat{b}, \hat{L}$ in (21)-(22), one has

$$\bar{A}_i\boldsymbol{x}_i^* - \bar{b}_i - L_{\mathbb{G}_i^R}\boldsymbol{z}_i^* = 0, \quad i \in \mathbf{m}. \tag{42}$$

This along with the definitions of $\bar{A}_i, \bar{b}_i$ in (9) leads to

$$\begin{bmatrix} a_{i1}x_{i1}^* \\ a_{i2}x_{i2}^* \\ \vdots \\ a_{in}x_{in}^* \end{bmatrix} - \begin{bmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{in} \end{bmatrix} - L_{\mathbb{G}_i^R}\boldsymbol{z}_i^* = 0, \quad i \in \mathbf{m}. \tag{43}$$

Pre-multiplying by $\mathbf{1}_n'$ on both sides of (43) yields

$$\sum_{j=1}^{n}(a_{ij}x_{ij}^* - b_{ij}) - \mathbf{1}_n' L_{\mathbb{G}_i^R}\boldsymbol{z}_i^* = 0, \quad i \in \mathbf{m}.$$

This and $\mathbf{1}_n' L_{\mathbb{G}_i^R} = 0$ imply

$$\sum_{j=1}^{n}(a_{ij}x_{ij}^* - b_{ij}) = 0. \tag{44}$$

which is the row conservation in (3).

From equation (41), the definition of $\hat{L}_C$ and P, one has

$$L_{\mathbb{G}_j^C}\begin{bmatrix} x_{1j}^* \\ x_{2j}^* \\ \vdots \\ x_{mj}^* \end{bmatrix} = 0, \quad j \in \mathbf{n} \tag{45}$$

which implies the column consensus in (4) that $x_{1j}^* = x_{2j}^* = \cdots = x_{mj}^*,\ j \in \mathbf{n}$. This completes the proof. ■

V. Simulation

We utilize the 12-agent network in Fig. 1 to solve the linear equation $A\boldsymbol{x} = b$, where

$$A = \begin{bmatrix} 3 & -2 & 1 & 5 \\ 2 & 3 & 2 & 0 \\ -2 & 2 & 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} -6 \\ 15 \\ 4 \end{bmatrix} \tag{46}$$

Since A has rank 3, the equation is clearly solvable. Suppose each agent ij knows a_{ij} of A and b_{ij} with $b_{11} = -6,\ b_{21} = 15,\ b_{31} = 4$ and all others are zero. We employ the updates (5) and (6) with arbitrary initializations. Let

$$V(t) = \frac{1}{2}\sum_{i=1}^{3}\left\| \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \\ x_{i4}(t) \end{bmatrix} - \boldsymbol{x}^* \right\|_2^2$$

where $\boldsymbol{x}^* = \begin{bmatrix} 1.216 & 3.941 & 0.373 & -0.428 \end{bmatrix}'$ is a solution to $Ax = b$. Then $V(t)$ measures the closeness of all agent states to the solution $\boldsymbol{x}^*$.

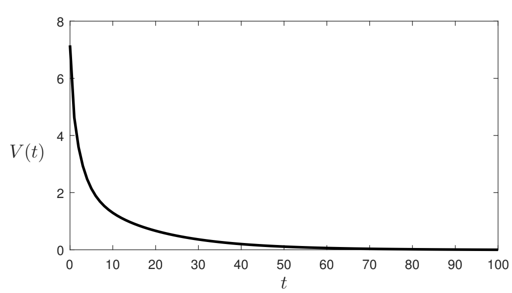

Fig. 2. Evolution of $V(t)$ under the proposed updates (5)-(6)

Simulations shown in Fig. 2 suggest that $V(t)$ converges exponentially fast to 0, which indicates all $x_{ij}(t)$ converge exponentially fast to constant vectors x_{ij}^* that satisfies (3)-(4). This is in accord with Theorem 1.

VI. Conclusion

In this paper, we have developed a distributed update for solving linear equations by multi-agent networks, in which each agent only knows two scalar entries and controlled two scalar states. Given the underlying row-neighbor networks and column-neighbor networks to be connected and undirected, the proposed algorithms enables agents in the networks to collaboratively achieve a solution to the overall equation exponentially fast. Our future work includes the generalization of the proposed update to time-varying directed networks; to the case that each agent controls more than a scalar entry of the equation; and to study the possibility of using each agent's historical data to accelerate the convergence rate.

VII. Appendix

Proof of Lemma 1: Let λ denote any eigenvalue of M with a non-zero eigenvector $\operatorname{col}\{\boldsymbol{u}, \bar{\boldsymbol{u}}\}$. Then

$$M\begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} \tag{47}$$

with

$$M = \begin{bmatrix} -M_1'M_1 - M_2 & M_1'M_3 \\ M_1 & -M_3 \end{bmatrix}.$$

Let $\bar{M} = \begin{bmatrix} I & 0 \\ 0 & M_3' \end{bmatrix} M$. Then one has

$$\bar{M} = \begin{bmatrix} -M_1'M_1 - M_2 & M_1'M_3 \\ M_3'M_1 & -M_3'M_3 \end{bmatrix}$$

which can be written as

$$\bar{M} = -\begin{bmatrix} M_1' \\ -M_3' \end{bmatrix}\begin{bmatrix} M_1 & -M_3 \end{bmatrix} - \begin{bmatrix} M_2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{48}$$

Thus $\bar{M}$ is negative semi-definite. Premultiplying by $\begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & M_3' \end{bmatrix}$ on both sides of (47), one has

$$\begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \bar{M} \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & M_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} \tag{49}$$

First, we prove that λ must be real by contradiction. Suppose $\lambda = \alpha + \beta\boldsymbol{i}$ where $\beta \neq 0$. Since $\bar{M}$ is negative semi-definite, then the imaginary part of the left-hand side of (49) is 0. So is the imaginary part of the right-hand side. It follows that

$$\beta \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & M_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = 0$$

Since $\beta \neq 0$ there follows

$$\boldsymbol{u}'\boldsymbol{u} + \bar{\boldsymbol{u}}'M_3\bar{\boldsymbol{u}} = 0.$$

Recall that M_3 positive semi-definite. Hence

$$\boldsymbol{u} = 0, \quad M_3\bar{\boldsymbol{u}} = 0$$

Taken with (47) and noting $\lambda \neq 0$ since $\beta \neq 0$, one has $\bar{\boldsymbol{u}} = 0$. This and the assumption that $\boldsymbol{u} = 0$ contradict to the fact that $\operatorname{col}\{\boldsymbol{u}, \bar{\boldsymbol{u}}\}$ is non-zero. Thus $\beta = 0$. Therefore, λ is real. From this, (49), $\bar{M}$ is negative semi-definite and M_3 is positive semi-definite, one has

$$\lambda \leq 0.$$

Second, if $\lambda = 0$ is an eigenvalue of M, we prove that it must be non-defective by contradiction. Suppose $\lambda = 0$ is defective, then there exists a non-zero vector $\operatorname{col}\{\boldsymbol{v}, \bar{\boldsymbol{v}}\}$ such that

$$M \begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} \tag{50}$$

and

$$M \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = 0 \tag{51}$$

Premultiplying by $\begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & M_3' \end{bmatrix}$ on both sides of (50), one has

$$\begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix}' \bar{M} \begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix} = (\boldsymbol{u}'\boldsymbol{u} + \bar{\boldsymbol{u}}' M_3' \bar{\boldsymbol{u}}) \tag{52}$$

Premultiplying by $\begin{bmatrix} I & 0 \\ 0 & M_3' \end{bmatrix}$ on both sides of (51), one has

$$\bar{M} \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = 0 \tag{53}$$

This and the fact that $\bar{M}$ is symmetric imply that the left hand side of (52) is 0. Then

$$(\boldsymbol{u}'\boldsymbol{u} + \bar{\boldsymbol{u}}' M_3' \bar{\boldsymbol{u}}) = 0 \tag{54}$$

from which, using the fact that M_3 is positive semi-definite, one has

$$\boldsymbol{u} = 0, \quad M_3'\bar{\boldsymbol{u}} = 0. \tag{55}$$

Premultiplying by $\begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix}' \begin{bmatrix} I & 0 \\ 0 & M_3' \end{bmatrix}$ on both sides of (50) one has

$$\begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix}' \bar{M} \begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}'\boldsymbol{u} \\ \bar{\boldsymbol{v}}' M_3' \bar{u} \end{bmatrix}$$

The right-hand side is 0 by (55). Thus

$$\begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix}' \bar{M} \begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix} = 0, \tag{56}$$

Together with (76), this yields

$$M_1\boldsymbol{v} - M_3\bar{\boldsymbol{v}} = 0, \quad M_2\boldsymbol{v} = 0. \tag{57}$$

From this and the definition of M, one has

$$M \begin{bmatrix} \boldsymbol{v} \\ \bar{\boldsymbol{v}} \end{bmatrix} = 0,$$

By (50), this yields $\operatorname{col}\{\boldsymbol{u}, \bar{\boldsymbol{u}}\} = 0$, contradicting the assumption that $\operatorname{col}\{\boldsymbol{u}, \bar{\boldsymbol{u}}\}$ is a non-zero eigenvector. Thus, $\lambda = 0$ is non-defective. ■

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