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On the Effects of Heterogeneous Logical Interdependencies in a Multi-Dimensional Opinion Dynamics Model

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Abstract— **In this paper, we investigate a recently proposed opinion dynamics model which considers a network of individuals simultaneously discussing a set of logically interdependent topics. The logical interdependence between the topics is captured by a "logic matrix". Previous works have investigated the model under the assumption that all individuals have the same logic matrix, or that individuals have different logic matrices but each individual has some stubbornness, which are restrictive assumptions. In contrast, we investigate heterogeneous logic matrices for the individuals, and assume that no stubborn individuals are present. We show that such heterogeneity can lead to a stable system with persistent disagreement among the final opinions. This indicates heterogeneity in individuals' logical interdependence structures, and not just the stubbornness of individuals (as in the Friedkin–Johnsen model), may explain the phenomenon of strong diversity of opinions often observed in a strongly connected network: the opinions at equilibrium are not at a complete consensus and opinions in any cluster are similar but not equal.**

I. Introduction

Recently, there has been great interest in the study of agent-based network models of *opinion dynamics* that describe how individuals' opinions on a topic evolve over time as they interact [1], [2]. The seminal discrete-time French–Harary–DeGroot model [3]–[5] (or DeGroot model for short) assumes that each individual's opinion at the next time step is a convex combination of his/her current opinion and the current opinions of his/her neighbours. For networks satisfying mild connectivity conditions, the opinions reach a consensus, i.e. the opinion values are equal for all individuals. This model captures social influence, where individuals exert a conforming influence on each other so that over time, opinions become more similar.

Since then, and to reflect real-world networks, much focus has been placed on developing models in which the opinions converge to final values where there is some disagreement, and not consensus at a single value. The Hegselmann–Krause model [6]–[8] introduces the concept of bounded confidence, which is used to capture homophily, i.e. the phenomenon whereby individuals only interact with those individuals whose opinion values are similar to their own. In the limit, the opinions may converge to distinct clusters, where the opinion values within each cluster are at consensus, but are different between clusters. The Altafini model [9]–[11] introduces negative edge weights to model antagonistic interactions between individuals (perhaps arising from mistrust). If the network is "structurally balanced" [9], then the opinions converge to two polarised clusters of opinions where opinions within each cluster are equal in value. These models capture *weak diversity* [12], where there is no difference between opinions in the same cluster.

There has therefore been a growing interest to study models which are able to capture *strong diversity*, as frequently observed in the real world, in networks which remain connected (in bounded confidence models, the network becomes disconnected into subgroups associated with the clusters) [12]. Strong diversity is where the opinions eventually converge to a configuration of persistent disagreement, with a diverse range of opinion values (and there may be clusters of opinions with similar, but *not equal*, values within a cluster). If social influence is acting to bring opinions closer together, some other process must be at work in connected networks to generate strong diversity. At least two models can display this behaviour. The Friedkin–Johnsen model shows that strong diversity may occur due to an individual's stubborn attachment to his/her initial opinion [13]. The paper [14] assumes an individual's susceptibility to interpersonal influence is dependent on the individual's current opinion; strong diversity can then arise, but only in a special case. In contrast to these works, we identify that strong diversity can arise due to *heterogeneity in the individuals' logical interdependence structure for a set of topics* (the next paragraph will explain the meaning of "logical interdependence").

In [15], a multi-dimensional extension to the Friedkin–Johnsen (and also the DeGroot model, which is a special case of the Friedkin–Johnsen model with no stubborn individuals) is proposed for individuals who simultaneously discuss a set of *logically interdependent* topics. The set of interdependent topics forms a "belief system" [16]. That is, an individual's position on Topic A may influence his/her position on Topic B due to his/her view of constraints or relations between the two topics. A "logic matrix" is used to capture this interdependence. The focus in [15] was on obtaining comprehensive conditions on the network topology and the logic matrix (assumed to be the same for all individuals) for convergence of opinions to steady values. Apart from minor remarks, little consideration was given to the effects of the logic matrix on the final opinion values. In [17], heterogeneous logic matrices were considered, but at least

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one individual was required to exhibit stubbornness in order to obtain a stability result. The focus of [17] was to apply the model to explain the shift in opinion values over time of the US population on the 2003 US-led invasion of Iraq.

In this paper, we study the special case of the model in [15] with no stubborn individuals, i.e. the multi-dimensional DeGroot model; this is in order to highlight the significant effects of the logical interdependence between topics. In particular, we focus on lower triangular heterogeneous logic matrices, i.e. individuals may have different logic matrices, and there is a cascade structure to the logical interdependence. We explain that, perhaps surprisingly, lower triangular logic matrices capture many situations of relevance in the problem context. Furthermore, we explicitly detail the final opinion values as a function of the individuals' logic matrices and the network topology. Most importantly and as a major conclusion, we identify that, for the simplest type of interdependence, *strong diversity* in the final opinion values can be a direct consequence of a network having individuals with *competing logical interdependence* between topics. Two individuals i and j are said to have competing logical interdependence if the entries in their logic matrices at the same position have opposing signs. For complex interdependences, heterogeneity of the logic matrix entries is typically enough to result in strong diversity.

These results combine to give a new, illuminating perspective in explaining how *strong diversity* in well connected networks can exist. Experimental studies are inconclusive with regards to the existence of ubiquitous and persistent antagonistic interpersonal interactions (there might be limited occurrences in the network over short time spans) [15], while it is unlikely that an individual would maintain the same level of stubborn attachment to his/her initial opinion value for months or years. This paper uses competing and heterogeneous logical interdependence to explain how strong diversity can *last for extended periods of time* as in the real world. We expect that it is in general much more difficult to change an individual's logic structure on a set of topics (which may be built upon years of experience and education), than to change the opinions themselves via interpersonal influence. In other words, opinion diversity may not only be due to the stubborn attachment of an individual to his/her initial opinion but may also be a result of heterogeneity of logical interdependence in belief systems.

The rest of the paper is structured as follows. In Section II, we provide notations, an introduction to graph theory and the opinion dynamics model. At the same time, a formal problem statement is given. The main results are presented in Section III, and conclusions are drawn in Section IV.

II. Background and Formal Problem Statement

We begin by introducing some mathematical notation used in the paper. The $(i,j)^{th}$ entry of a matrix $\boldsymbol{M}$ is denoted m_{ij}. A matrix $\boldsymbol{A}$ is said to be nonnegative (respectively positive) if all a_{ij} are nonnegative (respectively positive). We denote $\boldsymbol{A}$ as being nonnegative and positive by $\boldsymbol{A} \geq 0$ and $\boldsymbol{A} > 0$, respectively. An $n \times n$ matrix $\boldsymbol{A} \geq 0$ is said to be row-substochastic (respectively row-stochastic) if, for all i, there holds $\sum_{j=1}^n a_{ij} \leq 1$ (respectively $\sum_{j=1}^n a_{ij} = 1$). The transpose of a matrix $\boldsymbol{M}$ is denoted by $\boldsymbol{M}^\top$. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors of all ones and all zeros. The $n \times n$ identity matrix is given by $\boldsymbol{I}_n$. The spectral radius of $\boldsymbol{A}$ is given by $\rho(\boldsymbol{A})$. We now introduce some definitions and results to be used in the sequel.

Definition 1 (Primitivity, [18, Definition 1.12])**.** *A square matrix $\boldsymbol{A} \geq 0$ is primitive if $\exists k \in \mathbb{N}$ such that $\boldsymbol{A}^k > 0$.*

Lemma 1. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a given row-substochastic matrix. Suppose $\boldsymbol{A}$ has at least one row with row sum strictly less than one, i.e. $\exists\, i : \sum_{j=1}^n a_{ij} < 1$. Suppose further that $\boldsymbol{A}$ is irreducible. Then, $\rho(\boldsymbol{A}) < 1$.*

Proof. An immediate consequence of [19, Lemma 2.8]. □

A. Graph Theory

The interaction between individuals in a social network is modelled using a weighted directed graph, denoted as $\mathcal{G}[\boldsymbol{A}] = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$. Each individual is a node in the finite, nonempty set of nodes $V = \{v_1, \ldots, v_n\}$, with index set $\mathcal{I} = \{1, \ldots, n\}$. The set of ordered edges is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge as $e_{ij} = (v_i, v_j) \in \mathcal{E}$, and because the graph is directed, in general the existence of e_{ij} does not imply existence of e_{ji}. An edge e_{ij} is said to be outgoing with respect to v_i and incoming with respect to v_j. The presence of an edge e_{ij} connotes that individual j learns of, and takes into account, the opinion value of individual i when updating j's own opinion. Self-loops are allowed, i.e. e_{ii} may be in $\mathcal{E}$. The matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}[\boldsymbol{A}]$ captures the edge weights. More specifically, $a_{ij} \neq 0$ if and only if $e_{ji} \in \mathcal{E}$. If $\boldsymbol{A}$ is nonnegative, then all edges e_{ij} have positive weights, while a generic $\boldsymbol{A}$ may be associated with a signed graph $\mathcal{G}[\boldsymbol{A}]$, having both positive and negative edge weights.

A directed path is a sequence of edges of the form $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), \ldots$ where $v_{p_i} \in \mathcal{V}, e_{p_i p_{i+1}} \in \mathcal{E}$. Node i is reachable from node j if there exists a directed path from v_j to v_i. A graph is said to be strongly connected if every node is reachable from every other node. The matrix $\boldsymbol{A}$ is irreducible if and only if the associated graph $\mathcal{G}[\boldsymbol{A}]$ is strongly connected. A directed cycle is a directed path that starts and ends at the same vertex, and contains no repeated vertex except the initial (which is also the final) vertex. The length of a directed cycle is the number of edges in the directed cyclic path. A directed graph is *aperiodic* if there exists no integer $k > 1$ that divides the length of every directed cycle of the graph [18]. Note that any graph with a self-loop is aperiodic. The following is a useful result.

Lemma 2 ([18, Proposition 1.35])**.** *The graph $\mathcal{G}[\boldsymbol{A}]$ is aperiodic and strongly connected if and only if $\boldsymbol{A}$ is primitive.*

B. The Multi-Dimensional DeGroot Model

In this paper, we investigate a recently proposed extension to the DeGroot and Friedkin-Johnsen models [15], [17]; the new model considers the *simultaneous discussion of logically* *interdependent topics*. In order to place the focus on the effects of the logical interdependence, this paper will assume that there are no stubborn individuals in the network, i.e. we study the extension to the DeGroot model.

Formally, the model considers a network of n individuals discussing simultaneously their opinions on m topics, with topic index set $\mathcal{J} = \{1, \ldots, m\}$. Individual i's opinions on the m topics, at time $t = 0, 1, \ldots$, are denoted by $\boldsymbol{x}_i(t) = [x_i^1(t), \ldots, x_i^m(t)]^\top \in \mathbb{R}^m$, and evolves according to

$$\boldsymbol{x}_i(t+1) = \sum_{j=1}^{n} w_{ij} \boldsymbol{C}_i \boldsymbol{x}_j(t), \tag{1}$$

where the nonnegative scalar $w_{ij} \leq 1$, which is the $(i,j)^{th}$ entry of the influence matrix $\boldsymbol{W}$ associated with the influence network $\mathcal{G}[\boldsymbol{W}]$, represents the influence weight individual i accords to the vector of opinions of individual j. It is assumed that $\sum_{j=1}^{n} w_{ij} = 1$ for all $i \in \mathcal{I}$, which implies that $\boldsymbol{W}$ is row-stochastic. The matrix $\boldsymbol{C}_i$, with $(p,q)^{th}$ entry $c_{pq,i}$, is termed the logic matrix, and represents the logical interdependence between the m topics. We explore this matrix and its properties in more detail below, but we note here that the $\boldsymbol{C}_i$ are assumed to be heterogeneous (i.e. $\exists i, j : \boldsymbol{C}_i \neq \boldsymbol{C}_j$). Indeed, a critical aspect of this paper is to show how heterogeneity, and specifically competing $\boldsymbol{C}_i$ (the formal definition of which will be introduced in the sequel), can generate persistent disagreement in the network. For convenience, denote the vector of opinions for the entire influence network as $\boldsymbol{x} = [\boldsymbol{x}_1(t)^\top, \ldots, \boldsymbol{x}_n(t)^\top]^\top \in \mathbb{R}^{nm}$.

For completeness and to aid discussion, we record the Friedkin–Johnsen variant to Eq. (1), which is given as

$$\boldsymbol{x}_i(t+1) = \lambda_i \sum_{j=1}^{n} w_{ij} \boldsymbol{C}_i \boldsymbol{x}_j(t) + (1 - \lambda_i) \boldsymbol{x}_i(0). \tag{2}$$

Here, the parameter $\lambda_i \in [0,1]$ represents individual i's susceptibility to interpersonal influence, while $1 - \lambda_i$ represents the level of stubborn attachment by individual i to his/her initial opinion $\boldsymbol{x}_i(0)$. Thus, when we say that there are no stubborn individuals in the network, we mean that $\lambda_i = 1$ for all $i \in \mathcal{I}$, and in this case Eq. (2) is equivalent to Eq. (1).

Supposing that the logic matrices were indeed homogeneous, i.e. $\boldsymbol{C}_i = \boldsymbol{C}_j = \boldsymbol{C}$ for all $i, j \in \mathcal{I}$, then one could write the compact influence network dynamics, with no stubborn individuals, as

$$\boldsymbol{x}(t+1) = (\boldsymbol{W} \otimes \boldsymbol{C}) \boldsymbol{x}(t), \tag{3}$$

and limiting behaviour is easy to characterise. The following is a result from [15].

Theorem 1 ([15, Theorem 3])**.** *The system Eq. (3) converges if and only if* $\lim_{k\to\infty} \boldsymbol{C}^k \triangleq \boldsymbol{C}^\infty$ *exists, and either* $\boldsymbol{C}^\infty = \boldsymbol{0}_{m\times m}$ *or* $\lim_{k\to\infty} \boldsymbol{W}^k = \boldsymbol{W}^\infty$ *exists. Moreover, the system converges to* $\lim_{t\to\infty} \boldsymbol{x}(t) = (\boldsymbol{W}^\infty \otimes \boldsymbol{C}^\infty)\boldsymbol{x}(0)$.

Remark 1. *The paper [15] mainly focuses on the considerable challenge of obtaining complete convergence results for the Friedkin-Johnsen model (including with logical interdependence between topics as captured by a homogeneous* $\boldsymbol{C}$*), and aside from some short remarks, does not investigate the effect of* $\boldsymbol{C}$ *on the final opinion distribution (if the opinions do in fact converge to a steady state). In contrast, one of the key advances of this paper is to identify and characterise the effects of heterogeneous* $\boldsymbol{C}_i$ *on the final opinion distribution. The paper [17] does obtain convergence results for heterogeneous* $\boldsymbol{C}_i$*, but makes an assumption that there is at least one individual* i *with* $\lambda_i < 1$*. Importantly, [17] only secures a convergence result, and does not investigate the effect of heterogeneous* $\boldsymbol{C}_i$ *on the final opinion distribution.*

C. The $\boldsymbol{C}$ Matrix and Its Properties

In [15], [17], the authors elucidate that $\boldsymbol{C}_i$ captures how individual i views the logical interdependence between the m topics being discussed, and is used by individual i to obtain a consistent belief system on a set of topics by adjusting any inconsistencies. We illustrate with a simple example, and in doing so, show that certain constraints, not explicitly discussed in [15], [17], must be imposed on $\boldsymbol{C}_i$ due to the problem context. In this paper, we adopt a standard definition of an opinion [17]. In particular, $x_i^p(t) \in [-1, 1]$ is individual i's certainty on the truth of statement p. Suppose that there are two topics. Topic 1: The exploration of Space is important to mankind's future. Topic 2: The exploration of Space should be privatised. For individual i, $x_i^1 = 1$ represents i's maximal certainty of the importance of Space exploration, while $x_i^1 = -1$ represents maximal certainty that Space exploration is *not* important. A mild assumption is placed on $\boldsymbol{C}_i$ in the sequel to ensure that $x_i^p(t) \in [-1, 1]$ for all $t \geq 0$. Now, suppose that individual i has $\boldsymbol{x}_i(0) = [1, -0.2]^\top$, i.e. i initially believes Space exploration is important and initially believes with some certainty that Space exploration should not be privatised[1]. Let

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}. \tag{4}$$

That is, i's opinion on the importance of Space exploration is unaffected by i's opinion on whether Space exploration should be privatised. On the other hand, i's opinion on Topic 2 depends positively on i's opinion on Topic 1, perhaps because i believes privatised companies are more effective. In the absence of opinions from other individuals, individual i's opinions evolves as

$$\boldsymbol{x}_i(t+1) = \boldsymbol{C}_i \boldsymbol{x}_i(t), \tag{5}$$

which yields $\lim_{t\to\infty} \boldsymbol{x}_i(t) = [1, 1]^\top$, i.e. i eventually believes that Space exploration should be privatised. Thus, i's belief system moves from $\boldsymbol{x}_i(0) = [1, -0.2]^\top$, where i's opinions are inconsistent with the logical interdependence as captured by $\boldsymbol{C}_i$, to the final state $\boldsymbol{x}_i(\infty) = [1, 1]^\top$, which is consistent with the logical interdependence. In general, one expects that eventually, a belief system will become consistent. We exclude the scenario of $\rho(\boldsymbol{C}_i) < 1$, which in Eq. (5) would lead to the non-generic case of $\boldsymbol{x}_i(\infty) = \boldsymbol{0}_m$. Also note that for a topic p which is independent of all other

[1]Note that we do not require $\boldsymbol{C}_i$ to be row-stochastic and nonnegative, though the $\boldsymbol{C}_i$ of this example is.

topics, one expects that $x_i^p(t+1) = x_i^p(t)$ when individual i has no neighbours. Combining the above arguments, we conclude that generically, Eq. (5) must converge to $\boldsymbol{x}_i(\infty) \neq \mathbf{0}_m$, and thus impose the following assumption:

Assumption 1. *The matrix $\boldsymbol{C}_i$, for all i, has a semi-simple*[2] *eigenvalue 1. All other eigenvalues of $\boldsymbol{C}_i$ have modulus strictly less than 1, and for all $i \in \mathcal{I}$ and $p \in \mathcal{J}$, $\sum_{q=1}^m |c_{pq,i}| = 1$. If topic p is independent of all other topics, i.e. $c_{pq,i} = 0, \forall q \neq p$, then $c_{pp,i} = 1$. The diagonal entries are nonnegative, i.e. $c_{pp,i} \geq 0, \forall i \in \mathcal{I}$ and $p \in \mathcal{J}$.*

The assumption that $c_{pp,i} \geq 0$ simply means topic p is not negatively correlated with itself. The special case where topics are totally independent is $\boldsymbol{C}_i = \boldsymbol{I}_m$, while $x_i^p(t) \in [-1,1]$ for all $t \geq 0$ holds if $x_i^p(0) \in [-1,1]$, and $\sum_{q=1}^m |c_{pq,i}| = 1$ for all $i \in \mathcal{I}$ and $p \in \mathcal{J}$ (see [15]). It is clear that if we have homogeneous $\boldsymbol{C}$, then Assumption 1 is consistent with the requirement on $\boldsymbol{C}$ in Theorem 1. Note that heterogeneity of $\boldsymbol{C}_i$ may arise for many different reasons, such as education, background, or expertise in the topic. For example, if the set of topics is related to sports, a competitive athlete may have very different weights in $\boldsymbol{C}_i$ compared to someone that does not pursue an active lifestyle.

D. Formal Problem Statement

In this paper, we investigate topics whose logical interdependence gives rise to reducible $\boldsymbol{C}_i$, for all i. That is, the logic matrices of all individuals are expressed in a lower triangular form, and in the problem context, implies a cascade logical interdependence structure among the topics. Specifically, we place the following assumption on $\boldsymbol{C}_i$.

Assumption 2. *There exists a common permutation matrix $\boldsymbol{P}$ such that, for all $i \in \mathcal{I}$, $\boldsymbol{P}^\top \boldsymbol{C}_i \boldsymbol{P}$ is lower triangular. Without loss of generality, it is then assumed that the topics $p \in \mathcal{J}$ are ordered such that, for every $i \in \mathcal{I}$,*

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & & & \\ c_{21,i} & c_{22,i} & & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \\ c_{n1,i} & c_{n2,i} & \cdots & c_{nn,i} \end{bmatrix} \tag{6}$$

Such an assumption is not necessarily restrictive. For many individuals, their logical interdependence structure might be obtained by sequentially building upon an axiom or axioms in their thought processes. In the extended journal version, we aim to investigate a more general class of $\boldsymbol{C}_i$ which includes block lower triangular, and irreducible structures. We now give a definition of "competing logical interdependencies", which will crystallise one objective of this paper.

Definition 2 (Competing Logical Interdependence)**.** *An influence network is said to contain individuals with competing logical interdependencies on topic p if there exist individuals i, j such that $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$, for some $q \in \mathcal{J}$ and $q \neq p$, have nonzero entries $c_{pq,i}$ and $c_{pq,j}$ that are of opposite signs.*

[2]By semi-simple, we mean that the geometric and algebraic multiplicities are equal. Equivalently, the Jordan blocks of the eigenvalue 1 are all 1×1.

In other words, individuals with competing logical interdependencies are those who, *when having the same opinion on topic q*, move in opposite directions on the opinion spectrum for topic p. Such occurrences are widely prevalent in society, especially on politically contentious issues; [20] showed that different people could interpret the same fact to draw different conclusions. Using the example in Section II-C, one might have an individual j with

$$\boldsymbol{C}_j = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}, \tag{7}$$

because j considers that private companies are profit-driven, and therefore cannot be trusted with the exploration of Space. Then, from Eq. (5), one has that $\boldsymbol{x}_j(\infty) = [1, -1]^\top$, i.e. j firmly believes Space exploration should not be privatised. In particular, $x_j^2(\infty) = -x_i^2(\infty)$.

We place a further assumption on the interaction topology.

Assumption 3. *The influence network $\mathcal{G}[\boldsymbol{W}]$ is strongly connected and aperiodic.*

This assumption implies $\boldsymbol{W}$ is primitive from Lemma 2. An extended version of this paper will consider relaxation of this assumption to directed graphs $\mathcal{G}[\boldsymbol{W}]$ which are not strongly connected. From the Perron-Frobenius Theorem [21], we conclude that $\boldsymbol{W}$ has a simple eigenvalue at 1, and all other eigenvalues have modulus less than 1. Moreover, $\boldsymbol{W}$ has a left eigenvector $\boldsymbol{\gamma}^\top$ with strictly positive entries, associated with the eigenvalue 1. We assume $\boldsymbol{\gamma}^\top$ is normalised to satisfy $\boldsymbol{\gamma}^\top \mathbf{1}_n = 1$. In addition, $\lim_{k\to\infty} \boldsymbol{W}^k = \mathbf{1}_n \boldsymbol{\gamma}^\top$ [18].

III. Main Results

We begin by introducing a coordinate transform in order to aid in the analysis of the opinion dynamical system. In particular, define $\boldsymbol{y}_k(t) = [y_k^1(t), \dots y_k^n(t)]^\top = [x_1^k(t), \dots, x_n^k(t)]^\top$, for $k \in \mathcal{J}$ as the vector of all individuals' opinions on the i^{th} topic. Then, $\boldsymbol{y}(t) = [\boldsymbol{y}_1(t)^\top, \dots, \boldsymbol{y}_m(t)^\top]^\top$ captures all of the n individuals' opinions on the m topics. One obtains that

$$\boldsymbol{y}_k(t+1) = \sum_{j=1}^m \operatorname{diag}(c_{kj}) \boldsymbol{W} \boldsymbol{y}_j(t), \tag{8}$$

where $\operatorname{diag}(c_{kj}) = \operatorname{diag}(c_{kj,1}, \dots, c_{kj,n})$, i.e. the i^{th} diagonal element of $\operatorname{diag}(c_{kj})$ is the $(k,j)^{th}$ entry of $\boldsymbol{C}_i$, $c_{kj,i}$. It follows that

$$\boldsymbol{y}(t+1) = \begin{bmatrix} \operatorname{diag}(c_{11})\boldsymbol{W} & \cdots & \operatorname{diag}(c_{1m})\boldsymbol{W} \\ \vdots & \ddots & \vdots \\ \operatorname{diag}(c_{m1})\boldsymbol{W} & \cdots & \operatorname{diag}(c_{mm})\boldsymbol{W} \end{bmatrix} \boldsymbol{y}(t), \tag{9}$$

and denote the matrix in Eq. (9) as $\boldsymbol{A}$, with block matrix elements $\boldsymbol{A}_{pq} = \operatorname{diag}(c_{pq})\boldsymbol{W}$. Note that the assumption that the $\boldsymbol{C}_i$ are simultaneously lower triangular implies that $\boldsymbol{A}$ is reducible (block lower triangular). We now present a convergence result for the opinion dynamical system.

Theorem 2. *Suppose that the opinion vector of each individual $i \in \mathcal{I}$ updates according to Eq. (1), and suppose further that Assumptions 1, 2 and 3 hold, and that $\boldsymbol{C}_i$ is of* *the form in Eq. (6) for all $i \in \mathcal{I}$. Then, $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*$ exponentially fast, where $\boldsymbol{x}^*$ is the vector of final opinions.*

Proof. For convenience of analysis, we work with variables $\boldsymbol{y}$ and prove that $\boldsymbol{y}(t)$ converges exponentially fast to the equilibrium of Eq. (9), with $\boldsymbol{x}^*$ obtained by recalling the definition of $\boldsymbol{y}$ above Eq. (8). Note that because $\boldsymbol{A}$ is block lower triangular, the eigenvalues of $\boldsymbol{A}$ are the eigenvalues of all the $\text{diag}(c_{pp})\boldsymbol{W}$, $p \in \mathcal{J}$.

For some $p \in \mathcal{J}$, suppose that $c_{pp,i} = 1$ for all i, that is, every individual in the network considers topic p independent of all other topics. Then,

$$\boldsymbol{y}_p(t+1) = \boldsymbol{W}\boldsymbol{y}_p(t), \tag{10}$$

and from Assumption 3 one has that $\lim_{k\to\infty} \boldsymbol{W}^k = \boldsymbol{1}_n\boldsymbol{\gamma}^\top$, which implies that $\lim_{t\to\infty} \boldsymbol{y}_p(t) = (\boldsymbol{\gamma}^\top\boldsymbol{y}_p(0))\boldsymbol{1}_n$. In other words, a consensus is reached on the p^{th} topic. From Eq. (6), one has that $\lim_{t\to\infty} \boldsymbol{y}_1(t) = (\boldsymbol{\gamma}^\top\boldsymbol{y}_1(0))\boldsymbol{1}_n$. For convenience, denote $\boldsymbol{y}_1^* = \lim_{t\to\infty} \boldsymbol{y}_1(t)$.

Consider topic 2, and suppose that there $\exists j \in \mathcal{I}$ such that $c_{pp,j} < 1$ (otherwise, topic 2 is independent of all other topics by the reducible structure of $\boldsymbol{C}_i$, as given in Eq. (6)). Then, $\text{diag}(c_{22})\boldsymbol{W}$ is row-substochastic with at least one row sum strictly less than 1 (in particular, the j^{th} row). Lemma 1 then indicates that $\rho(\text{diag}(c_{22})\boldsymbol{W}) < 1$ and thus $(\text{diag}(c_{22})\boldsymbol{W})^\infty = \boldsymbol{0}_{n\times n}$. One obtains that

$$\boldsymbol{y}_2(t+1) = \boldsymbol{A}_{21}\boldsymbol{y}_1(t) + \boldsymbol{A}_{22}\boldsymbol{y}_2(t). \tag{11}$$

Notice that Eq. (11) can be considered as a forced system $\boldsymbol{z}(t+1) = \boldsymbol{F}\boldsymbol{z}(t) + \boldsymbol{G}\boldsymbol{u}(t)$, with $\rho(\boldsymbol{F}) = \rho(\boldsymbol{A}_{22}) < 1$ and with bounded input $\boldsymbol{G}\boldsymbol{u}(t) = \boldsymbol{A}_{21}\boldsymbol{y}_1(t)$ that converges exponentially fast to the constant $\boldsymbol{A}_{21}\boldsymbol{y}_1^*$. Thus, $\boldsymbol{y}_2(t)$ converges exponentially fast to the equilibrium $\boldsymbol{y}_2(\infty) \triangleq \boldsymbol{y}_2^*$, which obeys $\boldsymbol{y}_2^* = \boldsymbol{A}_{22}\boldsymbol{y}_2^* + \boldsymbol{A}_{21}\boldsymbol{y}_1^*$. This can be rearranged to obtain $\boldsymbol{y}_2^* = (\boldsymbol{I}_n - \boldsymbol{A}_{22})^{-1}\boldsymbol{A}_{21}\boldsymbol{y}_1^*$.

Exploiting the lower triangular structure of $\boldsymbol{A}$, and by recursion, we conclude that for all topics $p \in \mathcal{J}$, with $p \neq 1, 2$, the dynamics of $\boldsymbol{y}_p(t)$ is either given by Eq. (10) (in which case $\boldsymbol{y}_p(\infty) = (\boldsymbol{\gamma}^\top\boldsymbol{y}_p(0))\boldsymbol{1}_n$) or is given by

$$\boldsymbol{y}_p(t+1) = \sum_{j<p} \text{diag}(c_{pj})\boldsymbol{W}\boldsymbol{y}_j(t) + \text{diag}(c_{pp})\boldsymbol{W}\boldsymbol{y}_p(t),$$

where $\rho(\text{diag}(c_{pp})\boldsymbol{W}) < 1$ and $\sum_{j=1}^{j<p} \text{diag}(c_{pj})\boldsymbol{W}\boldsymbol{y}_j(t)$ is a bounded input that converges exponentially fast to the constant $\sum_{j=1}^{j<p} \text{diag}(c_{pj})\boldsymbol{W}\boldsymbol{y}_j^*$. In the second case, it follows that $\boldsymbol{y}_p(t)$ converges exponentially fast to the point

$$\boldsymbol{y}_p^* = (\boldsymbol{I}_n - \boldsymbol{A}_{pp})^{-1}\big(\sum_{j<p} \boldsymbol{A}_{pj}\boldsymbol{y}_j^*\big). \tag{12}$$

This completes the proof. □

Having proved the stability of the opinion dynamical system, we now turn to study of the equilibrium $\boldsymbol{y}^*$, and highlight the role of competing and heterogeneous $\boldsymbol{C}_i$ in creating persistent disagreement in the final opinions. Clearly if topic p is independent, then $\boldsymbol{y}_p^*$ reaches a consensus, i.e. $\boldsymbol{y}_p^* = \alpha\boldsymbol{1}_n$ for some $\alpha \in \mathbb{R}$. In the following result, we assume topics $p = 2, \ldots, m$ are dependent on at least one topic $q < p$. That is, for all topics $p \in \{2, \ldots, m\}$, there exist $r < p$ and $i \in \mathcal{I}$ such that $c_{pr,i}$ is nonzero. We remark that because $x_i^k(t) \in [-1, 1]$ for all $t \geq 0$, $i \in \mathcal{I}$ and $k \in \mathcal{J}$, then for any topic p which happens to achieve consensus, i.e. $\boldsymbol{y}_p^* = \alpha\boldsymbol{1}_n$, there holds $|\alpha| \leq 1$. The following result does not identify all equilibria of Eq. (9) but rather focuses on equilibria in which consensus of opinions might be reached; in some cases, other equilibria can indeed exist.

Theorem 3. *Suppose that the hypotheses in Theorem 2 hold. For topic $p \in \{2, \ldots, m\}$, let $\mathcal{J}_{p,i}$ be the nonempty set of topics on which topic p is dependent in the logic structure $\boldsymbol{C}_i$ of individual i, i.e., $c_{pq,i\neq 0}$ for $q \in \mathcal{J}_{p,i}$. Suppose further that all topics $q_{j,i} \in \mathcal{J}_{p,i}$ have reached a consensus, i.e. $\boldsymbol{y}_{q_{j,i}}^* = \alpha_{q_{j,i}}\boldsymbol{1}_n$, $\alpha_{q_{j,i}} \in \mathbb{R}$. Then the following hold.*

1) *Suppose that $\mathcal{J}_{p,i} = \{q\}, \forall i$, i.e. for every individual, topic p is dependent only on topic q. Then, $\boldsymbol{y}_p^* = \alpha_p\boldsymbol{1}_n$ if and only if there do not exist individuals $i, j \in \mathcal{I}$ with competing logical interdependencies on topic p, with $\alpha_p = \alpha_q$ if $c_{pq,i} > 0$ and $\alpha_p = -\alpha_q$ if $c_{pq,i} < 0$.*
2) *Suppose that for every individual, topic p is dependent on the same set of topics $\mathcal{J}_p = \{q_1, \ldots, q_s\}$, $s \geq 2$. If $c_{pk,i} = c_{pk,j}$ for all $k < p$ and $i, j \in \mathcal{I}$, then, $\boldsymbol{y}_p^* = \alpha_p\boldsymbol{1}_n$; moreover,*

$$\alpha_p = \frac{1}{1 - c_{pp}} \sum_{j=1}^{s} c_{pq_j}\alpha_{q_j}. \tag{13}$$

If $c_{pq_k,i} \neq c_{pq_k,j}$ for some $k \in \{1, \ldots, s\}$ and $i, j \in \mathcal{I}$, then $\boldsymbol{y}_p^* = \alpha_p\boldsymbol{1}_n$ if and only if[3], for every $i \in \mathcal{I}$ there holds

$$\sum_{j=1}^{s} \alpha_{q_j}c_{pq_j,i} = \kappa(1 - c_{pp,i}), \tag{14}$$

where $\kappa \in \{-1, 0, 1\}$. Moreover, $\alpha_p = \kappa$.

Proof. Statement 1): For the proof of sufficiency, recall that $\boldsymbol{y}_q^* = \alpha_q\boldsymbol{1}_n$ by hypothesis and from Eq. (12) one then obtains

$$\boldsymbol{y}_p^* = \alpha_q\boldsymbol{R}_{pp}^{-1}\,\text{diag}(c_{pq})\boldsymbol{1}_n. \tag{15}$$

where $\boldsymbol{R}_{pp} = \boldsymbol{I}_n - \text{diag}(c_{pp})\boldsymbol{W}$. This is because $\boldsymbol{W}$ is row-stochastic and thus $\text{diag}(c_{pq})\boldsymbol{W}\boldsymbol{1}_n = \text{diag}(c_{pq})\boldsymbol{1}_n$. Notice that $(\boldsymbol{I}_n - \text{diag}(c_{pp}))\boldsymbol{1}_n = \boldsymbol{R}_{pp}\boldsymbol{1}_n$. Since there are no competing logical interdependencies then $\boldsymbol{I}_n - \text{diag}(c_{pp}) = \beta\,\text{diag}(c_{pq})$, where $\beta = 1$ if $c_{pq,i} > 0$ and $\beta = -1$ if $c_{pq,i} < 0$. This is because Assumption 1 has $\sum_{j=1}^{m} |c_{pj,i}| = 1$ for all $p \in \mathcal{J}$ and $i \in \mathcal{I}$. It follows that $(\boldsymbol{I}_n - \text{diag}(c_{pp})\boldsymbol{W})\boldsymbol{1}_n = \beta\,\text{diag}(c_{pq})\boldsymbol{1}_n$ and substituting this into Eq. (15) yields $\boldsymbol{y}_p^* = \beta\alpha_q\boldsymbol{R}_{pp}^{-1}\boldsymbol{R}_{pp}\boldsymbol{1}_n = \beta\alpha_q\boldsymbol{1}_n$.

For the proof of necessity, suppose that there are competing logical interdependencies. Without loss of generality, assume that $c_{pq,1}$ and $c_{pq,2}$ have opposite signs, and in particular that $c_{pq,1} < 0$ and $c_{pq,j} > 0$ for $j = 2, \ldots, n$. Then, $\boldsymbol{I}_n - \text{diag}(c_{pp}) = \text{diag}(\bar{c}_{pq})$ where $\text{diag}(\bar{c}_{pq}) = \text{diag}(|c_{pq,1}|, c_{pq,2}, \ldots, c_{pq,n})$. That is, $\text{diag}(\bar{c}_{pq})$ is equal to

[3] See Remark 2 below for an interpretation of Eq. (14).

$\operatorname{diag}(c_{pq})$ except the first diagonal entry is of equal magnitude but opposite sign. Then clearly, $\boldsymbol{R}_{pp}^{-1} \operatorname{diag}(c_{pq})\mathbf{1}_n \neq \tau\mathbf{1}_n$ for some $\tau \in \mathbb{R}$ and thus $\boldsymbol{y}_p^*$ is not at a consensus.

Statement 2): First, we examine the scenario where $c_{pq_k,i} = c_{pq_k,j}$ for all $k = \{1,\ldots,s\}$ and $i,j \in \mathcal{I}$, i.e. where the logical interdependencies for row p of $\boldsymbol{C}_i$ are the same for all $i \in \mathcal{I}$. Clearly, $\operatorname{diag}(c_{pq_k}) = c_{pq_k}\boldsymbol{I}_n$. It follows from Eq. (12) that

$$\boldsymbol{y}_p^* = (\boldsymbol{I}_n - c_{pp}\boldsymbol{W})^{-1}\sum_{j=1}^{s} c_{pq_j}\alpha_{q_j}\mathbf{1}_n, \tag{16}$$

since $\boldsymbol{y}_{q_j}^* = \alpha_{q_j}\mathbf{1}_n \Rightarrow \boldsymbol{W}\boldsymbol{y}_{q_j}^* = \alpha_{q_j}\mathbf{1}_n$. Recall that $\boldsymbol{R}_{pp} = \boldsymbol{I}_n - c_{pp}\boldsymbol{W}$ and $\boldsymbol{R}_{pp}\mathbf{1}_n = (1-c_{pp})\mathbf{1}_n$. This in turn implies that $\boldsymbol{R}_{pp}^{-1}\mathbf{1}_n = (1-c_{pp})^{-1}\mathbf{1}_n$ because $\boldsymbol{R}^{-1}\boldsymbol{R}\mathbf{1}_n = \boldsymbol{R}^{-1}(1-c_{pp})\mathbf{1}_n = \mathbf{1}_n$. It follows from Eq. (16) that $\boldsymbol{y}_p^* = \big(\frac{1}{1-c_{pp}}\sum_{j=1}^{s} c_{pq_j}\alpha_j\big)\mathbf{1}_n$, giving Eq. (13).

Consider now the scenario where $c_{pq_k,i} \neq c_{pq_k,j}$ for some $k \in \{1,\ldots,s\}$ and $i,j \in \mathcal{I}$, i.e. there are heterogeneous logical interdependencies for row p of $\boldsymbol{C}_i$ among the individuals $i \in \mathcal{I}$. For the proof of sufficiency, observe that from Eq. (12), and because $\boldsymbol{y}_{q_j}^* = \alpha_{q_j}\mathbf{1}_n$, there holds

$$\boldsymbol{y}_p^* = (\boldsymbol{I}_n - \operatorname{diag}(c_{pp})\boldsymbol{W})^{-1}\big(\sum_{j=1}^{s}\alpha_{q_j}\operatorname{diag}(c_{pq_j})\big)\mathbf{1}_n = \beta\mathbf{1}_n,$$

where $\beta \in \{-1,0,1\}$. This is because if Eq. (14) holds, then $\sum_{j=1}^{s}\alpha_{q_j}\operatorname{diag}(c_{pq_j}) = \beta(\boldsymbol{I}_n - \operatorname{diag}(c_{pp}))$, and because below Eq. (15), we obtained $(\boldsymbol{I}_n - \operatorname{diag}(c_{pp}))\mathbf{1}_n = \boldsymbol{R}_{pp}\mathbf{1}_n$. For the proof of necessity, and in order to obtain a contradiction, suppose that Eq. (14) does not hold (and thus $\sum_{j=1}^{s}\alpha_{q_j}\operatorname{diag}(c_{pq_j}) \neq \beta(\boldsymbol{I}_n - \operatorname{diag}(c_{pp}))$) and $\boldsymbol{y}_p^* = \alpha_p\mathbf{1}_n$ for some $\alpha_p \in \mathbb{R}$. From Eq. (12), we obtain

$$\alpha_p\mathbf{1}_n = (\boldsymbol{I}_n - \operatorname{diag}(c_{pp})\boldsymbol{W})^{-1}\big(\sum_{j=1}^{s}\alpha_{q_j}\operatorname{diag}(c_{pq_j})\big)\mathbf{1}_n$$

Multiplying both sides by $\boldsymbol{R}_{pp} = \boldsymbol{I}_n - \operatorname{diag}(c_{pp})\boldsymbol{W}$ yields $\alpha_p(\boldsymbol{I}_n - \operatorname{diag}(c_{pp}))\mathbf{1}_n = \big(\sum_{j=1}^{s}\alpha_{q_j}\operatorname{diag}(c_{pq_j})\big)\mathbf{1}_n$, which delivers a contradiction. This completes the proof. □

Remark 2. *The condition in Eq. (14) holds only in special scenarios where α_{q_j} are all equal in modulus **and** the α_{q_j} have a special sign pattern that corresponds to the sign pattern of the $c_{pq,j}$ entries. In many (but not all) cases heterogeneity of $\boldsymbol{C}_i$ in row p is enough to ensure Eq. (14) does not hold, which results in disagreement in the equilibrium opinions in topic p, $\boldsymbol{y}_p^*$. Interestingly, when there are only 2 topics, heterogeneity in the entries of $c_{21,i}$ are not enough to prevent a consensus of opinions on topic 2; competing logical interdependences are required. This is a surprising, and non-intuitive result.*

IV. Conclusions

This paper investigated the effects of heterogeneity in individuals' logic structures as a social network simultaneously discusses opinions on a set of logically interdependent topics, whose interdependence structure was captured by a logic matrix. We focused on logic matrices that are simultaneously lower triangular. We derived several sufficient conditions for consensus to occur under heterogeneous logic matrices, showing that the existence of competing logical interdependencies is sufficient to prevent a consensus from being reached. Future work will focus on study of irreducible logic matrices, and further characterisation of conditions (both necessary, and sufficient) which lead to strong diversity of opinions at equilibrium.

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