23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

# Extended Abstract: Solving Network Linear Equations with Quantized Node Communications

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Consider the following linear algebraic equation:

$$\mathbf{z} = \mathbf{H}\mathbf{y} \tag{1}$$

with respect to variable $\mathbf{y} \in \mathbb{R}^m$, where $\mathbf{H} \in \mathbb{R}^{N\times m}$ and $\mathbf{z} \in \mathbb{R}^N$. The equation (1) has a unique exact solution if $\text{rank}(\mathbf{H}) = m$ and $\mathbf{z} \in \text{span}(\mathbf{H})$; an infinite set of solutions if $\text{rank}(\mathbf{H}) < m$ and $\mathbf{z} \in \text{span}(\mathbf{H})$; and no exact solutions if $\mathbf{z} \notin \text{span}(\mathbf{H})$. We denote

$$\mathbf{H} = \begin{pmatrix} \mathbf{h}_1^{\mathrm{T}} \\ \mathbf{h}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{h}_N^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

with $\mathbf{h}_i^{\mathrm{T}}$ being the $i$-th row vector of $\mathbf{H}$.

We consider a network with nodes indexed in the set $\mathrm{V} = \{1,\dots,N\}$. Each node $i$ has access to the value of $\mathbf{h}_i$ and $z_i$ without the knowledge of $\mathbf{h}_j$ or $z_j$ from other nodes. The network interaction structure is described by a connected undirected graph $\mathrm{G} = (\mathrm{V},\mathrm{E})$. Time is slotted at $k = 0, 1, 2, \dots$. Each node $i$ at time $k$ holds a state $\mathbf{x}_i(k) \in \mathbb{R}^m$ and exchanges this state information with other neighboring nodes in the set $\mathrm{N}_i := \{j : \{i,j\} \in \mathrm{E}\}$. Distributed algorithms that solve the equation (1) under this problem settings have been investigated in [1]–[3], with a close relation to the framework of distributed gradient optimization [4], [5]. The aim of this paper is to develop algorithms that us *uantized* node communications [6], [7].

### A. The Algorithm

*Definition 1 (Quantization Function):* A standard uniform quantizer is given by the function $Q_K(\cdot) : \mathbb{R} \to \{-K,\dots,-1,0,1,\dots,K\}$ where

$$Q_K(z) = \begin{cases} 0, & \text{if } -1/2 < z \le 1/2, \\ i, & \text{if } \frac{2i-1}{2} < z \le \frac{2i+1}{2},\ i = 1,\cdots,K, \\ K, & \text{if } z > \frac{2K+1}{2}, \\ -Q_K(-z), & \text{if } z \le -1/2. \end{cases} \tag{2}$$

With slight abuse of notation, we define $Q_K(\mathbf{a})$ for a vector $\mathbf{a} = (a_1,\cdots,a_m)^T \in \mathbb{R}^m$ by

$$Q_K(\mathbf{a}) = (Q_K(a_1),\cdots,Q_K(a_m))^T.$$

Note that to transmit the quantized information from the quantizer $Q_K(\mathbf{a})$ for $\mathbf{a} \in \mathbb{R}^m$ , we need to transmit $m\log_2(2K+1)$-bit information.

We propose a pair of encoder and decoder for each node to generate codes for its state, and to estimate the neighbors' states. More precisely, we suppose each node has a global scaling function $s(k)$ that monotonely decreases to zero at a suitable rate. We use $s(k)$ to zoom-in each node's quantization error to increase the accuracy of state estimation as time evolves, and the fact that the quantizer is not time-invariant (which is perhaps surprising) allows us to achieve the consensus requirement. We continue to use $\mathbf{x}_i(k)$ to denote the exact state of node $i$ at time $k$, whose evolution will be specified at a later stage.

**[Encoder]** The encoder of node $j \in V$ recursively generates quantized outputs $\{\mathbf{q}_j(k)\}$ and internal states $\{\mathbf{b}_j(k)\}$ from the exact state sequence $\{\mathbf{x}_j(k)\}$ as follows for any $k \ge 1$:

$$\begin{aligned} \mathbf{q}_j(k) &\triangleq Q_K\left(\frac{1}{s(k-1)}(\mathbf{x}_j(k) - \mathbf{b}_j(k-1))\right), \\ \mathbf{b}_j(k) &\triangleq s(k-1)\mathbf{q}_j(k) + \mathbf{b}_j(k-1). \end{aligned} \tag{3}$$

where the initial value $\mathbf{b}_j(0) = 0$. At time $k$, node $j$ sends the quantized state $\mathbf{q}_j(k)$ to each of its neighboring nodes $i \in \mathrm{N}_j$.

**[Decoder]** When node $i \in \mathrm{N}_j$ receives the quantized data $\mathbf{q}_j(k)$ from node $j$, a decoder recursively generates an estimate $\hat{\mathbf{x}}_{ji}(k)$ for $\mathbf{x}_j(k)$ by the following for any $k \ge 1$:

$$\hat{\mathbf{x}}_{ji}(k) \triangleq s(k-1)\mathbf{q}_j(k) + \hat{\mathbf{x}}_{ji}(k-1), \tag{4}$$

where the initial value $\hat{\mathbf{x}}_{ji}(0) \triangleq 0$.

**[Algorithm]** Motivated by the "consensus + projection" flow presented in [2] but reflecting the presence of quantized signals, we propose the following recursion for $\mathbf{x}_i(k)$:

$$\begin{aligned} \mathbf{x}_i(k+1) = \mathbf{x}_i(k) &+ h\sum_{j\in\mathrm{N}_i}\left(\hat{\mathbf{x}}_{ji}(k) - \mathbf{b}_i(k)\right) \\ &- \gamma\left(\frac{\mathbf{h}_i\mathbf{h}_i^\top}{\mathbf{h}_i^\top\mathbf{h}_i}\mathbf{x}_i(k) - \frac{z_i\mathbf{h}_i}{\mathbf{h}_i^\top\mathbf{h}_i}\right), \end{aligned} \tag{5}$$

where $h$ and $r$ are positive constants.

The algorithm (5) clearly relies on quantized node communication only since $\mathbf{q}_j(k)$ takes values in

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$\{-K, \ldots, -1, 0, 1, \ldots, K\}$ only. From the second equation of (3), from (4) and the assumed initial conditions of zero for $\hat{\mathbf{x}}_{ji}(0)$ and $\mathbf{b}_j(0)$, we have the following for any $k \geq 0$:

$$\hat{\mathbf{x}}_{ji}(k) = \mathbf{b}_j(k) \quad j \in V,\ i \in \mathrm{N}_j. \tag{6}$$

### B. Convergence Result

We introduce a few useful notations as follows:

$$\begin{aligned}
&\mathbf{x}(k) = col\{\mathbf{x}_1(k), \cdots, \mathbf{x}_N(k)\}, \\
&\mathbf{q}(k) = col\{\mathbf{q}_1(k), \cdots, \mathbf{q}_N(k)\}, \\
&\mathbf{w}_i(k) = \mathbf{x}_i(k) - \mathbf{y}^*, \\
&\mathbf{w}(k) = col\{\mathbf{w}_1(k), \cdots, \mathbf{w}_N(k)\}, \\
&\mathbf{e}_i(k) = \mathbf{x}_i(k) - \mathbf{b}_i(k), \\
&\mathbf{e}(k) = col\{\mathbf{e}_1(k), \cdots, \mathbf{e}_N(k)\}, \\
&\mathbf{H_{sd}} = diag\left\{\frac{\mathbf{h}_1\mathbf{h}_1^\top}{\mathbf{h}_1^\top\mathbf{h}_1}, \cdots, \frac{\mathbf{h}_N\mathbf{h}_N^\top}{\mathbf{h}_N^\top\mathbf{h}_N}\right\} \in \mathbb{R}^{mN\times mN}.
\end{aligned} \tag{7}$$

Define

$$\mathbf{P}_{\gamma,h} := \mathbf{I}_{mN} - \left(h\left(\mathbf{L}\otimes\mathbf{I}_m\right) + \gamma\mathbf{H_{sd}}\right).$$

We impose the following two assumptions.

**A1** There exists a unique solution satisfying (1): $\mathrm{rank}(\mathbf{H}) = m$ and $\mathbf{z} \in \mathrm{span}(\mathbf{H})$.

**A2** $\max_i \|\mathbf{x}_i(0)\|_\infty \leq C_x$ and $\max_i \|\mathbf{w}_i(0)\|_\infty \leq C_w$, where $C_x$ and $C_w$ are known constants.

Note that both the Laplacian $\mathbf{L}$ and $\mathbf{H_{sd}}$ are positive semi-definite. With the assumption **A1** and the condition that the undirected graph G is connected, the matrix $h\left(\mathbf{L}\otimes\mathbf{I}_m\right) + \gamma\mathbf{H_{sd}}$ turns out to be positive definite [3]. As a result, we can well define a nonempty set

$$\Xi := \Big\{(h,\gamma) : 0 < \lambda < 1,\ \forall\lambda\in\sigma\big(\mathbf{P}_{\gamma,h}\big)\Big\}.$$

For any $(h,\gamma)\in\Xi$, the corresponding eigenvalues of $\mathbf{P}_{\gamma,h}$ are sorted in an ascending order as $0 < \lambda_1 \leq \cdots \leq \lambda_{mN} < 1$. Then there exists a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}^T\mathbf{P}_{\gamma,h}\mathbf{U} = diag\{\lambda_1, \cdots, \lambda_{mN}\} \triangleq \Lambda.$$

Thus,

$$\left(\mathbf{P}_{\gamma,h}\right)^k = \left(\mathbf{U}\Lambda\mathbf{U}^T\right)^k = \mathbf{U}\Lambda^k\mathbf{U}^T. \tag{8}$$

Define $K_{\alpha,h,\gamma} := \lceil M_{\alpha,h,\gamma}\rceil$ for some $\alpha \in (\lambda_{mN}, 1)$ with

$$\begin{aligned}
M_{\alpha,h,\gamma} :=& \frac{\|\mathbf{I}_N + h\mathbf{L}\|_\infty}{2\alpha} + \frac{h\sqrt{mN}\|\mathbf{L}\|_2}{2\alpha(\alpha-\lambda_{mN})}\|h\left(\mathbf{L}\otimes\mathbf{I}_m\right) \\
&+ \gamma\mathbf{H_{sd}}\|_\infty.
\end{aligned} \tag{9}$$

We are now ready to present our main result on the performance of the algorithm (5) as an exact solver of the linear equation (1).

*Theorem 1:* Suppose **A1** and **A2** hold. Let $s(k) \triangleq s(0)\alpha^k\ \forall k\geq 0$ for some $\alpha\in(\lambda_{mN},1)$, and $(h,\gamma)\in\Xi$. Then for any $K \geq K_{\alpha,h,\gamma}$, along the algorithm (5) with the encode-decoder given by (3) and (4) there hold

$$\lim_{k\to\infty} x_i(k) = \mathbf{y}^* \quad \forall i\in V, \text{ and} \tag{10}$$

$$\lim_{k\to\infty}\frac{\|\mathbf{w}(k)\|_2}{\alpha^k} \leq \frac{hs(0)\sqrt{mN}\|\mathbf{L}\|_2}{2\alpha(\alpha-\lambda_{mN})} \tag{11}$$

provided that $s(0)$ satisfies

$$\begin{aligned}
s(0) > \max\Bigg\{&\frac{C_x + \gamma\|\mathbf{H_{sd}}\|_\infty C_w}{K+\frac{1}{2}}, \\
&\frac{2(\alpha-\lambda_{mN})\left(\alpha C_w + hC_x\|\mathbf{L}\|_2\right)}{h\|\mathbf{L}\|_2}\Bigg\}.
\end{aligned} \tag{12}$$

The proof is established based on a key observation that with (12), the quantization function at each individual node will be restricted in the interval $[-K-1/2, K+1/2]$ along the entire state evolution. This allows for a compact node state space which ensures convergence to a consensus. Finally, this consensus value can only be the solution of the linear equation (1) due to the structure of the proposed algorithm.

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