Distributed Algorithm for Achieving Minimum $l_1$ Norm Solutions of Linear Equation

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Abstract—This paper proposes a distributed algorithm for multi-agent networks to achieve a minimum $l_1$-norm solution to a linear equation $Ax = b$ where $A$ has full row rank. When the underlying network is undirected and fixed, it is proved that the proposed algorithm drive all agents’ individual states to converge in finite-time to the same minimum $l_1$-norm solution. Numerical simulations are also provided as validation of the proposed algorithms.

I. INTRODUCTION

A significant amount of effort in the control community has recently been given to distributed algorithms for solving linear equations over multi-agent networks, in which each agent only knows part of the equation and controls a state vector that can be looked at as an estimate of the solution of the overall linear equations [1]–[5]. Numerous extensions along this direction include achieving solutions with the minimum Euclidean norm [6], [7], elimination of the initialization step [8], reduction of state vector dimension by utilizing the sparsity of the linear equation [9] and achieving least square solutions [10]–[15]. All these algorithms yield asymptotic convergence, but require an infinite number of sensing or communication events.

Solutions to underdetermined linear equations with the minimum $l_1$ norm are perhaps the most important in many engineering applications including earthquake location detection [16], analysis of statistical data [17], solving biogentic inverse problems [18], and so on. One most intriguing case among these applications is compressive sensing, which enables transmission of sparse data in a very efficient way [19]. The decoding process of compressive sensing requires solving of linear equations with a minimum number of non-zero entries of the solution vectors, which, however, is an NP-hard problem and usually computationally costly [20]. Thus researchers usually turn to achieve solutions with minimum $l_1$ norm instead for which the function to be minimized is convex [21], [22]. Most existing results for achieving minimum $l_1$ norm solutions are based on the idea of Lasso including Alternating Direction Method of Multipliers (ADMM) [24], the Primal-Dual Interior-Point Method [25], [26], Gradient Projection Methods [27], Homotopy Methods [28], Iterative Shrinkage-Thresholding Methods [29] and Proximal Gradient Methods [30].

In this paper we aim to develop distributed algorithms for multi-agent networks to achieve in finite time a solution of linear equations with the minimum $l_1$ norm. By distributed is meant that each agent only knows part of the overall linear equation and can communicate with only its nearby neighbors. The problem of interest is formulated in Section II. We introduce in section III the concepts to be employed in the paper including Filippov set-valued maps, Filippov solutions, generalized Lie derivatives, based on which a preliminary result is achieved. In Section IV, we will first propose a centralized update for achieving a solution with the minimum $l_1$ norm. Motivated by the projection-consensus flow proposed in [13] and the finite-time gradient flow for consensus devised in [31]–[34], we utilize a combination of the finite time consensus flow and the proposed centralized algorithm to develop a distributed linear equation solver for achieving a minimum $l_1$ norm solution, which is shown to converge in finite time. We provide simulations in Section V and concluding remarks in Section VI.

Notation: Let $r$ denote an arbitrary positive integer. Let $1_r$ denote the vector in $\mathbb{R}^r$ with all entries equal to 1s. Let $I_r$ denote the $r \times r$ identity matrix. We let $\text{col} \{A_1, A_2, \ldots, A_r\}$ be a stack of matrices $A_i$ possessing the same number of columns with the index in a top-down ascending order, $i = 1, 2, \ldots, r$. Let diag $\{A_1, A_2, \ldots, A_r\}$ denote a block diagonal matrix with $A_i$ the $i$th diagonal block entry, $i = 1, 2, \ldots, r$. By $M > 0$ and $M \geq 0$ are meant that the square matrix $M$ is positive definite and positive semi-definite, respectively. By $M^T$ is meant the transpose of a matrix $M$. Let ker $M$ and image $M$ denote the kernel and image of a matrix $M$, respectively. Let $\otimes$ denote the Kronecker product.

II. PROBLEM FORMULATION

Consider a network of $m$ agents, $i = 1, 2, \ldots, m$; inside this network, each agent can observe states of certain other agents called its neighbors. Let $N_i$ denote the set of agent $i$’s neighbors. We assume that the neighbor relation is symmetric, that is, $j \in N_i$ if and only if $i \in N_j$. Then all
these neighbor relations can be described by an \( m \)-node-\( \bar{m} \)-edge undirected graph \( \mathbb{G} \) such that there is an undirected edge connecting \( i \) and \( j \) if and only if \( i \) and \( j \) are neighbors. In this paper we only consider the case in which \( \mathbb{G} \) is connected, fixed and undirected.

Suppose that each agent \( i \) knows \( A_i \in \mathbb{R}^{n_i \times n} \) and \( b_i \in \mathbb{R}^{n_i} \) and controls a state vector \( y_i(t) \in \mathbb{R}^{n_i} \). Then all these \( A_i \) and \( b_i \) can be stacked into an overall equation \( Ax = b, \) where \( A = \text{col} \{ A_1, A_2, \ldots, A_m \}, \) \( b = \text{col} \{ b_1, b_2, \ldots, b_m \}. \) Without loss of generality for the problems of interest to us, we assume \( A \) to have full row-rank. Let \( \bar{x}^* \) denote a minimum \( l_1 \)-norm solution to \( Ax = b, \) that is,

\[
\bar{x}^* = \arg \min_{Ax=b} \|x\|_1 \tag{1}
\]

The problem of interest in this paper is to develop distributed algorithms for each agent \( i \) to update its state vector \( y_i(t) \) by only using its neighbors’ states such that all \( y_i(t) \) to converge in finite time to a common minimum \( l_1 \)-norm solution \( \bar{x}^* \).

III. KEY CONCEPTS AND PRELIMINARY RESULTS

Before proceeding, we introduce some key concepts and preliminary results for future derivation and analysis. Key references for the background we summarize are [35] and [36].

A. Filippov set-valued maps and Filippov Solutions

By a Filippov set-valued map \( F[f] : \mathbb{R}^r \rightarrow \mathcal{B} \subset \mathbb{R}^r \) associated with a function \( f : \mathbb{R}^r \rightarrow \mathbb{R}^r \) is meant

\[
F[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \overline{cv}\{f(B(x, \delta)) / S\} \tag{2}
\]

Here \( B(x, \delta) \) stands for the open ball on \( \mathbb{R}^r \), whose center is at \( x \) and has a radius of \( \delta \); \( \mu(S) \) denotes the Lebesgue measure of \( S \); and \( \overline{cv} \) stands for the convex closure. Let \( \text{sgn} \) \( : \mathbb{R}^r \rightarrow \mathbb{R}^r \) be a function with the \( k \)-th entry, \( k = 1, 2, \ldots, r \), defined as

\[
(\text{sgn} \ (x))_k = \begin{cases} 
1, & (x)_k > 0; \\
-1, & (x)_k < 0; \\
0, & (x)_k = 0.
\end{cases} \tag{3}
\]

It follows that the Filippov set-valued map \( F[\text{sgn}] \) \( (x) \) for \( x \in \mathbb{R}^r \) is defined entrywise as:

\[
(F[\text{sgn}] \ (x))_k = \begin{cases} 
1, & (x)_k > 0; \\
[-1,1], & (x)_k = 0; \\
-1, & (x)_k < 0.
\end{cases} \tag{4}
\]

for \( k = 1, 2, \ldots, r \). Note that even if \( (x)_i = (x)_j = 0 \), the \( i \)-th and \( j \)-th entries of a vector in \( F[\text{sgn}] \ (x) \) may not necessarily be equal since each of them could be chosen as arbitrary values in the interval \([-1,1]\). From the definition of \( F[\text{sgn}] \ (x) \), one can verify that

\[
q^T x = \|x\|_1, \quad \forall q \in F[\text{sgn}] \ (x) \tag{5}
\]

While for any \( w \in \mathbb{R}^r \), there holds

\[
q^T w \leq \|w\|_1 \tag{6}
\]

By a Filippov solution for \( \hat{x} \in F[f](x) \) is meant a Caratheodory solution \( x(t) \) such that \( \hat{x} \in F[f](x) \) for almost all \( t \), \( x(t) \) is absolutely continuous and can be written in the form of an indefinite integral. The following two lemmas treat existence of such a Filippov solution.

**Lemma 1:** (Proposition 3 in [35]) If \( f : \mathbb{R}^r \rightarrow \mathbb{R}^r \) is measurable and locally bounded, then for any initial point \( x_0 \in \mathbb{R}^r \), there exists a Filippov solution\(^1\) for \( \hat{x} \in F[f](x) \).

**Lemma 2:** (Theorem 8 in page 85 of [36]) Let a vector-valued \( f(t,x) \) be defined almost-everywhere in the domain \( G \) of time space \( (t,x) \). With \( f(t,x) \) measurable and locally bounded almost-everywhere in an open domain \( G \), let

\[
F[f](t,x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \overline{cv}\{f(t,B(x, \delta)) / S\}. \tag{7}
\]

Then for any point \( (t_0, x_0) \in G \), there exists a Filippov solution of \( \hat{x} \in F[f](t, x) \) with \( x(t_0) = x_0 \). Note that Lemma 2 establishes the existence of a solution for time-varying systems. This is more general than Lemma 1, which only guarantees the existence of solutions to time-invariant systems.

B. Generalized Gradients and Generalized Lie Derivatives

For a locally Lipschitz function \( w : \mathbb{R}^r \rightarrow \mathbb{R} \), the generalized gradient of \( w \) is

\[
\partial w(x) \triangleq \text{col} \{ \lim_{i \to \infty} \nabla w(x_i) : x_i \rightarrow x, x_i \notin S \cup \Omega_w \} \tag{8}
\]

where \( S \subset \mathbb{R}^r \) is an arbitrarily chosen set of measure zero, \( \Omega_w \) denotes the set of points at which \( w \) is not differentiable, and \( \text{co} \) denotes convex hull. Specially, for the function \( \|x\|_1 \), one computes the \( k \)-th element of its generalized gradient to be:

\[
(\partial \|x\|_1)_k = \begin{cases} 
1, & x_k > 0; \\
[-1,1], & x_k = 0; \\
-1, & x_k < 0.
\end{cases} \tag{9}
\]

It follows from this and the definition of \( F[\text{sgn}] \) \( (x) \) in (4) that

\[
F[\text{sgn}] \ (x) = \partial \|x\|_1 \tag{10}
\]

For a set-valued map \( F : \mathbb{R}^r \rightarrow \mathbb{B}(\mathbb{R}^r) \), the generalized Lie derivative of \( w \) is defined as

\[
\mathcal{L}_F w(x) = \{ q \in \mathbb{R} : \text{there exists } \alpha \in F(x) \text{ such that } \forall \beta \in \partial w(x), \beta \in \alpha \} \tag{11}
\]

The above definition of generalized Lie derivative implies that for each \( \alpha \in F(x) \), we check if the inner product \( \beta \alpha \) is a fixed value for all \( \beta \in \partial w(x) \). If so, this inner product is an element in \( \mathcal{L}_F w(x) \), but note that the set \( \mathcal{L}_F w(x) \) may be empty. Moreover, for locally Lipschitz and regular(see [37], p.39 and [38], p.3 for detailed discussion of regular function\(^2\)) functions \( w(x) \), one has the following lemma:

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\(^1\)There is no implication that the solution exists on an infinite interval

\(^2\)That a function \( w(x) : \mathbb{R}^r \rightarrow \mathbb{R} \) is called regular at \( x \in \mathbb{R}^r \) if

1) for all \( v \in \mathbb{R}^r \) there exists the usual right directional derivative \( w'_x(v) \).

2) for all \( v \in \mathbb{R}^r, w'_x(v, v) = w''(x, v). \)
Lemma 3: (Proposition 10 in [35]) Let \( x : [0, t_1] \rightarrow \mathbb{R}^r \) be a solution for \( \dot{x} \in F(x(t)) \), where \( F \) is any set-valued map. Then \( w(x(t)) \) is differentiable at almost all \( t \in [0, t_1] \); The derivative of \( w(x(t)) \) satisfies \( \frac{dw(x(t))}{dt} \in \hat{L}xw(x(t)) \) for almost all \( t \in [0, t_1] \).

Lemma 3 guarantees the existence of generalized Lie derivatives for functions that are locally Lipschitz and regular. If one focuses on a specific solution, one can show that \( \alpha \) in (11) is a special vector as summarized in the following lemma.

Lemma 4: (See Proof of Lemma 1 in [38]) Let \( x(t) \) denote a specific solution of a differential enclosure. Suppose \( w(x) \) is locally Lipschitz and regular. Let \( \mathcal{I} \subset [0, \infty) \) denote the time interval for which \( \dot{x}(t) \) exists. Then
\[
\frac{dw(x(t))}{dt} = \beta^T \dot{x}(t)
\]
where \( \beta \) is any vector in \( \partial w(x) \).

C. Preliminary Results

For any positive semi-definite matrix \( M \in \mathbb{R}^{r \times r} \), \( M \neq 0 \) one can define
\[
\Phi(M) = \{ q \in \mathbb{R}^r \mid \exists \phi \in F[\text{sgn} (q)], M \phi(q) = 0 \}
\]
and its compliment
\[
\Phi_c(M) = \{ q \in \mathbb{R}^r \mid \forall \phi \in F[\text{sgn} (q)], M \phi(q) \neq 0 \}. \tag{14}
\]
We impose a further requirement on \( M \), namely that \( \Phi_c(M) \) is nonempty. This is easily ensured, see below. Let
\[
\Lambda(M) = \{ \phi \mid \exists q \in F[\text{sgn} ](q), q \in \Phi_c(M) \}. \tag{15}
\]
Now \( F[\text{sgn} ](q) \) is a closed set for any fixed \( q \); also note that \( F[\text{sgn} ](q) \) can only be one of a finite number of different sets; hence it is easy to check for a given \( M \) whether \( \Phi_c(M) \) is nonempty, (and in a later use of the result, it proves easy to check). It further follows that \( \Lambda(M) \) is also a closed set. Consequently, the continuous function \( f(\phi) = \phi^T M \phi \) has a nonzero minimum on \( \Lambda(M) \). We denote
\[
\lambda(M) = \min_{\phi \in \Lambda(M)} f(\phi). \tag{16}
\]
From the definition of \( \Phi_c(M) \) and \( \Lambda(M) \), one has \( \lambda(M) > 0 \). To summarize, one has the following lemma:

Lemma 5: For any nonnegative-definite matrix \( M \), we let \( \Phi_c(M) \), \( \Lambda(M) \) and \( \lambda(M) \) defined as above. Suppose that \( \Phi_c(M) \) is nonempty. Then \( \lambda(M) \) is a positive constant.

For the \( m \)-node-\( m \)-edge graph \( \mathbb{G} \), we label all its nodes as 1, 2, \ldots, \( m \) and all its edges as 1, 2, \ldots, \( m \). Assign an arbitrary direction to each edge in \( \mathbb{G} \). Then the incidence matrix of \( \mathbb{G} \) denoted by \( H = [h_{ik}]_{m \times m} \) is defined as follows
\[
h_{ik} = \begin{cases} 
1, & i \text{ is the head of the } k \text{th edge;} \\
-1, & i \text{ is the tail of the } k \text{th edge; } \\
0, & \text{otherwise}. 
\end{cases} \tag{17}
\]
Since \( \mathbb{G} \) is connected, then \( \ker H' \) is the span of \( 1_m \) [39]. Moreover, one has the following lemma:

Lemma 6: Suppose \( A \) has full-row rank and \( \mathbb{G} \) is connected. Let \( \hat{P} = \text{diag} \{ P_1, P_2, \ldots, P_m \} \) where each \( P_i \) is the projection matrix to \( \ker A_i \). Let \( H = H' \otimes I_n \) with \( H \) the incidence matrix of \( \mathbb{G} \). Then one has
\[
\text{image } H \cap \ker \hat{P} = 0 \tag{18}
\]
and
\[
\text{image } H' \cap \Phi(H' \hat{P} H) = 0. \tag{19}
\]
For any \( q \in \text{image } H' \cap \Phi(H' \hat{P} H) \), there exists a vector \( p \) such that
\[
q = \hat{H}' p \tag{20}
\]
and a vector \( \phi \in F[\text{sgn} ](q) \) such that \( \hat{H}' \hat{P} \hat{H} \phi = 0 \). Note that \( \hat{P} \) is a projection matrix; then
\[
\hat{P} \hat{H} \phi = 0. \tag{21}
\]
From (18) one then has
\[
\hat{H} \phi = 0. \tag{22}
\]
From \( \phi \in F[\text{sgn} ](q) \) and (5), one has
\[
\|q\|_1 = \phi^T q
\]
which together with (20) and (22) implies \( \|q\|_1 = 0 \). Then one has \( q = 0 \) and (19) is true. \hfill \blacksquare

Now consider the system
\[
\dot{x} \in -M F[\text{sgn} ](x) \tag{23}
\]
with any positive semi-definite matrix \( M \in \mathbb{R}^{r \times r} \). The existence of a Filippov solution to (23) can be guaranteed by Lemma 1. The existence interval is \( t \in [0, \infty) \) because of the global bound on the right-hand side to (23). Let \( x(t) \) denote such a Filippov solution for any given \( x(0) \). Note that the function \( \|x\|_1 \) is locally Lipschitz and regular (The word was introduced early with reference). Then by Lemma 3, the time derivative of \( \|x(t)\|_1 \) exists for almost all \( t \in [0, \infty) \) and is in the set of generalized Lie derivatives. In other words, there exists a set \( \mathcal{T} = [0, \infty) \setminus \mathcal{T} \) of Lebesgue measure 0 such that
\[
\frac{d\|x(t)\|_1}{dt} \text{ exists for all } t \in \mathcal{I} \tag{24}
\]

Proposition 1: Let \( x(t) \) denote a Filippov solution to (23) for any given \( x(0) \in \mathbb{R}^r \). Then
\[
\frac{d\|x(t)\|_1}{dt} \leq 0, \quad t \in \mathcal{I}; \tag{25}
\]
there exists a finite time \( T \) such that
\[
x(T) \in \Phi(M); \tag{26}
\]
if it is further true that
\[
\frac{d\|x(t)\|_1}{dt} = 0, \quad t \in [T, \infty) \setminus \mathcal{T}, \tag{27}
\]
then one has
\[
x(t) = x(T) \quad t \in [T, \infty) \tag{28}
\]

Remark 1: Note that the right-hand side of (23) is a projection of the gradient flow of the potential function \( \|x(t)\|_1 \).
It is also standard that the gradient law of a real analytic function with a lower bound converges to a single point which is both a local minimum and a critical point of the potential function [40]. However, if the real analytic property does not hold, the convergence result may fail. Indeed, the function \( \|x(t)\|_1 \) here is obviously not real analytic, and one cannot immediately assert that (23) will drive \( \|x(t)\|_1 \) to its minimum, not to mention the finite time result in Proposition 1. Thus Proposition 1 is nontrivial and will serve as the foundation for devising finite-time distributed linear equation solvers in this paper.

IV. ALGORITHMS AND MAIN RESULTS

In this section, we first present a centralized update for minimum \( l_1 \)-norm solutions to \( Ax = b \) and then devise a distributed one for all agents in the network to achieve the same minimum \( l_1 \)-norm solution in finite time.

A. Finite-Time Centralized Update for Minimum \( l_1 \)-norm Solution

In this subsection, we will propose a centralized update for achieving a minimum \( l_1 \)-norm solution to \( Ax = b \). By noting that \( \|x\|_1 \) is convex, we conceive of using a negative gradient flow of \( \|x\|_1 \) subject to \( x \) remaining on the manifold \( Ax = b \) in order to achieve \( \hat{x}^* = \arg\min_{Ax=b} \|x\|_1 \). This leads us to the following update:

\[
\dot{y} = -PF[\text{sgn}](y), \quad Ay(0) = b. \tag{29}
\]

where \( P \) denotes the projection matrix onto the kernel of \( A \). Again by Lemma 1 one has there exists a Filippov solution to system (29) for \( t \in [0, \infty) \), which we denote by \( y(t) \). By Lemma 3, there exists a set \( \mathcal{I} = [0, \infty) \setminus \mathcal{T} \) with \( \mathcal{T} \) of measure 0 such that \( \frac{d[y(t)]}{dt} \) exists for all \( t \in \mathcal{I} \). Moreover, we have the following main theorem:

**Theorem 1:** With \( A \) of full row rank, the Filippov solution \( y(t) \) to (29) converges in finite time to a constant, which is a minimum \( l_1 \)-norm solution to \( Ax = b \).

B. Finite-Time Distributed Update for Minimum \( l_1 \)-norm Solutions

In this subsection we will develop a distributed update for a multi-agent network to achieve a minimum \( l_1 \)-norm solution to \( Ax = b \) in finite time. Of course, \( A \) is assumed to have full row rank, but not necessarily be square. Motivated by study a combination of the finite-time consensus flow [34] and the finite-time centralized update for minimum \( l_1 \)-norm solutions in (29), we propose the following update for agent \( i, i = 1, 2, \ldots, m \):

\[
\dot{y}_i = -k(t)P_i \phi_i - P_i \sum_{j \in \mathcal{N}_i} \phi_{ij} \tag{30}
\]

where \( \phi_i \in F[\text{sgn}](y_i), \phi_{ij} \in F[\text{sgn}](y_i - y_j) \). Since the graph is undirected we can make the following assumption

**Assumption 1:** Each pair of neighbor agents \( i \) and \( j \) takes the choice of \( \phi_{ij} \) and \( \phi_{ji} \) when \( y_i = y_j \) such that

\[
\phi_{ij} = -\phi_{ji} \tag{31}
\]

and take the initialization as

\[
A_i y_i(0) = b_i. \tag{32}
\]

We assume that \( k(t) \in \mathbb{R} \) is measurable and locally bounded almost everywhere for \( t \in [0, \infty) \), and

\[
\lim_{t \to \infty} k(t) = \delta \tag{33}
\]

\[
\int_0^\infty k(t)dt = \infty \tag{34}
\]

where \( \delta \) is a sufficiently small nonnegative number depending on the connection of the network and \( A \), note that 0 is always a feasible choice of \( \delta \). One example of a choice of \( k(t) \) is \( k(t) = \frac{3}{\sqrt{1 + t^2}} + \delta \). One simple case is choosing \( \delta = 1, \delta = 0 \), and resulting in \( k(t) = \frac{3}{\sqrt{1 + t^2}} \), obtained by taking \( \delta \) to be zero. This choice obviates the need to decide how small one has to be to meet a “sufficiently small” condition, but may result in rather slow convergence. Now from \( A_i y_i(0) = b_i \) and the fact that \( P_i \) is the projection to the kernel of \( A_i \) (which ensures \( y_i \in \ker A_i \)), one has

\[
A_i y_i(t) = b_i, \quad t \in [0, \infty) \tag{35}
\]

Let \( y = \col \{ y_1, y_2, \ldots, y_m \} \), \( \hat{P} = \text{diag} \{ P_1, P_2, \ldots, P_m \} \), and \( \hat{H} = H \otimes I_n \) with \( H \) the incidence matrix of \( \mathcal{G} \). From the updates in (30) and Assumption 1, one has

\[
\dot{y} = -k(t)\hat{P}F[\text{sgn}](y) - \hat{P}\hat{H}F[\text{sgn}](\hat{H}y) \tag{36}
\]

Note that \( \text{sgn}(y), k(t) \) and \( \hat{P}\hat{H}F[\text{sgn}](\hat{H}y) \) are measurable and locally bounded almost everywhere. Then by Lemma 2 there exists a Filippov solution to system (36) for any given \( y(0) \) satisfying (32), which we denote by \( y(t) = \col \{ y_1(t), y_2(t), \ldots, y_m(t) \} \). By Lemma 3, there exists a set \( \mathcal{I} = [0, \infty) \setminus \mathcal{T} \) with \( \mathcal{T} \) of Lebesgue measure 0 such that \( \frac{d[y(t)]}{dt} \) exists for all \( t \in \mathcal{I} \). Then one has the following theorem:

**Theorem 2:** Under Assumption 1 and the update (30) and with \( A \) of full row rank, all \( y_i(t), i = 1, 2, \ldots, m \) converge in finite time to the same value which is a minimum \( l_1 \)-norm solution to \( Ax = b \).

V. SIMULATION RESULT

In this section, we will report several simulations of the proposed algorithms for solving an underdetermined linear equation \( Ax = b \) in a four-agent undirected and connected network as in Figure 1.

Here, \( A \) and \( b \) are partitioned as \( A = [A'_1 A'_2 A'_3 A'_4] \) and \( b = [b'_1 b'_2 b'_3 b'_4]^T \), respectively. Each agent \( i \) knows
$A_i$ and $b_i$ with

\[
A_1 = \begin{bmatrix}
0.63 & 0.04 \\
0.58 & 0.60 \\
0.68 & 0.01 \\
0.22 & 0.51 \\
0.34 & 0.10 \\
0.49 & 0.23 \\
0.21 & 0.29 \\
0.62 & 0.25 \\
0.57 & 0.21 \\
0.71 & 0.66 \\
0.28 & 0.90
\end{bmatrix}, \quad A'_1 = \begin{bmatrix}
0.80 & 0.99 \\
0.25 & 0.65 \\
0.53 & 0.38 \\
0.79 & 0.12 \\
0.13 & 0.76 \\
0.34 & 0.55 \\
0.45 & 0.24 \\
0.10 & 0.55 \\
0.94 & 0.51 \\
0.78 & 0.58 \\
0.70 & 0.85
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
0.44 & 0.34 \\
0.66 & 0.94 \\
0.77 & 0.28 \\
0.16 & 0.41 \\
0.84 & 0.75 \\
0.62 & 0.56 \\
0.74 & 0.41 \\
0.26 & 0.89 \\
0.44 & 0.69 \\
0.28 & 0.23 \\
0.50 & 0.88 \\
0.38 & 0.63
\end{bmatrix}, \quad A'_2 = \begin{bmatrix}
0.05 & 0.23 \\
0.09 & 0.33 \\
0.65 & 0.92 \\
0.69 & 0.66 \\
0.94 & 0.92 \\
0.73 & 0.06 \\
0.51 & 0.13 \\
0.59 & 0.94 \\
0.76 & 0.40 \\
0.95 & 0.69 \\
0.59 & 0.24 \\
0.03 & 0.92
\end{bmatrix},
\]

\[
b'_1 = [0.47, 0.52], \quad b'_2 = [0.77, 0.34],
\]

\[
b'_3 = [0.63, 0.33], \quad b'_4 = [0.31, 0.65].
\]

**Example 1:** We employ the centralized update (29) with state vector $y(t)$ to achieve $\bar{x}^*$ which denotes a minimum $l_1$-norm solution to $Ax = b$. As shown in Figure 2, $\|y(t) - \bar{x}^*\|_1$ reaches 0 in finite time and maintains to be 0 afterwards. This indicates that a minimum $l_1$-norm solution $\bar{x}^*$ is achieved in finite time corresponding to Theorem 1. It is worth noting that one could observe multiple phases of convergence in Figure 2. This is because $F[\text{sgn} (\{y(t)\})]$ in the update (29) takes different values piece-wisely, and results in different convergence rates.

**Example 2:** Finally, we utilize the distributed update (30) to achieve a minimum $l_1$ solution to $Ax = b$ denoted by $\bar{x}^*$ in finite time. Here $k(t)$ is chosen to take the form $\frac{\delta}{\delta t} + \delta$ with $\delta$ and $\tilde{\delta}$ constants. We still let $y_i(t)$ denote the state of agent $i$ that is the estimate of agent $i$ to $\bar{x}^*$. Then $\|y(t) - 1_m \otimes \bar{x}^*\|_1$ measures the difference between all agents’ estimations and $\bar{x}^*$. As shown in Figure 3 and Figure 4, all $y_i(t)$ reach the same minimum $l_1$-norm solution in finite time regardless of different choices of $\delta$ and $\tilde{\delta}$. Moreover, by fixing $\delta$ and increasing the value of $\delta$ in $k(t)$, one achieves a significantly faster convergence as shown in Figure 3. Similarly, increasing $\delta$ with a fixed $\tilde{\delta}$ also leads to a faster convergence, although not that dramatically, as shown in Figure 4.

We also note from Figure 3 and Figure 4 that the convergence time required in this distributed way for minimum $l_1$-norm solutions is dramatically longer, roughly speaking, $\frac{1}{\delta}$ times longer, than that in the centralized case in Figure 2. The major reason for this is that the centralized update appearing in the distributed update (30) is scaled by $k(t)$, which is smaller than 1.

**VI. Conclusion**

We have developed continuous-time distributed algorithms for achieving minimum $l_1$-norm solutions to linear equations $Ax = b$ in finite time. The algorithms result from combination of the projection-consensus flow proposed in [13] and the finite-time gradient flow for consensus devised in [34], and work for fixed undirected multi-agent networks. Future work includes the generalization of the proposed update to networks that are directed and time-varying.

**References**

Fig. 4. Distributed solver for achieving minimum $l_1$ norm solution under update (30) where $k(t) = \bar{\delta} t + \delta$ with fixed $\delta = 0.01$ and different values of $\delta$.


