Leader Tracking of Euler-Lagrange Agents on Directed Switching Networks Using A Model-Independent Algorithm

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Abstract—In this paper, we propose a discontinuous distributed model-independent algorithm for a directed network of Euler-Lagrange agents to track the trajectory of a leader with non-constant velocity. We initially study a fixed network and show that the leader tracking objective is achieved semi-globally exponentially fast if the graph contains a directed spanning tree. By model-independent, we mean that each agent executes its algorithm with no knowledge of the parameter values of any agent’s dynamics. Certain bounds on the agent dynamics (including any disturbances) and network topology information are used to design the control gain. This fact, combined with the algorithm’s model-independence, results in robustness to disturbances and modeling uncertainties. Next, a continuous approximation of the algorithm is proposed, which achieves practical tracking with an adjustable tracking error. Last, we show that the algorithm is stable for networks that switch with an explicitly computable dwell time. Numerical simulations are given to show the algorithm’s effectiveness.

Index Terms—model-independent, euler-lagrange agent, directed graph, distributed algorithm, tracking, switching network

I. INTRODUCTION

COORDINATION of multi-agent systems using distributed algorithms has been widely studied over the past decade [1]. Of recent interest is the study of agents whose dynamics are described using Euler-Lagrange equations of motion, which from here onward will be referred to as Euler-Lagrange agents (in some literature known as Lagrangian agents). The non-linear Euler-Lagrange equation can be used to model the dynamics of a large class of mechanical, electrical and electromechanical systems [2]. Thus, there is significant motivation to study coordination problems with multiple Euler-Lagrange agents. The interacting agents collectively form a network, and may be modeled using a graph [1]. Directed networks capture unilateral interactions (e.g. sensing or communication) and are generally more plausible in real-world applications when compared to undirected networks.

To better place our results in context, two existing approaches for designing coordination algorithms for Euler-Lagrange networks are reviewed: model-dependent and adaptive algorithms. The aim is to give readers an idea of available works; the list is not exhaustive. The papers [3]–[5] study different coordination objectives, such as consensus or leader tracking, using algorithms that require exact knowledge of the agent models. Specifically, each agent’s algorithm requires knowledge of its own Euler-Lagrange equation in order to execute. The algorithms are therefore less robust to uncertainties in the model, e.g. some parameters in the Euler-Lagrange equation may be unknown or uncertain. Recently, the more popular approach is for each agent to use an adaptive algorithm. Specifically, an Euler-Lagrange equation can be linearly parametrized [2] with respect to a set of constant parameters of the equation, e.g. the mass of an arm on a robotic manipulator agent. This parametrization is then used in an adaptive algorithm to allow the agent to estimate its own set of constant parameters (which is assumed to be unknown) while simultaneously achieving the multi-agent coordination objective. Adaptive algorithms have been exploited to solve problems of containment control [6], [7], leaderless consensus [7], [8], and leader tracking [9]–[13].

In contrast to the above works, which rely on direct knowledge (or adaptive identification) of an agent model, there have been relatively few works studying model-independent algorithms, that is, algorithms for obtaining robust controllers. Furthermore, most results study model-independent algorithms on undirected networks. The pioneering work in [14] considered leaderless position consensus, with time-delay considered in [15]. Consensus to the intersection of target sets is studied in [16]. Leader-tracking algorithms are studied in [17]–[19]. Rendezvous to a stationary leader with collision avoidance is studied in [20]. For directed networks, several results are available. Passivity analysis in [21] showed that synchronization of the velocities (but not of the positions) is achieved on strongly connected directed networks. Rendezvous to a stationary leader and position consensus was studied in [22] and [23], respectively, but the papers assumed that the agents did not have a gravitational term in the dynamics. Leader tracking is studied in [24] but restrictive assumptions are placed on the leader. Preliminary work by the authors also appeared in [25], and is further analyzed below.

A. Motivation for Model-Independent Algorithms

Further study of model-independent algorithms is desirable for several reasons. Given a unique Euler-Lagrange equation, determining the minimum number of parameters in an adaptive
algorithm is difficult in general [26]. Moreover, the adaptive algorithms require knowledge of the exact equation structure; the algorithms can deal with uncertain constant parameters associated with the agent dynamics but are not robust to unmodelled nonlinear agent dynamics. Model-independent algorithms are reminiscent of robust controllers, which stand in conceptual contrast to adaptive controllers. Stability and indeed performance is guaranteed given limited knowledge of upper bounds on parameters of the multiagent system, and without use of any attempt to identify these parameters.

As will be shown in this paper, and similarly to [22], [23], model-independent controllers are exponentially stable, with a computable minimum rate of convergence. Exponentially stable systems are desired over systems which are asymptotically stable, but not exponentially so, because exponentially stable systems offer improved rejection to small amounts of noise and disturbance. Some algorithms requiring exact knowledge of the Euler-Lagrange equation have been shown to be exponentially stable [3], [5]. Further, adaptive controllers will yield exponential stability if certain conditions are satisfied, e.g. persistency of excitation. However, the above detailed works using adaptive algorithms have not verified such conditions.

B. Contributions of this paper

In this paper, we propose a discontinuous model-independent algorithm that allows a directed network of Euler-Lagrange agents to track a leader with arbitrary trajectory. First, we assume that the network is fixed and contains a directed spanning tree. Then, we relax this assumption to allow for a network with switching interactions. In order to achieve stability, a set of scalar control gains must be sufficiently large, i.e. satisfy a set of lower bounding inequalities. These inequalities involve limited knowledge of the bounds on the agent dynamic parameters, limited knowledge of the network topology, and a bound on the initial conditions (which may be arbitrarily large). This last requirement means the algorithm is semi-globally stable; a larger set of allowed initial conditions, which may be arbitrarily large, i.e. satisfy a set of lower bounding inequalities. These inequalities hold: (i) 

\[ \lambda_{\min}(A) \geq \lambda_{\max}(B) \]

(1)

where A is the system matrix, and B is a matrix associated with the agent dynamics but is not robust to unmodelled nonlinear agent dynamics. Model-independent algorithms are reminiscent of robust controllers, which stand in conceptual contrast to adaptive controllers. Stability and indeed performance is guaranteed given limited knowledge of upper bounds on parameters of the multiagent system, and without use of any attempt to identify these parameters.

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First, definitions of notation and several results are provided. The Kronecker product is denoted as \( \otimes \). Denote the \( p \times p \) identity matrix as \( I_p \) and the \( n \)-column vector of all ones as \( 1_n \). The \( l_1 \)-norm and Euclidean norm of a vector \( x \), and matrix \( A \), are denoted by \( \| x \|_1 \) and \( \| x \|_2 \), respectively. The signum function is denoted as \( \text{sgn}(\cdot) \), and for a vector \( x \), the function \( \text{sgn}(x) \) is defined element-wise. A matrix \( A = A^\top \) that is positive definite (respectively nonnegative definite) is denoted by \( A > 0 \) (respectively \( A \geq 0 \)). For symmetric matrices \( A, B \), \( A > B \) is equivalent to \( A - B > 0 \). For a matrix \( A = A^\top \), the minimum and maximum eigenvalues are \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) respectively. The following inequalities hold: (i) \( \lambda_{\min}(A) > \lambda_{\max}(B) \Rightarrow A > B \), (ii) \( \lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B) \), (iii) \( \lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B) \), and (iv) \( \lambda_{\min}(A)x^\top x \leq x^\top Ax \leq \lambda_{\max}(A)x^\top x \).

Definition 1. A function \( f(x) : \mathcal{D} \to \mathbb{R} \), where \( \mathcal{D} \subseteq \mathbb{R}^n \), is said to be positive definite in \( \mathcal{D} \) if \( f(x) > 0 \) for all \( x \in \mathcal{D} \), except \( f(0) = 0 \).

Lemma 1 (The Schur Complement [27]). Consider a symmetric block matrix, partitioned as

\[ A = \begin{bmatrix} B & C \\ C^\top & D \end{bmatrix} \]

(1)
Lemma 2. Suppose $A > 0$ is defined as in (1). Let a quadratic function with arguments $x$, $y$ be expressed as $W = [x, y] \cdot A [x, y]^T$. Define $F := B - CD^{-1}C^T$ and $G := D - C' B^{-1}C$. Then, there holds

$$\lambda_{\min}(F)x^T x \leq x^T F x \leq W$$

(2a)

and

$$\lambda_{\min}(G)y^T y \leq y^T G y \leq W$$

(2b)

Proof. We obtain (2b) by recalling Lemma 1 and observing that $W = y^T G y + [y^T C D^{-1} + x^T] B [B^{-1} C y + x]$. An equally straightforward proof yields (2a).

Lemma 3. Let $g(x, y)$ be a function given as

$$g(x, y) = ax^2 + by^2 - cxy^2 - dx y$$

for real positive scalars $a, b, c, d > 0$. Then for a given $X > 0$, there exist $a, b > 0$ such that $g(x, y)$ is positive definite for all $y \in [0, X]$ and $x \in [0, X]$.

Corollary 1. Let $h(x, y)$ be a function given as

$$h(x, y) = ax^2 + by^2 - cxy^2 - dx y$$

where the real positive scalars $a, b, c, d$ and two further positive scalars $\varepsilon, \vartheta$ are fixed. Suppose that for given $y, X'$ there holds $\exists \varepsilon > 0$, and $X' > \vartheta$. Define the sets $\mathcal{U} = \{x, y : x \in [X' - \vartheta, X'] \}, y > 0 \}$ and $\mathcal{V} = \{x, y : x > 0, y \in [\varepsilon, \varepsilon']\}$. Define the region $\mathcal{R} = \mathcal{U} \cup \mathcal{V}$. Then, there exist $a, b > 0$ such that $h(x, y)$ is positive definite in $\mathcal{R}$.

The results of Lemma 3 and Corollary 1 are almost intuitively obvious. However, the detailed statements in the proof (see [28, Section II-A]) lay out explicit inequalities for $a, b$. These inequalities will be used to show that for any given Euler-Lagrange network, control gains can always be found to ensure leader tracking is achieved.

B. Graph Theory

The agent interactions can be modeled by a weighted directed graph which is denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with the set of nodes $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$, and with a corresponding set of ordered edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed edge $e_{ij} = (v_i, v_j)$ is outgoing with respect to $v_i$ and incoming with respect to $v_j$, and implies that $v_j$ is able to obtain some information from $v_i$. The precise nature of this information will be made clear in the sequel. The weighted adjacency matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$ of $\mathcal{G}$ has nonnegative elements $a_{ij}$. The elements of $\mathcal{A}$ have properties such that $a_{ij} > 0 \Leftrightarrow e_{ij} \in \mathcal{E}$ while $a_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$ and it is assumed $a_{ii} = 0, \forall i$. The neighbor set of $v_i$ is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. The $(n+1) \times (n+1)$ Laplacian matrix, $L = \{l_{ij}\}$, of the associated digraph $\mathcal{G}$ is defined as $l_{ij} := -a_{ij}$ for $j \neq i$ and $l_{ii} = \sum_{k=1 \ldots n, k \neq i} a_{ik}$ if $j = i$. A directed spanning tree is a directed graph formed by directed edges of the graph that connects all the nodes, and where every vertex apart from the root has exactly one parent. A graph is said to contain a spanning tree if a subset of the edges forms a spanning tree. We make use of the following standard lemma.

Lemma 4 ([29]). Let $\mathcal{G}$ contain a directed spanning tree, and suppose there are no edges which are incoming to the root vertex of the tree, which without loss of generality, is set as $v_0$. Then, the Laplacian of $\mathcal{G}$ can be partitioned as

$$L = \begin{bmatrix} 0 & 0 \\ \mathcal{L}_{11} & \mathcal{L}_{22} \end{bmatrix}$$

(5)

and $\exists \Gamma > 0$ which is diagonal and $\Gamma \mathcal{L}_{22} + \mathcal{L}_{22}^\top \Gamma > 0$.

For future use, denote the $i^{th}$ diagonal element of $\Gamma$ as $\gamma_i$ and define $\bar{\gamma} \triangleq \max, \gamma_i$ and $\underline{\gamma} \triangleq \min, \gamma_i$.

C. Euler-Lagrange Systems

The $i^{th}$ Euler-Lagrange agent’s equation of motion is:

$$M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + g_i(q_i) + \zeta_i = \tau_i$$

(6)

where $q_i(t) \in \mathbb{R}^p$ is a vector of the generalized coordinates. Note that from here onward, we drop the time argument $t$ wherever there is no ambiguity. The inertia matrix is $M_i(q_i) \in \mathbb{R}^{p \times p}$, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p \times p}$ is the Coriolis and centrifugal force matrix, $g_i \in \mathbb{R}^p$ is the vector of (gravitational) potential forces and $\zeta_i(t)$ is an unknown, time-varying disturbance. It is assumed that all agents are fully-actuated, with $\tau_i \in \mathbb{R}^p$ being the control input vector. For each agent, the $k^{th}$ generalized coordinate is denoted using superscript $(k)$; thus $q_i = [q_i^{(1)}, \ldots, q_i^{(n)}]^T$. It is assumed that the systems described using (6) have the following properties given below:

P1 The matrix $M_i(q_i)$ is symmetric positive definite.

P2 There exist scalar constants $k_m, k_T > 0$ such that $k_m I_p \leq M_i(q_i) \leq k_T I_p, \forall i, q_i$. It follows that $\sup_{q_i} \|M_i\|_2 \leq k_T$ and $k_m \leq \inf_{q_i} \|M_i^{-1}\|_2$ holds $\forall i$.

P3 The matrix $C_i(q_i, \dot{q}_i)$ is defined such that $M_i - 2C_i$ is skew-symmetric, i.e. $M_i = C_i + C_i^\top$.

P4 There exist scalar constants $k_C, k_T > 0$ such that $\|C_i\|_2 \leq k_C \|q_i\|_2, \forall i, q_i$ and $\|q_i\|_2 < k_T q_i, \forall i$.

P5 There exists a constant $k_T$ such that $\|q_i\|_2 \leq k_T, \forall i$. Properties P1-P4 are standard and widely assumed properties of Euler-Lagrange dynamical systems, see [2] for details. Property P5 is a reasonable assumption on disturbances.

D. Problem Statement

The leader is denoted as agent 0, i.e. vertex $v_0$, with $q_0(t)$ and $\dot{q}_0(t)$ being its time-varying generalized coordinates and generalized velocity, respectively. The objective is to develop a model-independent, distributed algorithm which allows a directed network of Euler-Lagrange agents to synchronize and track the trajectory of the leader. The leader tracking objective is said to be achieved if $\lim_{t \to \infty} \|q_i(t) - \dot{q}_0(t)\|_2 = 0$ and $\lim_{t \to \infty} \|q_i(t) - q_0(t)\|_2 = 0$ for all $i = 1, \ldots, n$. By model-independent, we mean that the algorithm does not contain $M_i, C_i, g_i \forall i$ nor make use of an associated linear parameterization. Two mild assumptions are now given.

Assumption 1. The leader trajectory $q_0(t)$ is a $C^2$ function with derivatives $q_0$ and $\dot{q}_0$ which are bounded as $\|q_0\|_2 \leq k_p$ and $\|\dot{q}_0\|_2 \leq k_q$. The positive constants $k_p, k_q$ are known a priori.
Assumption 2. All possible initial conditions lie in some fixed but arbitrarily large set that is known. In particular, $\|q_0\|_2 \leq k_0/\sqrt{n}$ and $\|\tilde{q}_i\|_2 \leq k_0/\sqrt{n}$, where $k_0, k_0$ are known a priori.

These two assumptions are not unreasonable, as many systems will have an expected operating range for $q$ and $\tilde{q}$.

The follower agents’ capability to sense relative states is captured by the directed graph $G_A$ with an associated Laplacian $\mathcal{L}_A$. In Section III, we assume $G_A$ is fixed. Later in Section IV, it is assumed that $G_A$ is dynamic, i.e. time-varying. Thus, if $a_{ij} > 0$ then agent $i$ can sense $q_j - q_i$ and $\tilde{q}_i - \tilde{q}_j$. We denote the neighbor set of agent $i$ on $G_A$ as $N_Ai$. We further assume that agent $i$ can measure its own $q_i$ and $\tilde{q}_i$. A second weighted and directed time-varying graph $G_B(t)$, with the associated Laplacian $\mathcal{L}_B(t)$, exists between the followers to communicate estimates of the leader's state. Denote the neighbor set of agent $i$ on $G_B(t)$ at time $t$ as $N_Bi(t)$. Note that $v_j \in N_Bi(t)$ when agent $j$ communicates directly to agent $i$ its estimates of the leader’s state at time $t$ (the precise nature of this estimate is described in Section III-A). Further note that $G_A$ is not necessarily equal to $G_B$ and so $N_Ai \neq N_Bi$ in general. However the node sets of $G_A$ and $G_B$ are the same.

Remark 1 (Comparison of this paper to recent leader tracking results). Almost all mechanical systems will have trajectories which satisfy the mild Assumption 1. In comparison, more restrictive assumption are made on the leader trajectory in [13], [24]. In [13], [24], the leader trajectory is describable by an LTI system, with system matrix defined as $\mathbf{S}$. In [13], it is assumed that all eigenvalues of $\mathbf{S}$ are purely imaginary. In [24], it is assumed that $\mathbf{S}$ is marginally stable. More importantly, both [13] and [24] assume that $\mathbf{S}$ is known to all agents, which is a highly restrictive assumption. As will become apparent in the sequel, we use a distributed observer to allow every agent to obtain $\dot{q}_0(t)$ and $\dot{\tilde{q}}_0(t)$ precisely. The work [12] has similar assumptions to this paper, but uses an adaptive algorithm and is therefore fundamentally different to the model-independent controller studied in this paper.

III. LEADER TRACKING ON FIXED DIRECTED NETWORKS

A. Finite-Time Distributed observer

Before we show the main result, we detail a distributed finite-time observer developed in [30] which allows each follower agent to obtain $q_0$ and $\dot{q}_0$. Let $\tilde{r}_i$ and $\dot{\tilde{r}}_i$ be the $i^{th}$ agent’s estimated values for the leader position and velocity respectively. Agent $i \in \{1, \ldots, n\}$ runs the observer

$$\ddot{r}_i = \dot{v}_i - \omega_1 \text{sgn} \left( \sum_{j \in N_Ai(t)} b_{ij}(t) (\tilde{r}_i - \tilde{r}_j) \right)$$

(7a)

$$\dot{\tilde{v}}_i = -\omega_2 \text{sgn} \left( \sum_{j \in N_Bi(t)} b_{ij}(t) (\tilde{v}_i - \tilde{v}_j) \right)$$

(7b)

where $b_{ij}$ are the elements of the adjacency matrix associated with graph $G_B(t)$ and $\omega_1, \omega_2 > 0$ are internal gains of the observer. Clearly, if $a_{ij} > 0$ then agent $i$ can directly sense the leader, $v_0$ and thus learns of $q_0$ and $\dot{q}_0$. For such an agent $i$, we set $b_{ij} > 0$ and $\ddot{r}_0(t) = q_0(t)$ and $\dot{\tilde{v}}_0(t) = \dot{q}_0(t)$; agent $i$ still runs the distributed observer (7). We now give a theorem for convergence of the observer, and explain below why all followers execute (7) even if they learn of $q_0, \dot{q}_0$ from $G_A$.

Theorem 1 (Theorem 4.1 of [30]). Suppose that the leader trajectory $q_0(t)$ satisfies Assumption 1. If at every $t$, $G_B(t)$ contains a directed spanning tree, and $\omega_2 > k_0/n$ then, for some $T_1 < \infty$, there holds $\tilde{r}_i(t) = q_0(t)$ and $\tilde{v}_i(t) = \dot{q}_0(t)$ for all $i \in \{1, \ldots, n\}$, for all $t \geq T_1$.

The key reason for agent $i$ to run the distributed observer even if $a_{ij} > 0$ (and thus agent $i$ knows $q_0$ and $\dot{q}_0$) is to ensure robustness to network changes over time (e.g. switching topology due to loss of connection). We elaborate further.

Before we show the main result, we detail a distributed observer (7) acts as a filter for noisy measurements. While (7) allows each agent to obtain $\dot{q}_0(t)$ and $\dot{\tilde{q}}_0(t)$ in finite time, the leader tracking objective must still be achieved. Towards that end, we now propose a distributed model-independent algorithm and study its stability properties, using analysis that is entirely different from that used in [30] to establish Theorem 1.

B. Model-Independent Control Law

Consider the following algorithm for the $i^{th}$ agent

$$\tau_i = -\eta \sum_{j \in N_Ai} a_{ij} \left( (q_i - q_j) + \mu(q_i - \dot{q}_i) \right)$$

$$-\beta \text{sgn} (q_i - \tilde{r}_i) + \mu(q_i - \tilde{v}_i)$$

(8)

where $a_{ij}$ is the weighted $(i, j)$ entry of the adjacency matrix $A$ associated with the weighted directed graph $G_A$. The control gains $\mu, \eta$ and $\beta$ are positive constants and their design will be specified later. It is assumed that $\eta > 1$. Note that for all $i$, $\tilde{r}_i$ and $\tilde{v}_i$ are replaced with $q_0$ and $\dot{q}_0$, respectively, $\forall t > T_1$.

Let us denote the new error variable $\tilde{q}_i = q_i - q_0$. Let $q = [q_1, \ldots, q_n]^\top \in \mathbb{R}^{np}$ the stacked column vector of all $q_i$. The leader tracking objective is therefore achieved if $\tilde{q}(t) = \ddot{q}(t) = 0$ as $t \to \infty$. We denote $g = [g_1, \ldots, g_n]^\top$, $\zeta = [\zeta_1, \ldots, \zeta_n]^\top$, $\xi = [\xi_1, \ldots, \xi_n]^\top$, and $\bar{q} = [\bar{q}_1, \ldots, \bar{q}_n]^\top$ as the $np$-column vectors of all $g_i, \zeta_i, q_i, \xi_i$ respectively. Let $M(q) = \text{diag}(M_1(q_1), \ldots, M_n(q_n)) \in \mathbb{R}^{np \times np}$, and $C(q, \bar{q}) = \text{diag}(C_1(q_1, \bar{q}_1), \ldots, C_n(q_n, \bar{q}_n)) \in \mathbb{R}^{np \times np}$. Since $M_i \succ 0, \forall i$ then $M$ is also symmetric positive definite. Define an error vector, $e_i = \tilde{r}_i - q_0, \forall i = 1, \ldots, n$ and $e_i = \tilde{v}_i - \dot{q}_0$. Define $e = [e_1^\top, \ldots, e_n^\top]^\top \in \mathbb{R}^{np}$, $\dot{e} = [\dot{e}_1^\top, \ldots, \dot{e}_n^\top]^\top \in \mathbb{R}^{np}$.

The definition of $\tilde{q}_i$ yields $\dot{M}(\tilde{q}) = M_\dot{q}_i - M\dot{q}_0$ and combining the agent dynamics (6) and the control law (8),

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2018.2856298, IEEE Transactions on Control of Network Systems.
the closed-loop system for the follower network, with nodes $v_1, \ldots, v_n$, can be expressed as
\[
\ddot{\bar{q}} + \alpha \dot{\bar{q}} + (k_{\bar{q}} - \delta)\bar{q} + \eta \bar{q} + \mu \dot{\bar{q}} + g + \zeta + \beta sgn(a + \mu \delta) + M(1_n \otimes \dot{\bar{q}}_0 + C(1_n \otimes \dot{\bar{q}}_0))
\]
where $K$ denotes the differential inclusion, $a \in \mathbb{R}^n$ stands for “almost everywhere” and $s = \bar{q} - e$. Here, $L_{\bar{q}}$ is the lower block matrix of $L_A$ as partitioned in (5). Filippov solutions of $\dot{\bar{q}}$ and $\ddot{\bar{q}}$ for (9) exist because the signum function is measurable and locally essentially bounded, and $\ddot{\bar{q}}$ and $\dddot{\bar{q}}$ are absolutely continuous functions of time [31].

C. An Upper Bound Using Initial Conditions

Before proceeding with the main proof, we calculate an upper bound (which may not be tight) on the initial states expressed as $\|\bar{q}(0)\|_2 < \mathcal{X}$ and $\|\bar{q}(0)\|_2 < \mathcal{Y}$ using Assumption 2. In the sequel, we show that these bounds hold for all time, and exponential convergence results in. In keeping with the model-independent approach, define
\[
\dot{V}_\mu = \begin{bmatrix} \bar{q}^T & \bar{q} \end{bmatrix}^T \left[ \begin{array}{cc} \rho_1(\mu) & \frac{1}{2}\mu \gamma(k_{\bar{q}} - \delta)I_{np} \\ \frac{1}{2}\mu \gamma(k_{\bar{q}} - \delta)I_{np} & \rho_2(\mu) \end{array} \right] \begin{bmatrix} \bar{q} \\ \bar{q} \end{bmatrix}
\]
(10)
where
\[
X = (\Gamma L_{\bar{q}} + L_{\bar{q}}^T \Gamma) \otimes I_p
\]
and
\[
P = \frac{1}{2} \eta \max(X) I_{np} \mu^{-1}(k_{\bar{q}} - \delta)I_{np} = \left[ \begin{array}{c} \bar{q} \\ \bar{q} \end{array} \right]
\]
(11)
for all $t$. Next, define
\[
\rho_1 = \frac{1}{2} \eta \min\{X\} I_{np} \mu^{-1}(k_{\bar{q}} - \delta)I_{np}
\]
and
\[
\rho_2 = \frac{1}{2} \eta \min\{X\} I_{np} \mu^{-1}(k_{\bar{q}} - \delta)I_{np}
\]
(12)

Call the matrix in (12) $\mathcal{N}_\mu$. Similarly to above, Lemma 1 to show that $\mathcal{N}_\mu = 0$ for any $\mu \geq \mu_2$ where $\mu_2 > \sqrt{2}\gamma(k_{\bar{q}} - \delta)/\min(X)$. Set $\mu_3 = \max(\mu_1, \mu_2)$. Define
\[
\rho_1(\mu) = \eta \min\{X\} I_{np} \mu^{-1}(k_{\bar{q}} - \delta)I_{np}
\]
and
\[
\rho_2(\mu) = \frac{1}{2} \eta \min\{X\} I_{np} \mu^{-1}(k_{\bar{q}} - \delta)I_{np}
\]
(13a)
and (13b)

and verify that $\rho_1(\mu_3) > \rho_2(\mu_3)$. Note that for any $\mu \geq \mu_3$ there holds $\dot{V}_\mu \leq \dot{V}_\mu$ and $\rho_1(\mu_3) \leq \rho_1(\mu)$, $i = 1, 2$. Compute
\[
\dot{V}_\mu = \eta \max(X) k_a^2 + \frac{1}{2}(k_{\bar{q}} - \delta)k_b^2 + \mu^{-1}(k_{\bar{q}} - \delta)k_a k_b
\]
From Assumption 2, one has that $\|\bar{q}(0)\|_2 \leq k_a$ and $\|\bar{q}(0)\|_2 \leq k_b$. Thus, one concludes from (11) and the above
\[
\text{equation that there holds } \dot{V}_\mu(0) \leq \dot{V}_\mu^* \text{ for any } \mu \geq \mu_3.
\]

Because we assumed $\eta > 1$, it follows from Lemma 2 and (2a) that
\[
\|\dot{\bar{q}}(0)\|_2 \leq \sqrt{\frac{\dot{\bar{q}}(0)}{\rho_1(\mu_3)}} \leq \sqrt{\frac{\dot{\bar{q}}(0)}{\rho_1(\mu_3)}} \leq \mathcal{X}
\]
(14)

Following a similar method yields $\mathcal{Y}_1$. Next, compute $\dot{V}_\mu = \eta \max(X) k_a^2 + \frac{1}{2}(k_{\bar{q}} - \delta)k_b^2 + \mu^{-1}(k_{\bar{q}} - \delta)k_a k_b$

and observe that $\dot{V}_\mu \leq \dot{V}_\mu^*$. Lastly, compute the bound
\[
\mathcal{X} = \sqrt{\dot{V}_\mu^*/\rho_2(\mu_3)}
\]
(15)

and notice that $\|\dot{\bar{q}}(0)\|_2 \leq \mathcal{Y}_1 \leq \mathcal{X}$. Similarly, $\mathcal{Y}$ is obtained using (2b), with the steps omitted due to spatial limitations. Because both sides of (15) are independent of $\mu$, the values $\mathcal{Y}$ and $\mathcal{X}$ do not change for all $\mu \geq \mu_3$.

D. Stability Proof

**Theorem 2.** Suppose that the conditions in Theorem 1 are satisfied. Under Assumptions 1 and 2, the leader-tracking is achieved exponentially fast if 1) the network $G_A$ contains a directed spanning tree with the leader as the root node, and 2) the control gains $\mu, \eta, \beta$ satisfy a set of lower bounding inequalities. For a given $G_A$ containing a directed spanning tree, there always exists $\mu, \eta, \beta$ which satisfy the inequalities.

**Proof.** The proof will be presented in four parts. In Part 1, we study a Lyapunov-like candidate function $V$. In Part 2, we analyze $\dot{V}$ and show that it is upper bounded. Part 3 shows that the system trajectory remains bounded for all time, and exponential convergence is proved in Part 4.

**Part 1: Consider the Lyapunov-like candidate function**

\[
V = \frac{1}{2}\eta \bar{q}^T X \bar{q} + \mu^{-1} \bar{q}^T \Gamma_p M \bar{q} + \frac{1}{2} \bar{q}^T \Gamma_p M \bar{q} = V_1 + V_2 + V_3
\]
with $X$ given below (10), and $\Gamma_p = \Gamma \otimes I_p$. Observe that
\[
V = \left[ \begin{array}{c} \bar{q} \\ \bar{q} \end{array} \right]^T \left[ \begin{array}{cc} \frac{1}{2}\eta X & \frac{1}{2} \bar{q}^T \Gamma_p M \\ \frac{1}{2}\bar{q}^T \Gamma_p M & \frac{1}{2}\bar{q}^T \Gamma_p M \end{array} \right] \left[ \begin{array}{c} \bar{q} \\ \bar{q} \end{array} \right]
\]
(16)

Call the matrix in (16) $H_\mu$. From Lemma 1, and the assumed properties of $M_1$, there holds $H_\mu > 0$ if and only if $\eta X - \mu^{-2} \Gamma_p M > 0$, which is implied by $\lambda_{\min}(X) - \mu^{-2} k_{\bar{q}} > 0$. This is because $k_{\bar{q}} \geq \sup \eta \max(X)$, and we assumed that $\eta > 1$ and $\gamma = 1$. For any $\mu \geq \mu_3$, where
\[
\mu_3 > 2\sqrt{2\gamma(k_{\bar{q}} - \delta)/\min(X)}
\]
there holds $\mu < H_\mu > N_\mu > 0$ because $\mu_3 \geq \mu_3$ as defined below (12). Thus, although the eigenvalues $\lambda_i(H_\mu)$ depend on $q(t)$, there holds $\lambda_i(N_\mu) \leq \lambda_i(H_\mu) \leq \lambda_i(M_\mu)$ for all $i$, and for all $\mu > 0$. For, thus, any $\mu \geq \mu_3$, $V > 0$ and is radially unbounded. For simplicity, let $V(t)$ denote $V(\bar{q}(t), \dot{\bar{q}}(t))$ and observe that $V(t) \leq V_1(t)$, $\forall t$ because
\[
V(t) \leq \frac{1}{2} \eta \max(X) \|\bar{q}(t)\|_2^2 + \frac{1}{2} \bar{q}^T \dot{\bar{q}}(t)\bar{q}(t) + \mu^{-1}(k_{\bar{q}} - \delta)k_a k_b
\]
(18)

Note 1: $\nu$ is not a Lyapunov function.

Note 2: In Remark 2, we detail an approach for designing the gains.
Part 2: Let $\hat{V}$ be the set-valued derivative of $V$ with respect to time, along the trajectories of the system (9). We obtain
\[
\dot{V} \in -\mu^{-1}K_1 \gamma X q + \mu \beta \frac{\gamma}{2} X q - \gamma X \dot{\Gamma}_p M q + \gamma \dot{\Gamma}_p C \gamma q = -\mu^{-1} \tilde{K}_1 \gamma (x - y) - \mu \beta \frac{\gamma}{2} (x - y)
\]
where $\gamma = g + \mathcal{C} + M (1_n \times q_0)$ and $x = \tilde{q} + \mu \tilde{q}$ and $y = e + \mu e$ (recall that $s = \tilde{q} - e$). We refer the reader to [28, Theorem 2] for the detailed calculations to obtained (19).

Define $\hat{V}_A$ (absolutely continuous) and $\hat{V}_B$ (set-valued) as
\[
\dot{V}_A = -\mu^{-1}(\varphi(\mu, \eta)) (\tilde{\varphi} q \gamma q + 2 \mu \beta \frac{\gamma}{2} X q)\]
\[
\dot{V}_B = -\mu^{-1}(\varphi(\mu, \eta)) (\tilde{\varphi} q \gamma q + 2 \mu \beta \frac{\gamma}{2} X q) - k_c k_p (x - y)
\]
where $\varphi(\mu, \eta) = \frac{1}{2} \mu^2 \lambda_{\min}(X) - \mu k_c k_p - k_{\beta}$. Straightforward calculations (see [28, Theorem 2]) show that
\[
\dot{V} \leq V_A + V_B
\]

Part 3: In Part 3, we study $\hat{V}_A$ and $\hat{V}_B$ separately to establish negative definiteness properties. Then, Part 3.2 studies $\hat{V}_A + \hat{V}_B$ and proves a boundedness property.

Part 3.1: Consider the region of the state variables given by $x = \tilde{q}$ and $y = \tilde{q}$ in Section III-C. One can compute $\mu_0 \geq \mu^*$ and $\eta_0 \geq \eta^*$ such that $\varphi(\mu, \eta) > \lim_{\eta \to \infty} \varphi(\mu, \eta)$. Note that $L_{\mu} > H_{\mu} > N_{\mu} > 0$ continues to hold. Observe that $g(\tilde{q}, \tilde{q})$ is of the same form as $g(x, y)$ in Lemma 3 with $x = \tilde{q}$ and $y = \tilde{q}$ in (20a) of the same form as (21) in the section. The inequality holds for all $\mu_0 \geq \mu^*$ and $\eta_0 \geq \eta^*$ such that $\varphi(\mu_0, \eta_0) > (2 k_c k_p)^{\frac{\gamma}{2}} 2^\gamma \lambda_{\min}(X) + k_c X$. Recall from (10) and (15) that $\hat{V}^*$ is dependent on $\eta$, but independent of $\mu$ because $\mu_0 \geq \mu^*$. One could leave $\eta_0 = \eta^*$ and find a sufficiently large $\mu_0^*$ to satisfy (22). Alternatively, we could increase $\eta$. Notice that $\mu_0^* (\eta_0^*)$ and $\hat{V}^*$ are both of $O(\eta)$. Thus, as $\eta$ increases, $\hat{V}$ becomes dependent on $\eta$, whereas $\varphi = O(\eta)$. We conclude that there exists a sufficiently large $\eta_0^*$ satisfying (22), and for which $X_1, Y_1, X, Y$ need not be recomputed. With $\mu_0^*, \eta_0^*$ satisfying (22), $\hat{V}_A < 0$ in the aforementioned region.

Now consider $\hat{V}_B$ over two time intervals, $t_p = [0, T_1)$ and $t_Q = [T_1, T_2)$, where $T_1$ is given in Theorem 1 and $T_2$ is the infimum of those values of $t$ for which one of the inequalities $\|\tilde{q}(t)\| < X, |\tilde{q}(t)| < Y$ fails. In Part 3.2, we argue that without loss of generality, it is possible to take $T_2 > T_1$. In fact, we establish that the inequalities never fail; $T_2$ does not exist and thus $t_Q = [T_1, \infty)$.

Consider $t \in t_p$. Observe that the set-valued function $-\beta X \dot{\Gamma}_p (x - y)$ is upper bounded by the single-valued function $\beta(x - y)$. Recalling $\hat{V}_B$ in (20b) yields
\[
\hat{V}_B \leq (\sqrt{\gamma} + k_c k_p \gamma + \beta)(\mu_0 \|q\| + \|q\|) \leq \hat{V}_{\beta}
\]
because $\|q\| < \mu \|q\| < \beta$. Next, for $t \in t_Q$, Theorem 1 yields that $\varphi(t) = \varphi(t) = 0$, which implies that $y = 0$. Thus, the set-valued term $K[x \dot{\Gamma}_p (x - y)]$ in (20b) becomes the singleton $K[x \dot{\Gamma}_p (x - y)] = \{\|\dot{\Gamma}_p x\|\}$ (since $\dot{\Gamma}_p > 0$ is diagonal). It then follows that
\[
\hat{V}_B = -\mu^{-1}(\beta X \dot{\Gamma}_p (x - y)) + k_c k_p \gamma + \beta \leq \hat{V}_B
\]

In other words, $\hat{V}_B$ for $t \in t_Q$ is a continuous, single-valued function in the variables $q$ and $\dot{q}$. For $t \in t_Q$, we observe that $\hat{V}_B \leq \mu^{-1}(\beta X \dot{\Gamma}_p (x - y)) + k_c k_p \gamma + \beta \leq 0$ if
\[
\beta > (k_c k_p \gamma + \beta) / \gamma
\]

Part 3.2: To aid in this part of the proof, refer to Fig. 1. Consider firstly $\hat{V}$ for $t \in t_p$. Specifically, one can show that
\[
\hat{V}_p \triangleq \hat{V}_A + \hat{V}_B = -\mu^{-1} \hat{V}_B (x - y)
\]
where the function $\hat{V}_B (x - y)$ is of the form of $h(x, y)$ in Corollary 1 with $x = \|q\| \|y\|$ and $y = \|q\| \|y\|$. The coefficients of $\|q\| \|y\|$ are given by $a = \frac{1}{\mu} \lambda_{\min}(X)$, $b = \varphi(\mu, \eta)$, $c = k_c, d = 2 k_c k_p, e = (\sqrt{\gamma} + k_c k_p \gamma + \beta)$ and $f = \mu_0$. Note that $\hat{V} \leq \hat{V}_p$, i.e. $\hat{V}$ for $t \in t_p$ is a differential inclusion which is upper bounded by a continuous function. Thus, for some given $\theta, \epsilon, \chi, \gamma$ satisfying the requirements detailed in Corollary 1, one can find a $\eta$ such that $\hat{V}_p (x - y)$ is positive definite in the region $\Gamma$. Note that $\theta, \epsilon$ can be selected by the designer. Choose $\theta > X - \chi$ and $\epsilon > \gamma - \chi$, and ensure that $X - \theta, \gamma - \epsilon > 0$. Note the fact that $X > \chi$ and $\gamma > \chi$ implies $\theta, \epsilon > 0$.

Define the sets $U, \chi$ and $\gamma$ in Corollary 1 with $x = \|q\| \|y\|$ and $y = \|q\| \|y\|$. Define the union set $U = \{x = \|q\| \|y\|, y = \|q\| \|y\| \}$ and $V = \{x = \|q\| \|y\|, y = \|q\| \|y\| \}$. Define the compact region $S = U \cup \tilde{\chi} \cup V$, see Fig. 1 for a visualization of $S$. Note $\tilde{\chi} \subset \Gamma$. Using Corollary 1, and with precise calculation details given in [28, Theorem 2], one can find a pair of gains $\gamma$ and $\mu$ which ensures that $p(\|q\|, \|y\|)$ is positive definite in $\tilde{\chi}$. This implies $p(\|q\|, \|y\|)$ is positive definite in $\tilde{\chi}$. It follows that $\hat{V}_p$ is negative definite in $\tilde{\chi}$. Further define the region $\|q\| \|y\| \in [0, X - \theta], \|q\| \|y\| \in [0, Y - \epsilon)]$, again with visualization in Fig. 1.

Now we justify the fact that we can assume $T_2 > T_1$. In fact, we show that the existence of $T_2$ creates a contradiction; the trajectories of (9) remain in $\Gamma \cup S$ for all time. See Fig. 1 for a visualization. Although $V$ is sign indefinite in $T$ (i.e. $V(t)$ can increase), notice from (18) that, in $T$ there holds
\[
V(t) \leq \frac{1}{2} \|\mu_{\max}(X) (X - \theta)\|^2 + \frac{1}{2} \mu_{\max}(Y - \epsilon)^2 + \mu k_c (X - \theta)(Y - \epsilon) = \mathcal{E}
\]

Establishing that $T_2$ does not exist rules out the possibility of finite-time escape for system (6).
It can then be shown that all trajectories of (9) beginning in $T \cup S$ satisfy $V(t) \leq \max\{Z, V(0)\} < \hat{V}^*$ for all $t \leq T_2$ (see [28, Theorem 2] for the arguments leading to this conclusion).

On the other hand, and from Lemma 2, there holds at $T_2$

$$\|\hat{q}(T_2)\|_2 \leq \sqrt{\frac{V(T_2)}{\chi}} < \sqrt{\frac{\bar{V}^*}{\rho_2(\mu^*_2)}} = \bar{X}$$

(28)

where $\chi = \lambda_{\min}\left(\frac{1}{2} \eta X - \frac{1}{2} \mu^* M\right) > \rho_2(\mu^*_2)$. One can also show that $\|\hat{q}(T_2)\|_2 < \bar{X}$ using an argument paralleling the argument leading to (28); we omit this due to spatial limitations. The existence of (28) and a similar inequality for $\|\hat{q}(T_2)\|_2$ contradicts the definition of $T_2$. In other words, $T_2$ does not exist and $\|\hat{q}(t)\|_2 < \bar{X}$, $\|\hat{q}(t)\|_2 < \bar{Y}$ hold for all $t$.

Part 4: Observe that $\hat{V}_B$ changes at $t = T_1$ to become negative definite. Consider now $t \in t_2 = [T_1, T_2]$. Recalling that $\hat{V} \leq \hat{V}_A + \hat{V}_B$, we have

$$\hat{V} \leq -\mu^* \left[ \varphi(\mu, \eta) \|\hat{q}\|_2^2 + 2 under (17) that the eigenvalues of $H_\mu$ are uniformly upper and lower bounded. Specifically, there holds $\lambda_{\min}(N_\mu)\|\hat{q}^T, \hat{q}\|_2^2 \leq \hat{V} \leq \lambda_{\max}(L_\mu)\|\hat{q}^T, \hat{q}\|_2^2$. Because $D$ is compact, one can find a scalar $a_3 > 0$ such that $\hat{V} \leq -a_3\|\hat{q}^T, \hat{q}\|_2^2$. This follows that $V$ decays exponentially fast to zero, with a minimum rate $e^{-\alpha_3}$ (32). It follows that $\lim_{t \to \infty} \hat{q}(t) = 0$ and $\lim_{t \to \infty} \hat{\eta}(t) = 0$, exponentially and the leader tracking objective is achieved.

Remark 2 (Designing the gains). We summarize here the process to design $\mu, \eta, \beta$ to satisfy inequalities detailed in the proof of Theorem 2. First, one may select $\beta$ to satisfy (25). Then, $\mu$ should be set to satisfy (17). The quantities $X$ and $Y$ discussed in Section III-C are then computed with $\beta \geq 1$; we noted below (22) that $X$ and $Y$ are independent of $\eta$ as $\eta$ increases. Having computed $X$ and $Y$, the last step is to adjust $\eta$ to ensure $\|\hat{q}\|_2^2 + p(\|\hat{q}\|_2^2)$ are positive definite (see Part 3.1 of Theorem 2 proof). Details of the inequalities to ensure positive definiteness are found in the proofs of Lemma 3 and Corollary 1 in [28, Section II-A].

Remark 3 (Additional degree of freedom). In [25] we assumed that $\eta = 1$. There is more flexibility in this paper since we allow $\beta \geq 1$: one can adjust separately, or simultaneously, $\mu$ and $\eta$. While the interplay between $\mu, \eta, \beta$, and its effect on performance, is difficult to quantify, we observe from extensive simulations that in general, one should make $\beta$ as small as possible. Where possible, one should hold $\mu$ constant and increase $\eta$ to satisfy an inequality involving both, e.g (22). Notice that $\lambda_{\max}(H_\mu)$ and $a_3$ are both $\mathcal{O}(\eta)$. As $\mu$ increases $\lambda_{\max}(H_\mu)$ does not increase but $a_3$ does decrease. Thus, the convergence rate $a_3/\lambda_{\max}(L_\mu)$ is not negatively affected by increasing $\eta$ but is reduced by increasing $\mu$. If only $\mu$ is increased (as in [25]) then velocity consensus is quickly achieved but position consensus is achieved after a long time.

Remark 4 (Robustness). The proposed algorithm (8) is robust in several aspects. First, the exponential stability property implies that small amounts of noise produce small departures from the ideal. Moreover, the signum term in the controller offers robustness to the unknown disturbance $\zeta(t)$. In contrast, and as discussed in the introduction, adaptive controllers are not robust to unmodelled agent dynamics.

Remark 5 (Controller structure). Consider the controller (8). The term containing the sign function ensures exact tracking of the leader’s trajectory. Consider Fig. 1. For $t < T_1$, the signum term results in the region $T$, where $V$ is sign indefinite. This sign term can in fact drive an agent away from its neighbors due to the nonzero error term $e(\epsilon_2(t), t < T_1).$ However, for $t > T_1$, the linear terms of the controller (and in particular adjustment of the gains $\eta, \mu$) ensure that $V < 0$ in $S$. This ensures that the followers remain in the bounded region $S$ centered on the leader. Such a controller gives added robustness. For example, if $\mathcal{G}_B$ becomes temporarily disconnected, all agents will remain close to the leader so long as $\mathcal{G}_A$ has a directed spanning tree. When connectivity of $\mathcal{G}_B$ is restored, perfect tracking follows, as illustrated in the simulation below.

E. Practical Tracking By Approximating the Signum Function

Although the signum function term in (8) allows the leader-tracking objective to be achieved, it carries an offsetting dis-
advantage. Use of the signum function can cause mechanical components to fatigue due to the rapid switching of the control input. Moreover, weighing often results, which can excite the natural frequencies of high-order unmodelled dynamics. A modified controller is now proposed using a continuous approximation of the signum function and we derive an explicit upper bound on the error in tracking of the leader.

Consider the following continuous, model-independent algorithm for the $i^{th}$ agent, replacing (8):

$$
\tau_i = -\eta \sum_{j \in N_{AI}} a_{ij} ((q_i - q_j) + \mu (q_i - \hat{q}_j)) - \beta z_i ((q_i - \hat{r}_i) + \mu (q_i - \hat{v}_i)) \quad (30)
$$

where $z_i(x) \triangleq \frac{x}{\|x\|_2 + \epsilon}$ with $\epsilon > 0$ being the degree of approximation. The function $z_i(x)$ approximates $\text{sgn}(x)$ via the boundary layer concept [33]. The networked system is:

$$
M\ddot{\bar{q}} + C\dot{\bar{q}} + \eta (L_{22} \otimes I_p)(\bar{q} + \mu \bar{q}) + g + \zeta + \beta z(s + \mu \hat{s}) + M(1_n \otimes \bar{q}_0) + C(1_n \otimes \bar{q}_0) = 0 \quad (31)
$$

where $z(s + \mu \hat{s}) = [z_1(s_1 + \mu \hat{s}_1)^T, \ldots, z_n(s_n + \mu \hat{s}_n)^T]^T$. Note that $\|z_i(x_i)\| < 1$ for any $\epsilon > 0$. The computation of $\chi, \gamma$ in Section III-C is unchanged. Because of similarity, we provide only a sketch. Consider the same Lyapunov-like function in (16), with $\mu$ sufficiently large to ensure $H > 0$.

Let $t_p$ and $t_q$ be defined as in Part 3.1 of the proof of Theorem 2. For $t \in t_p$, the derivative of (16) with respect to time along the trajectories of (31), can be shown to obey

$$
\dot{V} \leq -\mu^{-1} p(\|\bar{q}\|_2, \|\bar{q}\|_2) \quad (32)
$$

with $p(\cdot, \cdot)$ defined in (26) (we refer the reader to [28, Section III-E] for detailed calculations on $V$ and its bound). Thus, any $\mu, \eta$ which ensures that the system (9) remains in $S \cup T$, will also ensure that the system (31) remains in $S \cup T$ for all time.

Consider now $t \in t_q$. Using calculations that we detail in [28, Section III-E], one can show that

$$
\dot{V} \leq -\mu^{-1} \left[ \varphi(\nu, \mu) \frac{\bar{q}}{\|\bar{q}\|_2}^2 + \frac{1}{\eta} \lambda_{\text{min}}(X) \|\bar{q}\|_2^2 - 2kCk_p \|\bar{q}\|_2^2 - \sum_{i=1}^n (kCk_p + \beta) \|x_i\|_2^2 + \beta \gamma \sum_{i=1}^n \|x_i\|_2^2 + \epsilon \right] \quad (33)
$$

If $\beta$ satisfies (25) then there holds

$$
\beta \gamma \sum_{i=1}^n \|x_i\|_2^2 + \epsilon - \frac{1}{\eta} \lambda_{\text{min}}(X) \|\bar{q}\|_2^2 < -\beta \gamma \epsilon \quad (34)
$$

because $\|x_i\|_2^2/(\|\bar{q}\|_2^2 + \epsilon) < 1$ for all $\epsilon > 0$ (we refer the reader to [28, Section III-E] for detailed calculations used to obtain this inequality). From this, we conclude that $V \leq V_A + \beta \gamma \epsilon$. Recall that any $\mu, \eta$ for which $p(\cdot, \cdot)$ is positive definite in $\tilde{S}$ also ensures that $V_A$ is negative definite in $D$. Similar to Part 4 of the proof of Theorem 2, one has $V \leq \psi V + \beta \mu \psi$ for some $\psi > 0$. We conclude using [32, Lemma 3.4 (Comparison Lemma)] that $V(t) \leq V(0) e^{-\psi t} + \beta \mu \psi \int_0^t e^{-\psi(t-\tau)} d\tau$, which

We do not consider an approximation for (7) because the observer involves computing state estimates, as opposed to the physical control input for (6).
\[ a_{3,j}/\lambda_{\max}(L_{\mu,j}) \] where \( a_{3,j} \) was computed below (29). It follows that \( \Lambda = \min_{j \in J} a_{3,j}/\lambda_{\max}(L_{\mu,j}) \), and one can also obtain that \( \kappa = \max_{j \in J} \lambda_{\max}(L_{\mu,j})/\min_{j \in J} \lambda_{\min}(N_{\mu,j}) \).

**Theorem 4.** Under Assumptions 1 and 2, with dynamic topology given by \( G_A(t) = G_{A,n(t)} \), the leader tracking objective is achieved using (8) if 1) the control gains \( \mu, \eta, \beta \) satisfy a set of lower bounding inequalities, and 2) the dwell time \( \pi_d \) satisfies the inequality \( \pi_d > \log(\kappa)/\Lambda \), where \( \kappa, \Lambda \) are as defined in the immediately preceding paragraph.

**Proof.** By selecting \( \mu = \max_{j \in J} \mu_j, \eta = \max_{j \in J} \eta_j, \) and \( \beta = \max_{j \in J} \beta_j \), we guarantee each \( j \)th subsystem (38) is exponentially stable, and also guarantee the boundedness of the trajectories of (36) before the finite-time observer has converged. After convergence of the finite-time observer, application of Theorem 3 using the quantities of \( \kappa, \Lambda \) outlined above delivers the conclusion that (36) is exponentially stable, i.e. the leader tracking objective is achieved. \( \square \)

**V. SIMULATIONS**

A simulation is now provided to demonstrate the algorithm (8). Each agent is a two-link robotic arm and there are five follower agents. The equations of motion are given in [26, pp. 259-262]. The generalized coordinates for agent \( i \) are \( q_{i} = [q_{i}^{(1)}, q_{i}^{(2)}]^T \), which are the angles of each link in radians. The agent parameters are given in Table I, and are chosen arbitrarily. Several aspects of the simulation are designed to highlight the robustness of the algorithm. First, the topology is assumed to be switching, with the graph \( G_A(t) \) switching periodically between the three graphs indicated in Fig. 2, at a frequency of 1 Hz. Graph \( G_B(t) \) switches between the three graphs indicated in Fig. 3, also at a frequency of 1 Hz. Moreover, if \( G_A(t) = G_{A,i} \) then \( G_B(t) = G_{B,i} \) for \( i = 1, 2, 3 \). Additionally, \( G_B(t) \) is entirely disconnected for \( t \in [10, 20) \) of the simulation. Last, each agent has a disturbance \( \xi(t) = [\sin(i \times 0.1t), \cos(i \times 0.1t)]^T \) for \( i = 1, \ldots, 5 \). All edges of \( G_A(t) \) and \( G_B(t) \) have edge weights of 5. The control gains are set as \( \mu = 1.5, \eta = 16, \beta = 25 \); they are first computed using the inequalities and then adjusted because the inequalities can lead to conservative gain choices. For the observer, set \( \omega_1 = 1, \omega_2 = 5 \).

The leader trajectory is

\[
q_0(t) = \begin{bmatrix}
0.5 \sin(t) - 0.2 \sin(0.5t) \\
0.4 \left(2 \sin(t) + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \frac{\sin(4t)}{4}\right)
\end{bmatrix}
\]

Figure 4 shows the generalized coordinates \( q^{(1)} \) and \( q^{(2)} \). The generalized velocities, \( \dot{q}^{(1)} \) and \( \dot{q}^{(2)} \) are shown in Fig. 5. The well studied observer results are omitted. Consider Fig. 4. Clearly, \( q_i(t) \) has almost tracked the leader by \( t = 10 \), but the distributed observer graph \( G_B(t) \) disconnects for \( t \in [10, 20) \). As discussed in Remark 5, the controller (8) has robustness to network failure, since the linear term in (8) ensures the trajectories remain bounded as long as \( G_B(t) \) is disconnected (thus followers do not possess accurate knowledge of \( q_0, \dot{q}_0 \)). In the simulation, we observe leader tracking is achieved once \( G_B(t) \) reconnects at \( t > 20 \). In [28], additional simulations show the effect of increasing the gain \( \mu \) (see Remark 3), and the continuous approximation (30).

**VI. CONCLUSION**

In this paper, a distributed, discontinuous model-independent algorithm was proposed for a directed network of Euler-Lagrange agents. It was shown that the leader tracking objective is achieved semi-globally exponentially fast if the directed graph contains a directed spanning tree, rooted at the leader, and if three control gains satisfied a set of lower bounding inequalities. The algorithm was shown to be robust to agent disturbances, unmodelled agent dynamics and modeling uncertainties. A continuous approximation of the algorithm was proposed to avoid chattering, and we then extended the result to include switching topologies. A numerical simulation illustrated the algorithm’s effectiveness.

**REFERENCES**


Table I

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<td>0.08</td>
<td>0.15</td>
<td>0.08</td>
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<td>Agent 3</td>
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<td>0.2</td>
<td>0.15</td>
<td>0.1</td>
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<td>0.3</td>
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<td>Agent 4</td>
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<td>0.45</td>
<td>0.8</td>
<td>0.2</td>
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<td>0.15</td>
<td>0.5</td>
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</tr>
<tr>
<td>Agent 5</td>
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</table>


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