2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Distance-based rigid formation control with signed area constraints

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Abstract— **This paper discusses a formation control problem in which a target formation is defined with both distance and *signed* area constraints. The control objective is to drive spatially distributed agents to reach a unique target rigid formation shape (up to rotation and translation) with desired inter-agent distances. We define a new potential function by incorporating both distance terms and signed area terms and derive the formation system as a gradient system from the potential function. We start with a triangle formation system with detailed analysis on the equilibrium and convergence property with respect to a weighting gain parameter. We then examine the four-agent double-triangle formation and provide conditions to guarantee that both triangles converge to the desired side distances and signed areas.**

I. INTRODUCTION

This paper deals with an aspect of formation shape control that has been largely untreated in the literature to this point, that is, to stabilize a target rigid formation shape with both distance and signed area constraints. To explain this aspect, and to state our contribution, we first recall the two broad approaches to shape control, viz., displacement-based and distance-based approaches [1]. In the linear displacement-based approach, the desired formation is specified by a certain set of inter-agent relative positions which means that the orientation of the final formation is implicitly fixed. The angular orientation of the formation is determined by the data, as is the signed area of a triangle formed by any three agents in the formation. The displacement-based approach has been discussed in e.g. [2] and [3].

In contrast, for the nonlinear distance-based approach, the desired formation is specified by a certain set of inter-agent distances, and the orientation of the target formation is not implicitly or explicitly defined. Actually, if a certain condition known as rigidity is satisfied by the underlying graph [4], [5], then there are a finite number of noncongruent formation orbits (and sometimes a unique orbit) which will achieve the distances, where a formation orbit is a set of formations differing simply by one or more of translation, angular rotation, or reflection. We refer the reader to the recent survey [1] for more discussions and comparisons of these two formation control approaches. We note that in most papers on distance-based formation control, signed area constraints are not considered.

Some attempts have been made to solve problems that are in some sense intermediate between these two classes. For example, as discussed in [6], one can consider the distance-based control approach and superimpose a requirement that a particular angular orientation be achieved, and indeed one or possibly two relative positions alone can be controlled, along with the distances associated with enough edges to guarantee rigidity, to achieve the objective. We note that the formation control strategy involving orientation constraints in [6] still has not fully solved the formation reflection issue, unless the initial formation shape is assumed to be sufficiently close to the target one. In [7], this issue has been tackled by considering the equilibrium set including the freedom of translation and rotation but excluding reflection, namely the special Euclidean group. However, global convergence has not been discussed. A more relevant paper is [8], which considered a similar distance-based formation control problem by including both distance and planar (or volume) restrictions in a performance index function. The planar (or volume) constraint is defined by the relative angular information in certain selected edges, which helps to exclude symmetric counterparts of a target formation if a gradient formation control system derived from the performance function is applied. However, the discussions and main results in [8] established only a local convergence result based on convergence results of gradient systems, and a detailed analysis on the equilibrium set and its properties is still lacking. Another closely related paper is [9], which proposed a novel distance-based potential function with the use of angular information between agents to reduce the likelihood of unwanted formation patterns. Moreover, the approach in [9] also eliminates the local minimum generated by the control laws to reach the desired formation configuration in the case of three agents.

In this paper, we do not seek to control the angular orientation, but we do seek to control the reflection variable. More precisely, in the case of a triangular formation, we seek to control the *sign* of the area of the triangle, as well as the lengths of its sides. There are clear motivations to control a

This work was supported by the Australian Research Council under grant DP130103610 and DP160104500. Z. Sun was supported by the Australian Prime Minister's Endeavour Postgraduate Award from Australian Government. T. Sugie is supported by JSPS KAKENHI Grant Number JP16K14284. S. Azuma was supported by JSPS KAKENHI Grant Number 15H00814. K. Sakurama was supported by JSPS KAKENHI Grant Number 15K06143.

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978-1-5090-2872-6/17/$31.00 ©2017 IEEE

target formation with preference of one signed area over the other. In general, such a preference might arise if the agents are in some way heterogeneous. Suppose for example that in a triangle formation three agents are fixed-wing unmanned aerial vehicles (UAVs), undertaking surveillance. It may be that each agent is equipped with a camera with limited angular field of view whose look-direction is a priori fixed relative to the direction of forward motion of the UAV, and the three cones associated with the look directions should have a common area of intersection in order to e.g., locate a target. If the three agents were to be interchanged in their positions but otherwise the interagent distances were the same (thus were there to be a change of the sign of the triangle area), then the three agents may no longer be able to simultaneously see a target. Thus in this case the sign of the triangle becomes critical in securing such a formation property.

Of course, the general aim is of broader applicability than just for a single triangle, and we illustrate its applicability with a four agent formation also in the paper. This strongly suggests the ideas could be extended to more general formations. For a four agent formation with five edge lengths specified, we can specify two triangle signed errors to pin down a desired formation that is unique up to translation and angular rotation, and we aim to provide a control law to converge to it. We note that the collision avoidance issue is not considered in the approach, which will be discussed in future research.

The remaining parts of this paper are organized as follows. We start with a triangular formation in Section II, derive the control law from a modified index function involving both distance errors and signed area, and present a detailed analysis on the equilibrium properties and convergence of the formation control system. For an extension, in Section III we treat a four agent formation comprising two triangles with five interagent distances specified, and show that it is possible to choose the performance index so that from all but a thin set of initial conditions, one will asymptotically obtain a formation where both triangles have the desired orientation and the distances are correct. The final Section IV offers concluding remarks.

II. Securing a triangle shape and signed area

In this section, we set out in a detailed way the problem of formation shape control with signed area for a triangular (three agent) formation.

In the usual approach to shape control (with no account taken of signed area, see e.g. [10], [11]), three desired lengths for the triangle sides which should satisfy the standard triangle inequality are specified. Denote the positions of the three agents as p_i with $i = 1, 2, 3$. Given a triangular formation with three desired lengths, there is a manifold of such formations formed by translation and rotation (see e.g., the discussions in [12]). But in addition, there is a mirror image manifold: a triangle with corners defined by p_1, p_2, p_3 is congruent to one defined by $-p_1, -p_2, -p_3$ and one triangle cannot be smoothly transformed to the other without encountering a collinearity of the three 'corners' at some intermediate point of the deformation process. There are evidently two branches in the set of desired formations. It is possible to think about these two branches in a quite systematic way, by distinguishing them on the basis of the sign of the enclosed area.

Indeed, as is well known, it is possible to define a signed area for a triangle, denoted by Z, as

$$Z = \frac{1}{2}\det \begin{bmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \end{bmatrix} \tag{1}$$

where 'det' denotes *determinant*. Note that the matrix after 'det' in (1) is a square matrix as each $p_i \in \mathbb{R}^2$. The quantity Z is positive or negative according as the ordering of p_1, p_2, p_3 around the boundary of the triangle is counterclockwise, or clockwise. This observation suggests that if we wish to achieve a particular formation shape with a prescribed cyclic ordering of the triangle vertices, i.e. a prescribed sign for the area of the triangle, we should incorporate a function reflecting that area into the performance index.

Let us now study how we can control both distances and sign of the triangle area. First, recall that with no account taken of the sign of the area, and three prescribed distances $d_{12}^*, d_{23}^*, d_{13}^*$, a commonly used index, see e.g. [10], has been

$$\begin{aligned} V(p_1,p_2,p_3) = \frac{1}{4}((||p_1-p_2||^2 - d_{12}^{*2})^2 + (||p_2-p_3||^2 - d_{23}^{*2})^2 \\ + (||p_3-p_1||^2 - d_{13}^{*2})^2) \end{aligned} \tag{2}$$

Now let Z^* denote the area (including sign) of the desired triangle. The magnitude of Z^* is of course determined from the d_{ij}^* but the sign is not. Then an adjustment to the index reflecting the area is evidently available as

$$\begin{aligned} V(p_1,p_2,p_3) = \frac{1}{4}((||p_1-p_2||^2 - d_{12}^{*2})^2 + (||p_2-p_3||^2 - d_{23}^{*2})^2 \\ + (||p_3-p_1||^2 - d_{13}^{*2})^2) + \frac{1}{2}(Z-Z^*)^2 \end{aligned} \tag{3}$$

Note that all four summands dimensionally involve distance raised to the fourth power. The information in terms of d_{ij}^* and Z^* is used to describe the target formation shape, for which we assume all agents have access to these terms before running the control law, or obtain them via communication during the formation control. Separately, we note that the index has the general properties of invariance to displacement and angular rotation. However, it is not invariant to reflection.

A. Triangle motion under the gradient descent law

In this subsection, we obtain equations of motion related to use of the index (3), and we derive several properties of the motion.

For notation simplicity we denote the distance error term e_{ij} for edge (i, j) associated with agents i and j as $e_{ij} = ||p_i - p_j||^2 - d_{ij}^{*2}$. The gradient descent law obtained from a performance index V is of the form $\dot{p}_i = -\frac{\partial V}{\partial p_i}$ which in the case of the common index of (2) becomes

$$\begin{aligned}\dot{p}_1 &= -\,e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3)\\ \dot{p}_2 &= -\,e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3)\\ \dot{p}_3 &= -\,e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2)\end{aligned} \tag{4}$$

It is not much harder to verify that the law obtained from (3) is, with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5}$$

given by

$$\begin{aligned}\dot{p}_1 &= -\,e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3) - (Z - Z^*)J(p_2 - p_3)\\ \dot{p}_2 &= -\,e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3) - (Z - Z^*)J(p_3 - p_1)\\ \dot{p}_3 &= -\,e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2) - (Z - Z^*)J(p_1 - p_2)\end{aligned} \tag{6}$$

We remark that, as is well known, to implement the law (4) each agent needs to be able to measure the relative position of its neighbors, but all agents can have their own coordinate bases, i.e. there is no requirement for a global coordinate basis to be available for each agent. Now we show the same is true for (6). Considering the last term in the first equation of (6) for example, observe that $p_2 - p_3 = (p_2 - p_1) - (p_3 - p_1)$ and from (1) there holds that

$$\begin{aligned} Z &= \frac{1}{2}\det\begin{bmatrix} 1 & 0 & 0 \\ p_1 & p_2 - p_1 & p_3 - p_1 \end{bmatrix}\\ &= \frac{1}{2}\det\begin{bmatrix} p_2 - p_1 & p_3 - p_1 \end{bmatrix}\end{aligned} \tag{7}$$

Evidently, the additional term in the law again involves relative positions. Therefore, all agents with control (6) do not require access to a global coordinate basis.

Our interest is in studying how the solutions of these equations evolve. As a preliminary observation, observe that, *with or without* the inclusion of the area term, there holds

$$\sum_{i=1}^{3} \dot{p}_i = 0 \tag{8}$$

which implies that the centroid of the triangle remains fixed. Two other observations are:

1) $p_1 = p_2 = p_3$ is an equilibrium point of both sets of equations. It is trivial to see that any incremental motion away from this equilibrium point reduces V, implying that the equilibrium point defines a maximum of V and accordingly is an unstable equilibrium point for a gradient descent algorithm.
2) If three agents are collinear and at most two are collocated, the trajectory of the system with no area term in the performance index retains collinearity of the agents, but this is not the case for the system involving an area term, as we now check in some detail.

To this end, we will now prove the following lemma.

Lemma 1. *Consider the motion of three agents according to equations (6), and suppose that at some point in time the three agents are collinear, with agent 2 between agents 1 and 3, or with agent 2 and agent 3 collocated. Suppose that Z^* is positive (negative). Then the components of motion of the agents at right angles to the line will be such that after an incremental motion the resulting triangle will have p_1, p_2 and p_3 occurring counterclockwise (clockwise), i.e. Z is positive (negative).*

Proof. Without loss of generality, suppose that the agents lie on the x-axis, with agent 1 the left most agent, and suppose that $Z^* > 0$. Suppose initially that no two of the three agents are collocated. Thus $x_1 < x_2 < x_3$. Then we can show that the y-components of $\dot{p}_1$ and $\dot{p}_3$ are positive and the y-component of $\dot{p}_2$ is negative. This is easily checked. In obvious notation, we have immediately from (6)

$$\begin{aligned}\dot{y}_1 &= -\,(0 - Z^*)(x_3 - x_2) > 0\\ \dot{y}_2 &= -\,(0 - Z^*)(x_1 - x_3) < 0\\ \dot{y}_3 &= -\,(0 - Z^*)(x_2 - x_1) > 0\end{aligned} \tag{9}$$

Next suppose that agents 2 and 3 are collocated, so that $x_2 = x_3$. Then it is easily seen that $\dot{y}_1 = 0, \dot{y}_2 < 0, \dot{y}_3 > 0$ and collocation is again broken. In both cases, when the collocation is broken there arises $Z > 0$, as claimed. □

Of course, equivalent results to those of the lemma occur with other orderings of the agents.

There is an immediate corollary to Lemma 1. Observe that if on a trajectory, the three agents define a triangle of area Z with the same sign as Z^*, they will never become collinear. This is evident from Lemma 1, which shows that movement from a collinear position is always in the direction which causes Z and Z^* to have the same sign.

Corollary 1. *Consider the motion of three agents according to equations (6), and suppose at some time the area of the triangle formed by the three agents, Z, has the same sign as the desired final area, Z^*. Then at subsequent times, the agents will never become collinear.*

B. Defining the equilibrium points

We now aim to study the remaining equilibrium points arising from the algorithm, especially the stable ones which result from local or global minima of V. It is well understood that the set of equilibria breaks into orbits, with an orbit being obtained through translation and rotation of an equilibrium point. Each orbit is evidently defined by the three inter-agent distances and sign of the enclosed area. We shall show that (up to rotation and translation) there is only one stable equilibrium point (i.e., more precisely there is one orbit of equilibrium points) for which the associated Z and the prescribed Z^* have the same sign, namely the equilibrium point at which all distances (and therefore also the area) equal the desired values. To analyse other cases, we shall investigate a generalization of the previous index. The generalization is obtained by varying the weighting of the area error relative to the distance errors, and with K a positive constant is given by

$$V(p_1,p_2,p_3) = \frac{1}{4}((||p_1-p_2||^2 - d_{12}^{*2})^2 + (||p_2-p_3||^2 - d_{23}^{*2})^2 + (||p_3-p_1||^2 - d_{13}^{*2})^2) + \frac{1}{2}K(Z-Z^*)^2 \quad (10)$$

The gradient descent law from the modified potential (10) is

$$\begin{aligned}
\dot p_1 &= -e_{12}(p_1-p_2) - e_{13}(p_1-p_3) - K(Z-Z^*)J(p_2-p_3)\\
\dot p_2 &= -e_{12}(p_2-p_1) - e_{23}(p_2-p_3) - K(Z-Z^*)J(p_3-p_1)\\
\dot p_3 &= -e_{13}(p_3-p_1) - e_{23}(p_3-p_2) - K(Z-Z^*)J(p_1-p_2)
\end{aligned} \quad (11)$$

Then at an equilibrium, there holds

$$\begin{aligned}
-e_{12}(p_1-p_2) - e_{31}(p_1-p_3) - K(Z-Z^*)J(p_2-p_3) &= 0\\
-e_{23}(p_2-p_3) - e_{12}(p_2-p_1) - K(Z-Z^*)J(p_3-p_1) &= 0\\
-e_{31}(p_3-p_1) - e_{23}(p_3-p_2) - K(Z-Z^*)J(p_1-p_2) &= 0
\end{aligned} \quad (12)$$

We will also use the observation of the following lemma in our calculations.

Lemma 2. *For a triangle with corners defined by vectors p_1, p_2, p_3, the triangle area Z is given by*

$$Z = -\frac{1}{2}(p_2-p_3)^\top J(p_1-p_2) = -\frac{1}{2}(p_2-p_3)^\top J(p_1-p_3) \quad (13)$$

The proof for the above equality is omitted as it can be verified via direct calculation. Now we consider the equilibrium points for system (11). The following result deals with the case when Z and Z^* have the same sign:

Theorem 1. *With notation as given above, consider the above set of equations (12) in which $Z^* > 0, Z > 0, K > 0$. Then there necessarily holds $e_{12} = e_{23} = e_{31} = 0$ and $Z = Z^*$.*

Proof. Our proof will be by contradiction. Suppose then there does exist an equilibrium with $Z > 0$, but without one at least of the distances or Z assuming correct values. We will use the consequence of the equilibrium equations (12) and divide the possibilities into three cases: $Z = Z^*, Z < Z^*$ and $Z > Z^*$. Taking these equations multiply successively by $-(p_2-p_3)^\top J, -(p_3-p_1)^\top J$ and $-(p_1-p_2)^\top J$, and use the above Lemma 2. There results

$$\begin{aligned}
-2(e_{12}+e_{31})Z - K(Z-Z^*)||p_2-p_3||^2 &= 0\\
-2(e_{23}+e_{12})Z - K(Z-Z^*)||p_3-p_1||^2 &= 0\\
-2(e_{31}+e_{23})Z - K(Z-Z^*)||p_1-p_2||^2 &= 0
\end{aligned} \quad (14)$$

Case 1: $Z = Z^*$. In this case, the equilibrium equations become identical to those applying when shape control is achieved using solely distances as the basis for the cost function. It is well known (see e.g. [13], [14]) that the only (noncollinear) equilibria are those where all distances are correct. Alternatively, we can see from (14) that $e_{12}+e_{23} = 0,\ e_{23}+e_{31} = 0,\ e_{31}+e_{12} = 0$, whence all three e_{ij} are zero.

Case 2: $Z < Z^*$. In this instance, there may be three, two, one or zero of the associated equilibrium distances $d_{ij} = ||p_i - p_j||$ which are less than the corresponding d_{ij}^*. Suppose firstly there are two or three such distances; without loss of generality, suppose $e_{12} < 0,\ e_{31} < 0$. Then the various sign constraints imply a contradiction of the first equation of (14). Next suppose there is just one distance d_{ij} with $d_{ij} < d_{ij}^*$, without loss of generality d_{12}, thus $e_{12} < 0,\ e_{23} \geq 0,\ e_{31} \geq 0$. Next, observe that from (12), when multiplying the equations successively by $(p_2-p_3)^\top,\ (p_3-p_1)^\top, (p_1-p_2)^\top$, there results

$$\begin{aligned}
-e_{12}(p_2-p_3)^\top(p_1-p_2) + e_{31}(p_2-p_3)^\top(p_3-p_1) &= 0\\
-e_{23}(p_3-p_1)^\top(p_2-p_3) + e_{12}(p_3-p_1)^\top(p_1-p_2) &= 0\\
-e_{31}(p_1-p_2)^\top(p_3-p_1) + e_{23}(p_1-p_2)^\top(p_2-p_3) &= 0
\end{aligned} \quad (15)$$

Now the sign of each of the inner products in these equations is determined by the cosine of the angle between the relevant vectors. The vectors themselves are the relative position vectors associated with the sides of the triangular formation; and taking account of directions, and noting that a triangle can have at most one obtuse angle, we see that at most one of the signs can be positive. Hence one at least of the first two equations has inner products which are both negative. Suppose without loss of generality, it is the first equation. Then the sign condition on the inner products coupled with the assumption that $e_{12} < 0,\ e_{31} > 0$ yields a contradiction.

Now we consider the case that none of the associated equilibrium distances are less than the corresponding d_{ij}^* distances, or equivalently, that $e_{12} > 0,\ e_{23} > 0,\ e_{31} > 0$. Note that any triangle can be classified as either obtuse, or right, or acute. First suppose the triangle is right-angled. Without loss of generality we assume that the angle opposite to the edge $(1,3)$ is a right angle, or equivalently, $(p_2-p_3)^\top(p_1-p_2) = 0$. From the first equation of (15) it leads to $e_{31} = 0$ which is a contradiction to the assumption that $e_{31} > 0$. Then suppose that the triangle is obtuse (with one obtuse angle), which indicates that one of the inner product terms in (15) is positive. However, by noting that all e_{ij} are positive, two equations in (15) could not hold due to the contraction of the sign conditions. Then we suppose the triangle is acute with all three angles being acute. However, there do not exist such equilibrium points satisfying $e_{12} > 0,\ e_{23} > 0,\ e_{31} > 0$ and $Z < Z^*$. [1]

In summary, there cannot be an equilibrium satisfying the condition of Case 2.

A very similar argument also shows there cannot be an equilibrium satisfying the condition of Case 3, i.e. $Z > Z^*$, and the theorem is then proved. □

Next, we will investigate equilibria when the associated Z is of opposite sign to Z^*. Our next result shows that if K is

[1] Denote the three sides and the corresponding angles in a triangle as $a,\ b,\ c$ and α, β, γ, respectively. According to the triangle area formula $Z = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$, and the cosine rule $a^2 = b^2 + c^2 - 2bc\cos\alpha$ (with the other two sets of equations, omitted here), it is obvious that the area Z is an increasing function of its three side lengths $a,\ b,\ c$ when $\alpha, \beta, \gamma \in (0, \pi/2)$.

chosen sufficiently large, there is no stable equilibrium with Z, Z^* of opposite signs.

Theorem 2. *With notation as given above, consider the above set of equations (12) in which $Z^* > 0, Z < 0, K > 0$. Then for K sufficiently large, there exists no solution of the equations. A sufficient condition on the magnitude of K for there to be no solution is*

$$K > \frac{d_{12}^{*2} + d_{23}^{*2} + d_{13}^{*2}}{\sqrt{3}Z^*} \tag{16}$$

We refer the readers to [15] for a detailed proof of the above theorem.

We make several observations. First, note that the inequality (16) is only a sufficient condition for there to be no equilibrium point with Z and Z^* of opposite sign. Thus it may be that a smaller value of K would suffice. In the case of an *equilateral* triangle, one can actually show that the bound from this inequality turns out to be exact. Second, observe using Weitzenbock's inequality that the right hand side of (16) is lower bounded by 4. Third, observe that there exist triangles with fixed perimeter and arbitrarily small area, implying the lower bound in K applying to such triangles can be arbitrarily large.

If the weighting K in the performance index is zero, we know that besides the two equilibrium points minimizing the performance index, corresponding to two triangles with correct distances and oppositely signed areas, there are saddle point equilibria (more strictly equilibrium orbits) "between" the two minimizing equilibria, at which the p_i are collinear. If K is small and nonzero, because there are two equilibria (strictly equilibrium orbits) known to be minima, there will be boundaries of the regions of attraction for each equilibrium which include points other than at infinity, and the set or sets of such boundary points itself forms an invariant set on which there will be an equilibrium. It will necessarily be a saddle however. For nonzero K, the set is obviously not straightforward to characterize.

Also, if K is taken sufficiently large that there is no equilibrium with incorrectly signed area, and if the initial condition for a trajectory is such that the area for that initial condition is incorrectly signed, it is clear that at some point in the motion, the area must pass through zero, i.e. the three agents will be collinear. Of course, they do not remain collinear.

III. Formations comprising two triangles

In this section, we will treat the case of a four agent formation with five edges, as a starting point of extensions to more general formations. In this case, there are precisely two triangles, with a common edge. To fix ideas, suppose that the two triangles are formed by agents $1, 2, 3$ and $2, 3, 4$, with a common edge $(2, 3)$. Taking into account the possible separate clockwise/anticlockwise orientations of the two triangles, we note that there are in all four possible formations (centroid and angular orientation being irrelevant in this classification). Two of these have agents 1 and 4 on the same side of edge $(2, 3)$, and are congruent but differ through the signs of the areas, one being obtainable from the other by replacing the four position vectors by their negatives. The other two have agents 1 and 4 on opposite sides of edge $(2, 3)$. Again, these two are congruent, and one can be obtained from the other by replacing the four position vectors by their negatives.

We shall show that it is possible to formulate a performance index through the inclusion of terms reflecting the signed area of each triangle to achieve a particular one of the four formations. Suppose that the desired distances are $d_{12}^*, d_{23}^*, d_{31}^*, d_{34}^*$ and d_{42}^*. Let the corresponding desired areas of triangle 123 and triangle 234 be respectively Z_A^*, Z_B^*. With no real loss of generality, take these as positive, so the counterclockwise orderings of the vertices of the desired triangles are 123 and 234, respectively. The performance index we use is:

$$\begin{aligned} V(p_1, p_2, p_3, p_4) =& \frac{1}{4}(e_{12}^2 + e_{23}^2 + e_{13}^2 + e_{24}^2 + e_{34}^2) \\ &+ \frac{1}{2}K\left((Z_A - Z_A^*)^2 + (Z_B - Z_B^*)^2\right) \end{aligned} \tag{17}$$

There would be no problem in having different weighting for $(Z_A - Z_A^*)^2$ and $(Z_B - Z_B^*)^2$ but no theoretical advantage appears to emerge.

For the case of a formation comprising a single triangle, we showed that use of a sufficiently large weighting K for the squared area error meant that there was no equilibrium with the associated triangle having an area with incorrect sign. For the case of a two triangle formation, the situation is slightly different; we shall show that if there is an equilibrium with a triangle having the wrong sign, the associated Hessian matrix has a negative eigenvalue, i.e. the equilibrium cannot be a minimum. Accordingly, for almost all initial conditions, an equilibrium with the correct signs of the triangle areas will be attained, and we can further show that indeed the distances are correct for such an equilibrium. To provide the necessary tools then, we first obtain the equilibrium equations and the Hessian.

A. Equilibrium equations and Hessian of the performance index

The equilibrium equations are simply obtained by setting $\nabla V = 0$ with V defined in (17). The calculations are little different to those applying with a single triangle. The end result is as follows:

$$\begin{aligned} &-e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3) - K(Z_A - Z_A^*)J(p_2 - p_3) \overset{(1)}{=} 0 \\ &-e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3) - e_{24}(p_2 - p_4) \\ &-K(Z_A - Z_A^*)J(p_3 - p_1) - K(Z_B - Z_B^*)J(p_3 - p_4) \overset{(2)}{=} 0 \\ &-e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2) - e_{34}(p_3 - p_4) \\ &-K(Z_A - Z_A^*)J(p_1 - p_2) - K(Z_B - Z_B^*)J(p_4 - p_2) \overset{(3)}{=} 0 \\ &-e_{24}(p_4 - p_2) - e_{34}(p_4 - p_3) - K(Z_B - Z_B^*)J(p_2 - p_3) \overset{(4)}{=} 0 \end{aligned} \tag{18}$$

Again, we remark that, by taking the gradient control law in the form $\dot{p} = -\nabla V$, each agent requires the measurement of relative positions with respect to its neighbors to implement the control. Write the performance index V as a sum $V_1 + V_2$, where V_1 contains the distance error terms and V_2 the area error terms. The Hessian of V_1 has been computed for an almost identical case in [16], and is given by

$$\nabla^2 V_1 = 2R^\top R + E(p) \otimes I_2 \tag{19}$$

where R is the 5×8 rigidity matrix (essentially the Jacobian of the mapping from agent positions to squares of edge lengths, with the edge ordering having immaterial effect on the Hessian), and E is the matrix

$$E = \begin{bmatrix} e_{12}+e_{13} & -e_{12} & -e_{13} & 0 \\ -e_{12} & e_{12}+e_{23}+e_{24} & -e_{23} & -e_{24} \\ -e_{13} & -e_{23} & e_{13}+e_{23}+e_{34} & -e_{34} \\ 0 & -e_{24} & -e_{34} & e_{24}+e_{34} \end{bmatrix}$$

A tedious calculation also delivers

$$\nabla^2 V_2 = \frac{1}{4}K\left\{ Y_A Y_A^\top + 2(Z_A - Z_A^*)\begin{bmatrix} 0 & J & -J & 0 \\ -J & 0 & J & 0 \\ J & -J & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + Y_B Y_B^\top + 2(Z_B - Z_B^*)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & J & -J \\ 0 & -J & 0 & J \\ 0 & J & -J & 0 \end{bmatrix}\right\} \tag{20}$$

with

$$Y_A = \begin{bmatrix} J(p_2 - p_3) \\ J(-p_1 + p_3) \\ J(p_1 - p_2) \\ 0 \end{bmatrix} \quad Y_B = \begin{bmatrix} 0 \\ J(p_3 - p_4) \\ J(-p_2 + p_4) \\ J(p_2 - p_3) \end{bmatrix} \tag{21}$$

Of course, $\nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2$.

B. Large area weighting implies no equilibria with incorrect triangle signs

In this subsection, we will establish the following result.

Theorem 3. *Consider the four agent control problem with specified and achievable interagent distances $d_{12}^*, d_{23}^*, d_{31}^*, d_{24}^*, d_{34}^*$ and let the associated signed triangle areas for agents 123 and 234 be $Z_A^* > 0, Z_B^* > 0$. Suppose that a control law is established as a gradient descent law using the performance index (17). Then for sufficiently large K, there can be no stable equilibrium in which the signs of the triangle areas are incorrect.*

In the proof we shall first show the effect of large K when at an equilibrium, at least one of the pairs Z_A, Z_A^* or Z_B, Z_B^* have differing signs. Then we shall consider the situation where one of Z_A, Z_B is zero at an equilibrium. Our aim is to establish that there can be no such stable equilibrium by investigating the condition on which the Hessian $\nabla^2 V$ has at least one negative eigenvalue. Due to space limit the proof is omitted here and will be provided in the journal version [15].

IV. Conclusions

In this paper we have discussed the formation control problem for rigid formation shapes with both distance and area constraints. For a rigid target formation, the signed area is incorporated in the performance index function to address the formation reflection issue. By taking the triangle formation as an example, we analyze the equilibrium property and provide conditions with a weighting parameter to guarantee the uniqueness of the desired equilibrium point and a global convergence of the formation to correct side lengths and signed area. The results are then extended to a four-agent formation shape comprising two triangles, which could be further generalized to more complex formations.

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