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IFAC PapersOnLine 50-1 (2017) 1742–1747

Asynchronous Agreement through Distributed Coordination Algorithms Associated with Periodic Matrices ⋆

Yuzhen Qin * **Ming Cao** * **Brian D. O. Anderson** **,***

* *Faculty of Science and Engineering, ENTEG, University of Groningen, Groningen, 9747 AG, the Netherlands. (e-mail: {y.z.qin, m.cao}@rug.nl)*
** *School of Automation, Hangzhou Dianzi University, Hangzhou, 310018, China.*
*** *Data61-CSIRO and Research School of Engineering, Australian National University, Canberra, ACT 2601, Australia. (e-mail: brian.anderson@anu.edu.au)*

Abstract: For the widely studied consensus-type distributed multi-agent algorithms, a standard discrete-time model is a linear system whose system matrix is stochastic, thus implementing the "averaging" updating rule for each agent. To ensure agreement among all the agents asymptotically, one usually requires the stochastic matrix to be indecomposable and aperiodic (SIA). In this paper, we show that in practice such requirements can be relaxed by allowing the matrix to be periodic if the agents update asynchronously. Such a relaxation is somewhat surprising since for synchronous updating, periodic matrices in general give rise to oscillations in the agents' states. The key step to prove reaching agreement is to use a stochastic Lyapunov function to prove almost sure convergence of the associated stochastic linear system. The results reveal the critical role that asynchrony may play for distributed network algorithms.

Keywords: Distributed Coordination, Periodic Matrix, Asynchronous Updating, Convergence

1. INTRODUCTION

Distributed coordination algorithms have played important roles in the studies of various distributed systems over the last decade (Jadbabaie et al. (2003); Cao et al. (2008); Ren et al. (2005)). When individuals coupled by a network repeatedly update their states to the weighted average of their neighbors' states and their own, they can reach *consensus* or *agreement* for the value of some variable of common interest. Various applications of distributed coordination algorithms have appeared, which include distributed optimization (Nedić et al. (2015)), the study of opinion dynamics in social networks (Hegselmann et al. (2002); Xia et al. (2016)), etc.

Distributed coordination algorithms can be described by a linear recursion equation, $x(k) = P(k)x(k-1)$, where $x(k) \in \mathbb{R}^n$ is the state vector, and $P(k) \in \mathbb{R}^{n\times n}$ is a stochastic matrix. When $P(k) = P$ is independent of time, the agreement problem is equivalent to studying whether the P^k converge to a rank-one matrix, i.e., whether P is an indecomposable and aperiodic stochastic matrix (Wolfowitz (1963); Seneta (2006)). If the matrix changes over time, the analysis of the convergence of the inhomogeneous products $P(k)P(k-1)\cdots P(2)P(1)$ is much more difficult. Results on the inhomogeneous products of stochastic matrices in Wolfowitz (1963) have been widely applied to establish the consensus of networked multi-agent systems (Jadbabaie et al. (2003); Ren et al. (2005); Liu et al. (2012)).

When agents update asynchronously, distributed coordination algorithms associated with a fixed P can also give rise to inhomogeneous products, in which case agents independently update their own states according to their own clocks. In Cao et al. (2008), it has been shown that agents can reach agreement if the underlying graphs of the updating sequence are repeatedly jointly rooted. Xia et al. have proved that agreement can be preserved even when there is no synchronous clock for agents to activate their update actions as long as P is scrambling (Xia et al. (2014)). However, most of the tools utilized for the analysis are those for deterministic updating dynamics.

An important application of these algorithms arises when the sequences driving the algorithms are stochastic in nature. Correspondingly, some results on the product of random stochastic matrices have already been presented in Tahbaz-Salehi et al. (2010), Touri et al. (2014) and Hendrickx et al. (2015). However, restrictive assumptions are made, without which the convergence will be difficult to establish. The results in Tahbaz-Salehi et al. (2010) require that the random sequences of stochastic matrices are generated by an ergodic stationary process; it is

⋆ The work of Qin was supported in part by China Scholarship Council (CSC). The work of Cao was supported by European Research Council (ERC-StG-307207) and Netherlands Organization for Scientific Research (NWO-vidi-14134). The work of Anderson was supported by the Australian Research Council under grant DP130103610 and DP160104500.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2017.08.502

assumed that the sequences $\{P(k)\}$ are strongly aperiodic in Touri et al. (2014); in Hendrickx et al. (2015), each $P(k)$ has positive diagonal elements. The randomized agreement problems studied in Shi et al. (2012, 2015) can be taken as problems with stochastic updating sequences of agents driven by independent Bernoulli processes. The convergence results have been obtained relying on the assumption that each agent has access to its own state while executing averaging actions at every time instant; in others words, the assumption implies that $P(k)$ always has positive diagonal entries. Therefore, the case when some individuals do not know or just ignore their own states while updating, which is called lack of self-confidence in social networks (Nowak (2015)), remains open.

In this paper, we consider a group of agents coupled by a fixed network described by a stochastic matrix P. Most existing works, e.g. Xia et al. (2014, 2017) and Chen et al (2017), require this stochastic matrix to be indecomposable and aperiodic to ensure agreement among all the agents asymptotically. However, we can relax this requirement and relax the matrix P to be periodic if the agents update asynchronously, even though systems with periodic matrices generally have oscillating behaviors.

Specifically, we assume that agents are activated by mutually independent clocks. At each time step, a random number of agents are activated and then update. In sharp contrast to the existing works, e.g. Shi et al. (2012, 2015) and Hendrickx et al. (2015), agents do not need to use their own states to update. In other words, the updating matrix $P(k)$ at each time step, which will be defined formally later, does not have to have all positive diagonal entries. A stochastic Lyapunov function is used to prove almost sure convergence of the associated stochastic linear system. The properties of *supermartingales* shown in Kushner (1965, 1972) and Durrett (2010) will be introduced to construct the proofs. The obtained results reveal that asynchrony can play a very important role in reaching consensus.

This paper is organized as follows. In section 2, the research problem is formulated. In section 3, the main results and corresponding proofs are provided. In section 4, we draw conclusions and briefly discuss future work.

2. PROBLEM FORMULATION

Consider a system consisting of n agents labeled by $1, 2, \cdots, n$, and let $x_i(k) \in \mathbb{R}$ denote the state of agent i, $i \in \{1, \cdots, n\}$, at time $k \in \mathbb{N}_{\geq 0}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}_{\geq 0}$ is the set of nonnegative integers. We study the following distributed coordination algorithm for this n-agent system

$$x_i\left(k+1\right) = \sum_{j=1}^{n} p_{ij} x_j\left(k\right), k \in \mathbb{N}_{\geq 0}, i = 1, 2, \cdots, n, \quad (1)$$

where the averaging weights $p_{ij} \geq 0$ and $\sum_{j=1}^{n} p_{ij} = 1$. The algorithm (1) can be written into a compact form

$$x\left(k+1\right) = Px\left(k\right), k \in \mathbb{N}_{\geq 0}, \quad (2)$$

where $x\left(k\right) = \left[x_1\left(k\right), x_2\left(k\right), \cdots, x_n\left(k\right)\right]^T$. We say a square matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is *nonnegative* if a_{ij} are all nonnegative; this matrix A is further called *stochastic* if $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1, 2, \cdots, n$. Thus, P in (2) is a stochastic matrix describing the coupling network structure. In fact, matrix P can be associated with a directed, weighted graph $\mathcal{G}_P = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \cdots, n\}$ is the vertex set and $\mathcal{E}$ is the edge set for which $(i, j) \in \mathcal{E}$ if $p_{ji} > 0$.

We say the states of system (1) reach *agreement* if $\lim_{k\to\infty} x(k) = \mathbf{1}\xi$, i.e.,

$$\lim_{k\to\infty} P^k x(0) = \mathbf{1}\xi, \forall x(0) \in \mathbb{R}^n, \quad (3)$$

where $\xi \in \mathbb{R}$, $\mathbf{1}$ is the column vector of all ones and $x(0)$ is the given initial state. It is known (3) is achieved if the P is indecomposable and aperiodic (such stochastic matrices are also referred to as SIA matrices) (Wolfowitz (1963); Seneta (2006)). However, the situations when the P is not SIA, a periodic matrix for example, have not been studied, although these situations appear in many fields such as social networks. The convergence problems of agents coupled by such networks, where the P's are not SIA, remain open. In fact, for such cases, people usually believe consensus cannot be reached, and oscillation or even divergence may appear.

This motivates us to consider a challenging agreement problem with P a periodic irreducible matrix. Note that a nonnegative matrix A is called *irreducible* if for any pair (i, j), there exists an $m \in \mathbb{N}^+$ such that $A_{ij}^{(m)} > 0$, where $A_{ij}^{(m)}$ is the (i, j)th element of matrix A^m and $\mathbb{N}^+$ denotes the set of positive integers. It is worthing noticing that an irreducible matrix corresponds to a *strongly connected* graph, i.e., there exists a directed path from any vertex i to any other vertex j in $\mathcal{G}_P$. Now let us define the periodicity of a stochastic matrix.

Definition 1. [Seneta (2006)] Consider an irreducible stochastic matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. An index $i \in \{1, 2, \cdots, n\}$ is said to have *period* $d(i)$ if $d(i)$ is the common divisor of those $m \in \mathbb{N}^+$ for which $A_{ii}^{(m)} > 0$. The matrix A is said to be *periodic* if $d(i) > 1$ for all i.

In the case of a periodic P, the agreement shown in (3) cannot be reached. Instead, the state of system (2) oscillates when all the agents in the network update their states simultaneously. The following example illustrates this phenomenon.

Example 1. For system (2), the initial state is given by $x(0) = [1, 2, 3, 4]^T$, and the matrix P is

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

By simple computation, one can check that $x(1) = [4, 1, 2, 3]^T, x(2) = [3, 4, 1, 2]^T, x(3) = [2, 3, 4, 1]^T, x(4) = [1, 2, 3, 4]^T = x(0)$. It is easy to see that the state equals the initial state after updating for four times. Then the same process will repeat again, which obviously implies a cyclic behavior instead of agreement.

The cyclic phenomenon results from the fact that all agents share a common clock to synchronize their updating actions. However, a more realistic case is that each agent updates according to its own clock, which can be different from others'. This is called *asynchronous* updating. Interestingly, if the agents in Example 1 update asynchronously in the sequence $\{4, 3, 2, 1\}$, one obtains $x(4) = [4, 4, 4, 4]^T$, which indicates an agreement is reached among the 4 agents. It implies that asynchronous updating among agents may result in agreement.

This observation motivates some crucial questions: will the agreement always be reached if asynchronous updating exists among agents? If not, what conditions does the asynchronous updating need to satisfy in order that agents can reach agreement? To answer these questions, in the next section, we prove that the agents' states converge to a common value almost surely if a random number of agents can update at any time step. Before providing the results, the definition of an asynchronous updating matrix is given for analysis to be presented.

Suppose there are m agents, $1 \le m \le n$, who update at time k, indexed by $k_1, k_2, \cdots, k_m$. They update using the states of their corresponding neighbors. The states of the rest stay unchanged. More specifically,

$$x_i(k+1) = \begin{cases} \sum_{j=1}^{n} p_{ij} x_j(k), i \in \{k_1, k_2, \cdots, k_m\}; \\ x_i(k), \text{otherwise}. \end{cases} \tag{4}$$

For concise illustration, we define the following matrices

$$P(k) = \left[e_1, \cdots, p_{k_1}^T, e_{k+1}, \cdots, p_{k_m}^T, \cdots, e_n\right]^T, \tag{5}$$

where $e_k \in \mathbb{R}^n$ is the kth column of the identity matrix I_n and $p_k \in \mathbb{R}^n$ denotes the kth row of P. We call $P(k)$ the *asynchronous updating matrix* at time k. A key feature of each $P(k)$ is that it does not have all positive diagonal entries. This asynchronous-updating setting results in

$$x(k) = P(k)x(k-1). \tag{6}$$

Thus, the problem becomes to study under what conditions the state of system (6) will converge to a scaled all-one vector. We address this in the next section.

3. MAIN RESULTS

In this section, we show the agreement of the agents' states can be reached if the agents activate their updating actions asynchronously. Before proceeding, we first define asynchronous updating precisely.

Since in practice it is difficult for all agents to have access to a common clock according to which they activate the updating actions. However, as a reflection of reality, it is legitimate to postulate that on occasions more than one, but not all, agent may update, see e.g. Fig 1. Assume that each agent is equipped with a clock, which can differ from others'. The state of each agent remains unchanged except when an activation event is triggered by its own clock. At the event times, agents update their states obeying the updating rule (4). Denote the set of event times of the ith agent by $\mathcal{T}^i = \{0, t_1^i, \cdots, t_k^i, \cdots\}, k \in \mathbb{N}^+$. The following assumption is important for the later analysis.

Assumption 1. For any agent i, the intervals between two event times, denoted by $h_k^i = t_k^i - t_{k-1}^i$, are such that

(i) h_k^i are bounded with probability 1 for all k and all i;
(ii) h_k^i for $k = 0, 1, 2, \ldots$ is a random sequence, with the sequences $h_k^1, h_k^2, \ldots, h_k^n$ *mutually independent.*

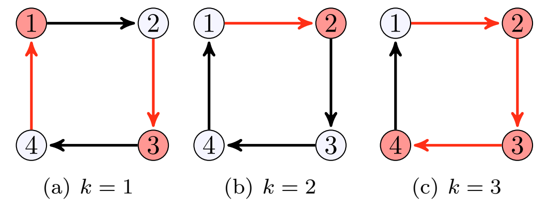

Fig. 1. Asynchronous updating: red nodes are the agents which are activated at each time.

Assumption 1 makes sure that an agent can be activated again within finite time after it is activated at t_{k-1}^i for all $k \in \mathbb{N}^+$, which implies that all agents will update their states for infinitely many times in the long run. In fact, Assumption 1 can be satisfied if the agents are activated by mutually independent Poisson clocks or at rates determined by mutually independent Bernoulli processes (Gallager (2012); Cassandras et al. (2009)).

Let $\mathcal{T} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\}$ denote all event times of all the n agents, in which the event times have been relabeled in a way such that $t_0 = 0$ and $t_\tau < t_{\tau+1}, \tau = \{0, 1, 2, \cdots\}$. This idea has been used in Bertsekas et al (1989) and Cao et al. (2008) to study asynchronous iterative algorithms. One situation may occur in which there exist some k's such that $t_k \in \mathcal{T}^i$ and $t_k \in \mathcal{T}^j$ for some i, j, which implies more than one agent is activated at some event times. This case is illustrated more clearly in the following figure. In fact, we consider this situation in this paper by assuming that a random number of agents update at each event time t_k.

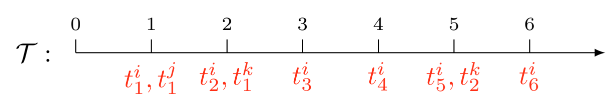

Fig. 2. Event times of all agents: one (or more) agents are activated simultaneously.

For simplicity of the later analysis, we rewrite the set of event times as $\mathcal{T} = \{0, 1, 2, \cdots, k, \cdots\}$. Then the system with asynchronous updating can be treated as one with discrete-time dynamics in which the agents are permitted to update only at certain event times $k, k \in \mathbb{N}_{\ge 0}$, with use of the updating rule (4) at each time k. Now, let us provide the main result of this paper.

Theorem 1. Agents coupled by a network described by a periodic stochastic matrix P given in (2) reach agreement *almost surely* if they update asynchronously under Assumption 1.

Before providing the proof of Theorem 1, we first present the following concepts and lemmas. We define a probability space $(\omega, \mathcal{F}, \mathbb{P})$ where ω is a set of outcomes; $\mathcal{F}$ is a set of events, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a function that assigns probabilities to events. $\mathcal{F}$ considered throughout this paper is a σ-field (Durrett (2010)). $\mathbb{E}(X)$ is the expectation of X, and $\mathbb{E}(X|\mathcal{F}_k)$ is the conditional expectation of X given $\mathcal{F}_k$, where $X \in \mathcal{F}, \mathcal{F}_k \subset \mathcal{F}$. We say X_k converges to X in probability if $\lim_{k\to\infty} \mathbb{P}(|X_k - X| > \varepsilon)=0$ for any ε, and X_k converges to X almost surely if $\mathbb{P}(\lim_{k\to\infty} X_k = X) = 1$.

We know for a given system with n agents, the number of all the possible asynchronous updating matrices at each time is finite and independent of time. We denote this by N, which can be computed as

$$N = \sum_{i=1}^{n} \frac{n!}{(n-i)!i!} = 2^n - 1.$$

Define the countable set of all the possible asynchronous updating matrices by $\mathcal{M} := \{P_1, P_2, \cdots, P_N\}$. The probability of the occurrence of each matrix in $\mathcal{M}$ at any time k is given by

$$\mathbb{P}\left(P\left(k\right) = P_i\right) = q_i, \forall k \in \mathbb{N}^+, i = 1, 2, \cdots, N. \quad (7)$$

It holds that $\sum_{i=1}^{N} q_i = 1$, and q_i can be computed from the corresponding stochastic processes, though the particular values are not of concern to us.

For known $P(k)$ and $x(k-1)$, $x(k)$ can be computed by (6). In order to study the convergence problem, we define $\bar{x}(k) := \max_{1 \le i \le n} x_i(k), \underline{x}(k) := \min_{1 \le i \le n} x_i(k)$, and

$$v_k = \bar{x}(k) - \underline{x}(k). \quad (8)$$

This scalar v_k has some important properties which are given by the following lemma.

Lemma 2. [Seneta (2006)] There always holds $v_k \le v_{k-1}$, i.e., $\bar{x}(k) - \underline{x}(k) \le \bar{x}(k-1) - \underline{x}(k-1)$, since $P(k)$ in (6) is always stochastic. In particular, $v_k < v_{k-1}$ if $P(k)$ is scrambling.

Note that a matrix is called scrambling if no two rows are orthogonal. It is easy to see that agreement can be reached if $v_k \to 0$. However, in this paper the $P(k)$'s in (6) are random matrices rather than deterministic ones, thus what we want to prove is that the system can reach agreement almost surely. Towards this end, we will show that $v_k \to 0$ almost surely by using the properties of supermartingales (Kushner (1965); Kumar et al. (2013)). The definition of a supermartingale and its properties are presented in the following lemma.

Lemma 3. [Durrett (2010)] The sequence X_k is said to be a supermartingale with respect to an increasing sequence of σ-fields $\mathcal{F}_k$ if it satisfies: (i) $\mathbb{E}X_n < \infty$; (ii) $X_k \in \mathcal{F}_k$ for all k; (iii) $\mathbb{E}\left(X_{k+1} \middle| \mathcal{F}_k\right) \le X_k$. If the sequence satisfies $X_k \ge 0$, then

$$X_k \xrightarrow{a.s.} X, k \to \infty, \quad (9)$$

and $\mathbb{E}X \le \mathbb{E}X_0$. Here X is a limit random variable and $a.s.$ stands for almost surely.

Subsequently, we define a stochastic Lyapunov candidate function v_k^2, and it will be shown in the following lemma that v_k^2 is exactly a supermartingale.

Lemma 4. The process $v_k^2, k \in \mathbb{N}_{\ge 0}$ is a supermartingale, i.e., it satisfies $\mathbb{E}\left(v_{k+1}^2 \middle| \mathcal{F}_k\right) \le v_k^2$, where v_k is defined in (8) and $\mathcal{F}_k = \sigma\left(P(1), P(2), \cdots, P(k)\right)$.

Proof. For a given $\mathcal{F}_k$, the sequence of asynchronous updating matrices $\{P(1), P(2), \cdots, P(k)\}$ is known. The state $x(k)$ can be calculated by iteration of (6). It is easy to see $v_k^2 \in \mathcal{F}_k$. One can also verify $\mathbb{E}(v_k^2) \le v_0^2 < \infty$ when the given $x(0)$ is bounded. Compute the conditional expectation of v_k^2:

$$\begin{aligned}
&\mathbb{E}\left(v_{k+1}^2 \middle| \mathcal{F}_k\right) = \mathbb{E}\left(\left(\bar{x}(k+1) - \underline{x}(k+1)\right)^2 | \mathcal{F}_k\right) \\
&= \sum_{i=1}^{N} q_i \left(\sum_{j=1}^{n} p_{I_i j} x_j(k) - \sum_{j=1}^{n} p_{T_i j} x_j(k)\right)^2 \\
&\le \sum_{i=1}^{N} q_i \left(\bar{x}(k) - \underline{x}(k)\right)^2 = v_k^2,
\end{aligned}$$

where I_i, T_i satisfy $\bar{x}(k+1) = x_{I_i}(k+1), \underline{x}(k+1) = x_{T_i}(k+1)$, respectively, for any $i = \{1, 2, \cdots, N\}$. Condition (i), (ii) and (iii) of Lemma 3 are satisfied, which implies that v_k^2 is a supermartingale. □

Now we know v_k^2 is a supermartingale. Consequently, it follows that $v_k^2 \xrightarrow{a.s.} V$ for some $V \in \mathbb{R}_{\ge 0}$ by Lemma 3. We then want to show that V must be 0, but the nonincreasing property of $\mathbb{E}\left(v_{k+1}^2 \middle| \mathcal{F}_k\right)$ with respect to v_k^2 is not sufficient to verify that. Looking at a longer sample run behavior of this conditional expectation, if we can find some integer $T \ge 1$ such that $\mathbb{E}\left(v_{k+T}^2 \middle| \mathcal{F}_k\right)$ decreases strictly with respect to v_k^2, then $V = 0$ can be proved.

Before providing the detailed proof, we first investigate the long run behavior of the system's state for a given $\mathcal{F}_k$. At time $k+t, t \ge 1$, the state is computed by

$$x^j(k+t) = \Phi^j(t,k)x(k) \quad (10)$$

where $\Phi^j(t,k) = P_{j_t} P_{j_{t-1}} \cdots P_{j_2} P_{j_1}$ is the transition matrix and $j_i \in \{1, 2, \cdots, N\}$ for all i. Under Assumption 1, we know there are various possibilities of $x^j(k+t)$ corresponding to the different possibilities of $\Phi^j(t,k)$. One can compute that the number of the possibilities is N^t, which implies j in (10) satisfies $j = 1, 2, \cdots, N^t$. The probability of each possible state at time $k+t$ is

$$\mathbb{P}\left(x\left(k+t\right) = x^j\left(k+t\right) \middle| \mathcal{F}_k\right) = \prod_{l=1}^{t} q_{j_l} := \hat{q}_j, \quad (11)$$

where (7) has been used.

We then want to show the probability of the occurrence of $\Phi^j(t,k)$ in the form of scrambling matrices is greater than zero, which will play a critical role in proving $\mathbb{E}\left(v_{k+T}^2 \middle| \mathcal{F}_k\right)$ is strictly less than v_k^2. The following lemma is important to show this property.

Lemma 5. There exists a $T \in \mathbb{N}^+$ such that some of the possible transition matrices, $\Phi^j(T,k), j = 1, 2, \cdots, N^t$, are scrambling. The probability of the occurrence of each of these scrambling matrices is greater than 0.

This lemma can be proved by considering a class of updating sequences. The transition matrix $\Phi^j(T,k)$ must be scrambling if the agents update following one of these sequences. These updating sequences will take place with non-zero probability. The reader can find the proof in the full version of this paper.

Let $\mathcal{M}_s$ be the set consisting of all the possible scrambling transition matrices among $\Phi^j(T,k), j = 1, 2, \cdots, N^T$. With the support of Lemma 5, we now move forward to Lemma 6.

Lemma 6. For a given $\mathcal{F}_k$, there exists a $T \in \mathbb{N}^+$ such that the longer run conditional expectation satisfies $\mathbb{E}\left(v_{k+T}^2 \middle| \mathcal{F}_k\right) < v_k^2$ for any $v_k \ne 0$.

This lemma follows from Lemma 5. The reasoning is similar to that in Lemma 4, which is omitted here.

We are now ready to prove Theorem 1.

Proof. From Lemma 4 one knows v_k^2 is a supermartingale, it holds that $v_k^2 \xrightarrow{a.s.} V$ and $\mathbb{E}V \le \mathbb{E}v_0^2$. By the fact that

$$v_k \le v_{k-1} \le \cdots v_1 \le v_0, \quad (12)$$

one can deduce $v_k^2 \le v_0^2$. For any bounded initial $x(0)$, it follows that v_k^2 is bounded. Applying the bounded convergence theorem in Durrett (2010) yields

$$\lim_{k\to\infty} \mathbb{E}v_k^2 = \mathbb{E}V. \tag{13}$$

According to Lemma 6, it is seen that $\mathbb{E}v_k^2 \to 0$ since $\mathbb{E}\left(v_{k+T}^2 \middle| \mathcal{F}_k\right) < v_k^2$ for any $v_k \neq 0$ (Kushner (1965)). Thus, we have $\mathbb{E}V = 0$. By use of the Chebyshev inequality, it follows straightforwardly from (13) that

$$Pr\left(v_k > \varepsilon\right) \le Ev_k^2/\varepsilon^2, \forall \varepsilon > 0. \tag{14}$$

Since $v_k > 0$, then $\lim_{k\to\infty} \mathbb{P}\left(v_k > \varepsilon\right) = 0$ which implies convergence to zero in probability, denoted $v_k \xrightarrow{p} 0$. It then follows that $v_k^2 \xrightarrow{p} 0$. Since $v_k^2 \xrightarrow{a.s.} V$ implies $v_k^2 \xrightarrow{p} V$, one can deduce

$$V = 0 \text{ almost surely.} \tag{15}$$

It follows that $v_k^2 \xrightarrow{a.s.} 0$ and subsequently $\mathbb{P}\left(v_k > \varepsilon \; i.o.\right) = 0$ for any $\varepsilon > 0$, where $i.o.$ stands for infinitely often. Consequently, one knows that $v_k \xrightarrow{a.s.} 0$, which means

$$\lim_{k\to\infty} \left(\bar{x}(k) - \underline{x}(k)\right) = 0 \tag{16}$$

almost surely. Then one can conclude the states of the agents in the system reach agreement almost surely, which completes the proof. □

The result we obtained in Theorem 1 implies the following interesting observations. Although the distributed coordination algorithm is associated with a periodic matrix, the individuals coupled by such a network converge to an identical state instead of oscillating, because they update their states asynchronously. In fact, asynchrony rather than synchrony may be more common in practice. The result we obtained may shed new light on understanding the consensus phenomenon that has been extensively studied.

4. SIMULATION

In this section, we demonstrate the obtained results by a numerical example. Meanwhile, the convergence rate is also of concern since it is very important for us to study the stochastic systems.

Consider system (2) with the following periodic matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding graph is given by Fig.4, which is strongly connected and cyclic. Let the initial state be $x(0) = [1.1, 4.2, 7.3, 3.4, 4.5, 5.6]^T$.

If the agents in the network have a common clock to synchronize the updating actions, the states of the agents cannot reach agreement, instead, a cyclic behavior takes place, as shown in Fig. 4.

However, if agents update according to their own clocks under Assumption 1, agreement can be reached. To illustrate this, we assume the clocks are driven by mutually in-

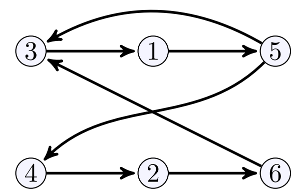

Fig. 3. Associated graph of P.

dependent Poisson processes in which the inter-arrival intervals (i.e., intervals between two successive event times) have the density functions

$$f_i(x) = \lambda_i e^{-\lambda_i x}, \text{for } x \ge 0,$$

where $i = 1, 2, \cdots, n$. Let $\lambda_i = 2$ for all i. The evolution of the agents' states is shown in Fig.5, which shows that the states converge to a common value instead of an oscillation even though the network is cyclic. Thus one knows that asynchrony has played a fundamental role.

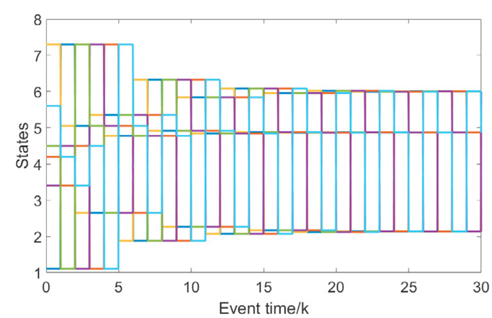

Fig. 4. Update synchronously: oscillation.

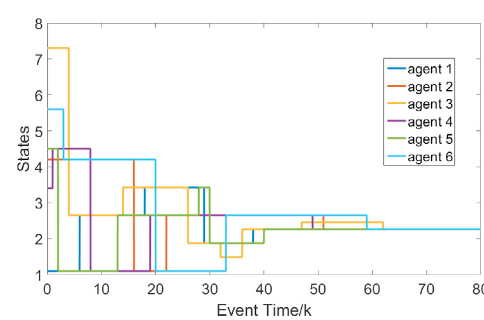

Fig. 5. Update asynchronously: agreement.

One may observe from Fig. 5 that the convergence is surprisingly fast, in that agreement is almost reached after only 65 event times. To get a more general view of the convergence rate of the stochastic system dynamics we have studied in this paper, we repeated the same trial 300 times. It is known v_k defined in (8) is a measurement of agreement. If v_k is sufficiently small, we can say the agreement among agents is almost reached. Let $k_c := \inf\{k \in \mathbb{N}^+ : v_k < 0.01\}$ be an estimate of the convergence time. we collected all the k_c's of the 300 trials.

The probability density and cumulative probability are estimated from the occurrence numbers of all the 300 k_c's (see Fig. 6). It can be seen that most of the k_c's are located in the interval $[20, 140]$, and the mean value of the 300 k_c's is approximately 60. Moreover, the probability of the agents reaching agreement within 220 iterations is almost 1. The observations reveal that the convergence is quite

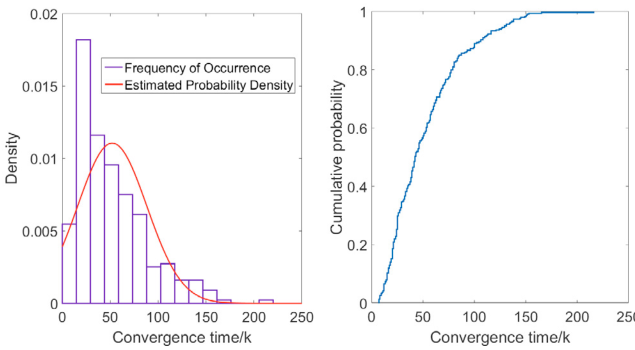

Fig. 6. Left: the bar graph indicates the frequency of occurrence of different k_c's, and the red line shows the distribution density estimated from the bars; Right: Cumulative probability of k_c's.

fast, which is perhaps unexpected in view of the stochastic nature of the system dynamics. It is of great interest to investigate the convergence rate of such randomized updating processes in our future work.

5. CONCLUSION

We have studied in this paper distributed coordination algorithms associated with periodic stochastic matrices P. We have shown in contrast to the cyclic behavior that takes place when agents update synchronously, the agents' states reach agreement almost surely with the existence of asynchrony. We have used the properties of supermartingale to construct the proofs. We are currently working on giving closed-form approximations of the convergence rate for such convergent randomized updating processes.

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