Asymptotic Enumeration of Tournaments Containing a Specified Digraph

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Abstract

This paper studies the probability that a random tournament with specified score sequence contains a specified subgraph. The exact asymptotic value is found in the case that the scores are not too far from regular and the subgraph is not too large. An n-dimensional saddle-point method is used. As a sample application, we
prove that almost all tournaments with a given score sequence (not too far from regular) have a trivial automorphism group.
1 Introduction

A tournament is a digraph such that between every pair of vertices there is exactly one arc. Throughout this paper, we fix the vertex set to be \( V = \{1, 2, \ldots, n\} \). Let \( d_j^+, d_j^- \) be the in-degree and out-degree of vertex \( j \) in a tournament. Define \( \delta_j = d_j^+ - d_j^- \) and call \( \delta_1, \delta_2, \ldots, \delta_n \) the excess sequence of the tournament. Let \( \delta = \max\{|\delta_1|, \ldots, |\delta_n|\} \).

Let \( H \) be a digraph with vertex set \( V \) and arc set \( A(H) \) such that between every pair of distinct vertices there is at most one arc. We use \( d_j^+(H) \) and \( d_j^-(H) \) to denote the out-degree and in-degree, respectively, of vertex \( j \) in \( H \). Define \( \delta_j(H) = d_j^+(H) - d_j^-(H) \), \( d_j(H) = d_j^+(H) + d_j^-(H) \), \( \delta(H) = \max\{|\delta_1(H)|, \ldots, |\delta_n(H)|\} \), and \( d(H) = \max\{d_1(H), \ldots, d_n(H)\} \).

Let \( T(H; \delta_1, \ldots, \delta_n) \) be the number of tournaments that contain a specified digraph \( H \) and have excess sequence \( \delta_1, \ldots, \delta_n \). As special cases, we have \( T(\delta_1, \delta_2, \ldots, \delta_n) \) to denote the number of all tournaments that have excess sequence \( \delta_1, \ldots, \delta_n \), and \( T(n) = T(0, 0, \ldots, 0) \) to denote the number of labelled regular tournaments with \( n \) vertices.

Spencer [1] evaluated \( T(n) \) to within a factor of \((1 + o(1))^n\) and obtained the estimate

\[
T(\delta_1, \ldots, \delta_n) = T(n) \exp\left(-\frac{1}{2} + o(1)\right) \sum_{j=1}^{n} \frac{\delta_j^2}{n}
\]

for tournaments close to regular. The asymptotic value of \( T(n) \) was obtained much later by McKay [2], who showed that

\[
T(n) \sim \left(\frac{2^{n+1}}{n\pi} \right)^{(n-1)/2} \left(\frac{n}{e}\right)^{1/2} (n \text{ odd}).
\]

Recently, McKay and Wang [3] obtained the asymptotic value of \( T(\delta_1, \ldots, \delta_n) \) for \( \delta = o(n^{3/4}) \). The following is an immediate consequence of [3, Theorem 4.4].

**Theorem 1** Suppose \( \delta = o(n^{2/3}) \). Then

\[
T(\delta_1, \ldots, \delta_n) \sim n^{1/2} \left(\frac{2^{n+1}}{n\pi} \right)^{(n-1)/2} \exp\left(-\frac{1}{2} - \left(\frac{1}{2n} - \frac{1}{n^2}\right) \sum_{j=1}^{n} \delta_j^2 \right)
- \frac{1}{12n^3} \sum_{j=1}^{n} \delta_j^4 - \frac{1}{4n^4} \left(\sum_{j=1}^{n} \delta_j^2\right)^2.
\]

McKay suggested that a similar argument can be used to obtain asymptotics for \( T(H; \delta_1, \ldots, \delta_n) \). We carry out this task in this paper. To simplify the analysis, we shall
restrict ourselves in the range \( \delta = o(n^{2/3}) \) and \( d(H) = O(n^{1/2-\epsilon'}) \), where \( \epsilon' \) is any positive constant.

For a given digraph \( H \) and a given excess sequence \( \delta_1, \delta_2, \ldots, \delta_n \), define

\[
\beta_1 = \frac{1}{2n} \sum_{1 \leq j \leq n} (2\delta_j \delta_j(H) - \delta_j^2(H)) + \frac{1}{3n^3} \sum_{1 \leq j \leq n} \delta_j^3 \delta_j(H)
\]

\[
= \frac{1}{n^3} \sum_{1 \leq j \leq n} \delta_j \sum_{1 \leq j \leq n} \delta_j \delta_j(H),
\]

and

\[
\beta_2 = -\frac{1}{2n^3} \sum_{(j,k) \in A(H)} (\delta_j - \delta_k - \delta_j(H) + \delta_k(H))^2.
\]

We shall prove

**Theorem 2** Suppose \( \delta = o(n^{2/3}) \), \( d(H) = O(n^{1/2-\epsilon}) \) and \( d(H) \delta = o(n) \), where \( \epsilon \) is any positive constant. Then

\[
T(H; \delta_1, \delta_2, \ldots, \delta_n)/T(\delta_1, \delta_2, \ldots, \delta_n) \sim 2^{-m} \exp(m/n + \beta_1 + \beta_2)
\]

uniformly as \( n \to \infty \).

The rest of the paper is organized as follows. In Section 2 we derive the asymptotic value of an integral. In Section 3, we use Cauchy’s Theorem to represent \( T(H; \delta_1, \delta_2, \ldots, \delta_n) \) as an integral and then apply the saddle point method and the results from Section 2 to obtain Theorem 2. In Section 4, we discuss several consequences of Theorem 2.

## 2 An Integral

In this section, we approximate the value of an \( n \)-dimensional integral we will need later. Define

\[ U_n(t) = \{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \mid |x_i| \leq t, i = 1, 2, \ldots, n \} \]

**Lemma 1** Let \( E, F \) and \( 0 < \epsilon < 1/20 \) be constants and let \( A_{jk}(n) \), \( B_k(n) \), \( C_{jk}(n) \), \( D_{jk}(n) \), \( \alpha_j(n) \) be real-valued functions.

Suppose

\[
(i) \quad \sum_{j=1}^{n-1} (|A_{jk} + A_{kj}|) = O(n^{1/2-3\epsilon})
\]

uniformly for all \( 1 \leq k \leq n - 1 \), and
\[(ii) \ B_k(n) = O(n^{-4\epsilon}), \ C_{jk}(n) = O(n^{-4\epsilon}), \ D_{jk}(n) = O(n^{1/2-5\epsilon}), \ \alpha_j(n) = O(n^{-1/2-6\epsilon}) \text{ uniformly for } 1 \leq j, k \leq n-1.\]

Define
\[
f(x) = \exp \left( -\frac{1}{2} (n-1) \sum_{1 \leq j \leq n-1} (1 - \alpha_j(n))x_j^2 + \frac{1}{2} \sum_{j \neq k} x_j x_k + nE \sum_{1 \leq k \leq n-1} x_k^4 \right.
\]
\[
+ F \left( \sum_{1 \leq k \leq n-1} x_k^2 \right)^2 + \sum_{j \neq k} A_{jk}(n)x_j x_k + i n \sum_{1 \leq k \leq n-1} B_k(n)x_k^3
\]
\[
+ i \sum_{j \neq k} C_{jk}(n)x_k^2 x_j + \sum_{j \neq k} D_{jk}(n)x_k^3 x_j + o(1) \bigg),
\]
where \(x = (x_1, x_2, \ldots, x_{n-1})\). Then
\[
\int_{U_{n-1}(n^{-1/2+\epsilon})} f(x) \, dx = n^{1/2} \left( \frac{2\pi}{n} \right)^{(n-1)/2} \exp \left( \sum_{1 \leq j \leq n-1} \alpha_j(n)/2 + 3E + F + o(1) \right).\]

**Proof.** Let \(I\) be the above integral and define the linear transformation \(T_1 : z_j = (1 - \alpha_j(n))^{1/2} x_j, \ 1 \leq j \leq n-1.\)

Let \(V_1\) be the image of \(U_{n-1}(n^{-1/2+\epsilon})\) under \(T_1\). It is clear that \(V_1\) is between \(U_{n-1}(n^{-1/2+\epsilon/2})\) and \(U_{n-1}(n^{-1/2+2\epsilon})\), and
\[
I = \prod_{1 \leq j \leq n-1} (1 - \alpha_j(n))^{-1/2} \times \int_{V_1} \exp \left( -\frac{1}{2} (n-1) \sum_{j \neq k} z_j^2 + \frac{1}{2} \sum_{j \neq k} z_j z_k \right)
\]
\[
+ nE \sum z_j^4 + F \left( \sum z_j^2 \right)^2 + \sum_{j \neq k} A'_{jk}(n)z_j z_k + i n \sum_{1 \leq k \leq n-1} B_k(n)z_k^3
\]
\[
+ i \sum_{j \neq k} C_{jk}(n)z_k^2 z_j + \sum_{j \neq k} D_{jk}(n)z_k^3 z_j + o(1) \bigg) \, dz,
\]
where \(A'_{jk}\) satisfies the same conditions as \(A_{jk}\).

Next we perform a second linear transformation \(T_2\) to diagonalise the major quadratic terms of the integrand:
\[
z_j = y_j - \beta \mu_1, \ 1 \leq j \leq n-1,
\]
where \(\beta = 1/(\sqrt{n} + 1)\) and \(\mu_m = \sum_{j=1}^{n-1} y_j^m\) for any \(m\). Let \(V_2\) be the image of \(V_1\) under \(T_2\). It is easily determined that the determinant of \(T_2\) is \(\sqrt{n}\), and so
\[
I = \sqrt{n} \prod_{1 \leq j \leq n-1} (1 - \alpha_j(n))^{-1/2} \times \int_{V_2} \exp \left( -\frac{1}{2} n \mu_2 + nE \mu_4 + F \mu_2^2 \right)
\]
5
+ \sum_{j \neq k} A'_{jk}(n)y_jy_k + in \sum_{1 \leq k \leq n-1} B_k(n)y_k^3
+ i \sum_{j \neq k} C_{jk}(n)y_k^2y_j + \sum_{j \neq k} D_{jk}(n)y_k^3y_j + o(1)\right)dy.

The region of integration \( V_2 \) is somewhat irregular, but by the same method as used in [2, Theorem 2.1], we can see that it can be replaced by \( U_{n-1}(n^{-1/2+\varepsilon}) \) with negligible change of value.

Finally we use an average technique introduced in [4, Lemma 3] to show that some unsymmetrical terms are negligible. Let \( f_0(y) = -\frac{1}{2}m\mu_2 + nE\mu_1 + F\mu_2^2 \) and let \( f(y) \) be the integrand of the previous integral.

For \( 1 \leq m \leq n \), define

\[
\psi_m(y) = \exp\left(f_0(y) + \sum_{k=m}^{n-1} \sum_{j=m}^{n-1} A'_{jk}(n)y_jy_k + in \sum_{k=m}^{n-1} B_k(n)y_k^3
+ i \sum_{k=1}^{n-1} \sum_{j=m}^{n-1} C_{jk}(n)y_k^2y_j + \sum_{k=m}^{n-1} \sum_{j=m}^{n-1} D_{jk}(n)y_k^3y_j \right),
\]

where \( A'_{jj}(n), C_{jj}(n) \) and \( D_{jj}(n) \) are interpreted as zero for \( 1 \leq j \leq n-1 \). Then

\[
\psi_1(y) = f(y) \exp(o(1)), \quad \psi_n(y) = \exp(f_0(y)),
\]

and

\[
\psi_m(y) = \psi_{m+1}(y)\exp(Z),
\]

with

\[
Z = \sum_{j=m}^{n-1} (A'_{jm}(n) + A'_{mj})y_jy_m + inB_m(n)y_m^3
+ i \sum_{k=1}^{n-1} \sum_{j=m}^{n-1} C_{nk}y_k^2y_m + \sum_{j=m}^{n-1} D_{jm}y_m^3y_j + \sum_{k=m}^{n-1} D_{nk}y_m^3y_j.
\]

Define

\[
\bar{\psi}_m(y) = \frac{1}{2}\left(\psi_m(y) + \psi_m(y_1, \ldots, y_{m-1}, -y_m, y_{m+1}, \ldots, y_{n-1})\right).
\]

Since \( U_{n-1}(n^{-1/2+\varepsilon}) \) is symmetric about the origin, we have

\[
\int_{U_{n-1}(n^{-1/2+\varepsilon})} \bar{\psi}_m(y)\,dy = \int_{U_{n-1}(n^{-1/2+\varepsilon})} \psi_m(y)\,dy.
\]
Using
\[(e^Z + e^{-Z})/2 = \exp(O(Z^2))\]
we obtain, for \(y \in U_{n-1} (n^{-1/2+\epsilon})\), that
\[\widetilde{\psi}_m(y) = \psi_{m+1}(y) \exp(O(n^{-1-2\epsilon}))\]
uniformly over \(m\), and hence
\[
\left| \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_m(y) \, dy - \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_{m+1}(y) \, dy \right|
= \exp(O(n^{-1-2\epsilon})) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_{m+1}(y)| \, dy.
\]
Applying the same argument to \(|\psi_m(y)|\), we obtain
\[
\int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_m(y)| \, dy = \exp(O(n^{-1-2\epsilon})) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_{m+1}(y)| \, dy.
\]
Therefore
\[
\int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_1(y)| \, dy = \exp(O(n^{-2\epsilon})) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_n(y)| \, dy,
\]
and finally
\[
\left| \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_1(y) \, dy - \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_n(y) \, dy \right|
= \exp(O(n^{-2\epsilon})) \int_{U_{n-1}(n^{-1/2+\epsilon})} |\psi_n(y)| \, dy.
\]
Putting these results together, we find that
\[I \sim \sqrt{n} \prod_{1 \leq j \leq n-1} (1 - \alpha_j(n))^{-1/2} \times \int_{U_{n-1}(n^{-1/2+\epsilon})} f_0(y) \, dy,\]
which is covered by [3, Theorem 2.1]. This gives the desired result on noting that
\[\prod_{1 \leq j \leq n-1} (1 - \alpha_j(n))^{-1/2} \sim \exp\left(\frac{1}{2} \sum_{1 \leq j \leq n-1} \alpha_j\right).\]
3 Proof of Theorem 2

Throughout this section, we assume $\delta = o(n^{2/3})$, $d(H) = O(n^{1/2} \epsilon')$, $d(H) \delta = o(n)$, and that $\epsilon' < 1/100$ is a positive constant.

For a given digraph $H$, define $\chi_{jk} = 1$ if $(j, k) \in A(H)$, and $\chi_{jk} = 0$ otherwise. Also define $\gamma_j = \delta_j - \delta_j(H)$. Let

$$G(x) = G(x_1, x_2, \ldots, x_n) = \prod_{1 \leq j < k \leq n} \left( x_j^{-1} x_k + x_j x_k^{-1} \right) \prod_{(j, k) \in A(H)} \frac{x_j x_k^{-1}}{x_j x_k^{-1} + x_j^{-1} x_k}.$$ 

Then $T(H; \delta_1, \ldots, \delta_n)$ is the coefficient of $x_1^{\delta_1} \cdots x_n^{\delta_n}$ in $G(x)$. Setting $x_j = r_j \exp(i \theta_j)$, we have by Cauchy’s Theorem that

$$T(H; \delta_1, \ldots, \delta_n) = (2\pi)^{-n} \prod_{1 \leq j \leq n} r_j^{-\delta_j} \int_{U_n(\pi)} G(r_1 e^{i \theta_1}, \ldots, r_n e^{i \theta_n}) \exp(-i \sum_{1 \leq j \leq n} \delta_j \theta_j) \, d\theta.$$ 

Define

$$T_{jk}(\theta) = \frac{r_j^2 \exp(i(\theta_j - \theta_k)) + r_k^2 \exp(i(\theta_k - \theta_j))}{r_j^2 + r_k^2},$$

$$g(\theta) = \exp\left(-i \sum_{1 \leq j \leq n} (\delta_j \theta_j)\right) \prod_{1 \leq j < k \leq n} T_{jk}(\theta) \prod_{(j, k) \in A(H)} \frac{e^{i(\theta_j - \theta_k)}/\theta_j}{T_{jk}(\theta)},$$

and

$$I = \int_{U_n(\pi/2)} g(\theta) \, d\theta. \quad (2)$$

Since $g(\theta)$ is invariant under the translation of any $\theta_j$ by $\pi$, we obtain

$$T(H; \delta_1, \ldots, \delta_n) = \pi^{-n} I \prod_{1 \leq j \leq n} r_j^{-\delta_j} \prod_{1 \leq j < k \leq n} (r_j/r_k + r_k/r_j) \times \prod_{(j, k) \in A(H)} r_j^2/(r_j^2 + r_k^2). \quad (3)$$

Since the integrand is invariant under a uniform translation of $\theta_j$ by $\theta_n$, and $\sum_{j=1}^n \delta_j = 0$, we have

$$I = \pi \int_{U_{n-1}(\pi/2)} g(\theta_1, \theta_2, \ldots, \theta_{n-1}, 0) \, d\theta',$$

where $\theta' = (\theta_1, \ldots, \theta_{n-1})$. For a positive constant $\epsilon$ satisfying $\epsilon < \epsilon'/6$, let $I_1$ be the contribution to $I$ from $\theta \in U_{n-1}(n^{-1/2+\epsilon})$. As in [3], we first estimate $I_1$ and then show
that $I_1 \sim I$. In the following analysis, we shall assume $\theta' \in U_{n-1}(n^{-1/2+\varepsilon})$ and $\theta_n = 0$. To apply the saddle point method, it is convenient to choose $r_j = (1 + b_j)/(1 - b_j)$, where

$$b_j = \gamma_j/n + d_j(H)\gamma_j/n^2 - \sum_{k=1}^{n}(\chi_{jk} + \chi_{kj})\gamma_k/n^2 + \gamma_j\sum_{k=1}^{n}\gamma_k^2/n^4. \quad (4)$$

It is important to note that $b_j = \gamma_j/n + o(1/n) = o(n^{-1/3})$ and $\sum_{1 \leq j \leq n} b_j = 0$. Let

$$a_{jk} = (r_j^2 - r_k^2)/(r_j^2 + r_k^2) = (b_j - b_k)/(1 - b_j b_k). \quad (5)$$

Using Taylor expansion, we have, for $\theta' \in U_{n-1}(n^{-1/2+\varepsilon})$, that

$$T_{jk}(\theta) = \exp\left(i a_{jk}(\theta_j - \theta_k) + \left(-\frac{1}{2} + \frac{1}{2} a_{jk}^2\right)(\theta_j - \theta_k)^2 + \frac{1}{3} a_{jk}i(\theta_j - \theta_k)^3 - \frac{1}{12}(\theta_j - \theta_k)^4 + O(n^{-2\varepsilon})\right). \quad (6)$$

Noting that $\sum_{k \geq 1}(\chi_{jk} + \chi_{kj}) = d_j(H) = O(n^{1/2-\varepsilon})$ and expanding the powers of $\theta_j - \theta_k$, we obtain

$$g(\theta) = \exp\left(i \sum_{1 \leq j \leq n} \left( \sum_{1 \leq k \leq n} a_{jk} - \delta_j + \delta_j(H) - \sum_{1 \leq k \leq n}(\chi_{jk} + \chi_{kj})a_{jk}\right)\theta_j^1 + \sum_{1 \leq j \leq n-1} \left(-\frac{1}{2}(n-1) + \sum_{1 \leq k \leq n} a_{jk}^2/2 + d_j(H)/2\right)\theta_j^2 + \frac{1}{2} \sum_{j \neq k} \theta_j\theta_k + \sum_{j \neq k} \left(-a_{jk}^2/2 - \chi_{jk}(1 - a_{jk}^2)\right)\theta_j\theta_k + i n \sum_{1 \leq j \leq n-1} O(n^{-1/3})\theta_j^3 + i \sum_{j \neq k} O(n^{-1/3})\theta_j\theta_k^2 + \frac{1}{12} n \sum_{1 \leq j \leq n-1} \theta_j^4 - \frac{1}{4} \left(\sum_{1 \leq j \leq n-1} \theta_j^2\right)^2 + \sum_{j \neq k} \theta_j\theta_k^2 + o(1)\right).$$

Using (4), (5) and the comment after (4), we have

$$\sum_{1 \leq k \leq n} a_{jk} = \gamma_j + d_j(H)\gamma_j/n - \sum_{1 \leq k \leq n}(\chi_{jk} + \chi_{kj})\gamma_k/n + o(n^{-2/3})$$

and

$$\sum_{1 \leq k \leq n} (\chi_{jk} + \chi_{kj})a_{jk} = d_j(H)\gamma_j/n - \sum_{1 \leq k \leq n}(\chi_{jk} + \chi_{kj})\gamma_k/n + o(n^{-2/3}),$$
and hence

\[ g(\theta) = \exp \left( \sum_{1 \leq j \leq n-1} \left( -\frac{1}{2}(n-1) + \sum_{1 \leq k \leq n} a_{jk}^2/2 + d_j(H)/2 \right) \theta_j^2 \right. \]
\[ + \frac{1}{2} \sum_{j \neq k} \theta_j \theta_k + \left. \sum_{j \neq k} \left( -a_{jk}^2/2 - \chi_{jk}(1 - a_{jk}^2) \right) \theta_j \theta_k \right) \]
\[ + i n \sum_{1 \leq j \leq n-1} O(n^{-1/3})\theta_j^3 + i \sum_{j \neq k} O(n^{-1/3})\theta_j \theta_k \]
\[ - \frac{1}{12} n \sum_{1 \leq j \leq n-1} \theta_j^4 - \frac{1}{4} \left( \sum_{1 \leq j \leq n-1} \theta_j^2 \right)^2 + \sum_{j \neq k} \theta_j \theta_k^3 + o(1) \right). \quad (7) \]

Applying Lemma 1 and using

\[ \frac{1}{2(n - 1)} \sum_{1 \leq j \leq n-1} \sum_{1 \leq k \leq n} a_{jk}^2 = \frac{1}{n^2} \sum_{1 \leq j \leq n} \delta_j^2 + o(1) \]

and

\[ \frac{1}{2(n - 1)} \sum_{1 \leq j \leq n-1} d_j(H) = \frac{m}{n} + o(1), \]

we obtain

\[ I_1 \sim \pi n^{1/2} \left( \frac{2\pi}{n} \right)^{(n-1)/2} \exp \left( m/n + \sum_{1 \leq j \leq n} \delta_j^2/n^2 - 1/2 \right) \exp \left( m/n + \sum_{1 \leq j \leq n} \delta_j^2/n^2 - 1/2 \right). \quad (8) \]

The proof of the fact that the contribution to \( I \) from the region other than that of \( I_1 \) is negligible is essentially the same as that of [3] and will be omitted.

Using (4) and (5) with some calculation, we obtain

\[ \prod_{1 \leq j < k \leq n} \left( \frac{r_j / r_k + r_k / r_j}{r_j} \right) \prod_{1 \leq j \leq n} r_j^{-\delta_j} \]
\[ = 2^{n(n-1)/2} \exp \left( -\frac{1}{2n} \sum_{1 \leq j \leq n} \delta_j^2 - \frac{1}{12n^3} \sum_{1 \leq j \leq n} \delta_j^4 - \frac{1}{4n^4} \left( \sum_{1 \leq j \leq n} \delta_j^2 \right)^2 \right. \]
\[ + \frac{1}{n^2} \sum_{(j,k) \in A(H)} (\delta_j(H)\gamma_k + \delta_k(H)\gamma_j) \]
\[ \left. + \frac{1}{2n} \sum_{1 \leq j \leq n} \delta_j^2(H) - \frac{1}{n^2} \sum_{1 \leq j \leq n} d_j(H)\delta_j(H)\gamma_j + o(1) \right) \quad (9) \]
\[
\prod_{(j,k) \in A(H)} \frac{\gamma_j^2}{(\gamma_j^2 + \gamma_k^2)} \\
= 2^{-m} \exp \left( \frac{1}{n} \sum_{1 \leq j \leq n} \delta_j(H) \gamma_j + \frac{1}{n^2} \sum_{1 \leq j \leq n} d_j(H) \delta_j(H) \gamma_j \right) \\
+ \frac{1}{3n^3} \sum_{1 \leq j \leq n} \delta_j(H) \delta_j^3 + \frac{1}{n^4} \sum_{1 \leq j \leq n} \delta_j(H) \delta_j \sum_{1 \leq j \leq n} \delta_j^2 \\
- \frac{1}{n^2} \sum_{(j,k) \in A(H)} \left( (\gamma_j - \gamma_k)^2 / 2 + \delta_j(H) \gamma_k + \delta_k(H) \gamma_j + o(1) \right). 
\] (10)

Now Theorem 2 follows from (3) and (8)–(10).

4 Consequences

From Theorem 2, we see that \( T(H; \delta_1, \delta_2, \ldots, \delta_n) \) usually depends on the structure of the digraph \( H \). However, it can have much simpler form in some special cases. Noting that

\[
\beta_1 = \frac{1}{2n} \sum_{1 \leq j \leq n} (2 \delta_j(H) \delta_j - \delta_j^2(H)) + o(1), \quad \beta_2 = -\frac{1}{2n^2} \sum_{(j,k) \in A(H)} (\delta_j - \delta_k)^2 + o(1)
\]

when

\[
\sum_{1 \leq j \leq n} |\delta_j(H)| = O(n),
\]

we obtain the following two corollaries.

**Corollary 1** Suppose \( \delta = o(n^{2/3}) \), \( d(H) = O(n^{1/2-\epsilon}) \), \( d(H) \delta = o(n) \) and \( \sum_{1 \leq j \leq n} |\delta_j(H)| = O(n) \). Then

\[
T(H; \delta_1, \delta_2, \ldots, \delta_n) / T(\delta_1, \delta_2, \ldots, \delta_n) \sim 2^{-m} \exp \left( \frac{m}{n} + \frac{1}{n} \sum_{1 \leq j \leq n} (2 \delta_j(H) \delta_j - \delta_j^2(H)) \right) \\
- \frac{1}{2n^2} \sum_{(j,k) \in A(H)} (\delta_j - \delta_k)^2 
\]

**Corollary 2** Suppose \( \delta = o(n^{2/3}) \), \( m = O(n^{1/2-\epsilon}) \), and \( \delta(H) \delta = o(n) \). Then

\[
T(H; \delta_1, \delta_2, \ldots, \delta_n) / T(\delta_1, \delta_2, \ldots, \delta_n) \sim 2^{-m} \exp \left( \frac{1}{n} \sum_{1 \leq j \leq n} \delta_j(H) \delta_j \right). 
\]
In particular, \( T(H; \delta_1, \delta_2, \ldots, \delta_n)/T(\delta_1, \delta_2, \ldots, \delta_n) \sim 2^{-m} \)
uniformly for all \( \delta = o(n^{2/3}) \) and \( m = O(n^{1/3}) \).

For regular tournaments, we have

**Corollary 3** Let \( T_n(H) \) be the number of regular tournaments with \( n \) vertices containing the digraph \( H \). Suppose \( d(H) = O(n^{1/2-\varepsilon}) \). Then, for odd \( n \),

\[
T_n(H) \sim \left( \frac{2n+1}{n\pi} \right)^{(n-1)/2} \left( \frac{n}{e} \right)^{1/2} \left( \frac{1}{2} \right)^m 
\times \exp \left( \frac{m}{n} - \frac{1}{2n} \sum_{1 \leq j \leq n} \delta_j^2(H) - \frac{1}{2n^2} \sum_{(j,k) \in A(H)} (\delta_j(H) - \delta_k(H))^2 \right).
\]

A simple application of Theorem 2 is the unsurprising fact that very few tournaments with \( \delta = o(n^{2/3}) \) have nontrivial automorphisms. This allows us to estimate the number of isomorphism types.

**Corollary 4** Suppose \( \delta = o(n^{2/3}) \). Then the number of unlabelled tournaments with excess sequence \( \delta_1, \delta_2, \ldots, \delta_n \) is asymptotically \( T(\delta_1, \delta_2, \ldots, \delta_n)/n! \).

**Proof.** Consider a random (labelled) tournament \( T \) with excess sequence \( \delta_1, \delta_2, \ldots, \delta_n \). It suffices to prove that the expected number of automorphisms of \( T \) is asymptotically 1.

We know that \( |\text{Aut}(T)| \) is odd, because \( T \) is a tournament. Let \( g \) be a non-trivial permutation of \( V \) of odd order. Define \( S = S(g) \) to be the set of vertices moved by \( g \), and let \( k = |S| \).

Consider the set \( E \) of pairs of distinct vertices defined by

\[
E = \{ \{i, j\}, \{i^g, j^g\} \mid i \in S, 1 \leq j - i \leq 12 \lfloor \ln n \rfloor \mod n \}.
\]

It is easy to see that \( E \) (considered as an undirected graph) has maximum degree at most \( 48 \lfloor \ln n \rfloor \), and that \( |E| = m \) for \( 6k \lfloor \ln n \rfloor \leq m \leq 48k \lfloor \ln n \rfloor \). Define a simple undirected graph \( G = G(E, g) \) whose vertices are the elements of \( E \) and whose edges are the pairs \( \{e, e^g\} \) for which both \( e \) and \( e^g \) are in \( E \). From the definition of \( E \), \( G \) has at most \( m/2 \) components.

Now consider digraphs \( H \) which are orientations of \( E \). Within each component of \( G \), there are only two orientations that are consistent with \( g \) being an automorphism of \( T \), and so there are at most \( 2^{m/2} \) possibilities for \( H \) with that consistency. From Theorem 2,
we have that each such \( H \) is a subgraph of \( T \) with probability less than \( 2^{-m} \exp(mn^{-1/3}) \) for sufficiently large \( n \). Consequently, the probability that \( g \) is an automorphism of \( T \) is at most
\[
2^{-m/2} \exp(mn^{-1/3}) \leq n^{-2k}
\]
for large \( n \).

There are less than \( n^k \) permutations of \( V \) that move exactly \( k \) vertices, so the total expected number of nontrivial automorphisms of \( T \) is asymptotically at most
\[
\sum_{k=3}^{n} n^{-k} = O(n^{-3}) = o(1).
\]

This completes the proof. Note that the bound \( O(n^{-3}) \) is much larger than the real value; we have been content to find a bound tending to 0.

References


