

THE SPECTRUM OF A GRAPH

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ABSTRACT

We survey the results obtained by a large number of authors concerning the spectrum of a graph. The questions of characterisation by spectrum, cospectral graphs and information derived from the spectrum are discussed.

1: INTRODUCTION

Our aim here is to review a large number of papers which deal with the spectrum of a graph, bringing the results together in one place for easy and convenient access. Prior to this, two similar surveys have been undertaken by Cvetković [16] and Wilson [74]. We attempt to bring these surveys up to date.

Throughout we will consider simple graphs on a finite set of vertices. The adjacency matrix $A(G)$ of a graph G on n vertices, is the $n \times n$ matrix (a_{ij}) , where $a_{ij} = 1$ if vertex i is adjacent to vertex j and $a_{ij} = 0$, otherwise. By $G(\lambda)$ we mean the characteristic polynomial of the graph G . This we define to be the characteristic polynomial, $\det(\lambda I - A(G))$. We note that some authors prefer $\det(A(G) - \lambda I)$, but that this clearly does not affect the results obtained. The eigenvalues of G are the eigenvalues of $A(G)$, and the set of all such eigenvalues is the spectrum of G . The spectral radius of G , $\text{spec. rad.}(G)$, is the largest eigenvalue of G .

The first paper published in this area appears to have been that by Collatz and Sinogowitz [11] in 1957. But obviously work had been done in this area long before, as one author, Sinogowitz, had already been dead for 13 years when the paper appeared. Further, some of the results of [11] were reviewed in [10].

Collatz and Sinogowitz put $G(\lambda) = \sum_{r=0}^n a_r \lambda^{n-r}$ and were able to obtain relations between some of the coefficients a_r and certain graphical properties. For instance, they proved

THEOREM 1.1. $a_0 = 1$; $a_1 = \text{tr}(A(G)) = 0$; $a_2 = -|E(G)|$; $a_3 = -2t$, where t is the number of triangles in G ; $a_4 = N(2P_2) - 2N(C_4)$; $a_5 = 2N(P_2 \cup C_3) - 2N(C_5)$, where $N(H)$ is the number of copies of H in G , and $\text{tr}(A(G))$ is the trace of the matrix $A(G)$. □

This result is generalised in [57] (see Theorem 5.1).

A related result of Collatz and Sinogowitz is

THEOREM 1.2. If G is connected bipartite, then $a_i = 0$ for all odd i . □

(This condition was shown to be sufficient by Gantmacher in [28]. He also proved that for G connected, G is bipartite if and only if $-\text{spec. rad.}(G)$ is an eigenvalue of G .)

In addition, Collatz and Sinogowitz determined the eigenvalues of P_n , C_n , K_n , all the graphs with fewer than 6 vertices, and all the trees with fewer than 9 vertices.

They also considered the spectral radius of a connected graph G , and achieved the following result.

THEOREM 1.3. (a) if G^* is obtained from G by adding a new edge, then $\text{spec. rad.}(G) < \text{spec. rad.}(G^*)$.

(b) Average degree of $G \leq \text{spec. rad.}(G) \leq$ maximum degree of G .

(c) $2 \cos \frac{\pi}{n+1} \leq \text{spec. rad.}(G) \leq n-1$. □

Many of the basic properties of the spectrum of G can be derived from elementary matrix theory. A number of these results are listed in [74] and we omit them here.

At first sight one might hope that the spectrum of a graph might somehow characterise it, but it is very easy to find graphs with the same spectrum. Examples were already available in [11]. Consequently we say that two non-isomorphic graphs are cospectral if they have the same characteristic polynomial. The smallest pair of cospectral graphs are $K_{1,4}$ and $C_4 \cup K_1$, a fact that seems to have

been discovered a number of times. Collatz and Sinogowitz also knew of a pair of cospectral trees on 8 vertices. We will pursue this line of investigation in Section 4.

The eigenvalues or characteristic polynomials for a number of graphs are known and are to be found in [5], [11]*, [13], [45], [53], [63], [71].

2: BASIC RESULTS

There are a number of straightforward results that follow directly from matrix theory and which we omit here due to lack of space. For instance, the eigenvalues of any graph are real and the spectrum of a disconnected graph is the union of the spectra of its components. Such results are readily available in [16] and [74].

We first give some results on methods of combining two graphs and the way in which the initial and final spectra are related.

THEOREM 2.1. If $G + H$ is the join of the graphs G, H on m, n vertices respectively, then

$$(G + H)(\lambda) = (-1)^n G(\lambda) \bar{H}(-\lambda - 1) + (-1)^m H(\lambda) \bar{G}(-\lambda - 1) \\ - (-1)^{m+n} \bar{G}(-\lambda - 1) \bar{H}(-\lambda - 1),$$

where \bar{G}, \bar{H} are the complements of G, H , respectively.

Proof: See [16]. When G and H are regular, the above result simplifies to that obtained by Finck and Grohmann [26]. Incidentally, if G is regular of degree r with n vertices,

$$(-1)^n \frac{\bar{G}(-\lambda - 1)}{(\lambda + r - n)} = \frac{G(\lambda)}{(\lambda - r)}. \quad \square$$

THEOREM 2.2. If the eigenvalues of G, H are λ_i, μ_j , respectively, then

* There are some errors here.

$$(G \times H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i - \mu_j),$$

$$(G \wedge H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i \mu_j),$$

and

$$(G * H)(\lambda) = \prod_i \prod_j (\lambda - \lambda_i \mu_j - \lambda_i - \mu_j).$$

Here $G \times H$ represents the cartesian product, $G \wedge H$ the conjunction and $G * H$ the strong product.

Proof: See [63]. The above theorem extends a result of Cvetković [14]. A formula for the characteristic polynomial of the generalized composition (Sabidussi's X-join), $G[H_1, H_2, \dots, H_m]$, is given in [63] for the case where each H_i is regular. \square

If $G \odot H$ is the graph obtained from G and H by joining a vertex v of VG and a vertex w of VH by an edge, then we have the following theorem.

THEOREM 2.3. $(G \odot H)(\lambda) = G(\lambda)H(\lambda) - G_v(\lambda)H_w(\lambda),$

where G_v, H_w are the subgraphs of G, H , respectively, induced by $VG \setminus \{v\}, VH \setminus \{w\}$, respectively.

Proof: See [63]. \square

A similar result can be achieved by identifying the vertices $v \in VG$ and $w \in VH$. This is called the coalescence of G and H and is denoted by $G \bullet H$.

THEOREM 2.4. $(G \bullet H)(\lambda) = G(\lambda)H_w(\lambda) + G_v(\lambda)H(\lambda) - \lambda G_v(\lambda)H_w(\lambda).$

Proof: See [63]. \square

We also have the following.

THEOREM 2.5. Suppose that VG can be partitioned into disjoint sets C_1, C_2, \dots, C_m , such that the number of vertices in C_j adjacent to a given vertex in C_i is independent of the choice of the vertex in C_i . Let this number be c_{ij} and let $C = (c_{ij})$. Then the characteristic polynomial of C divides that of G .

Proof: This is proved in [58], but is a direct consequence of a result in [34]. \square

An interesting result relating the characteristic polynomial of a graph, to those of certain of its subgraphs, is given below.

THEOREM 2.6. $\frac{d}{d\lambda} (G(\lambda)) = \sum_{i=1}^n G_{v_i}(\lambda)$, where $VG = \{v_1, v_2, \dots, v_n\}$.

Proof: See [9]. In this paper Clarke actually uses a different polynomial from the one we are using. However, it is straightforward to obtain the above result from his Corollary 3 to Theorem 5. □

COROLLARY. The vertex-deleted subgraphs of cospectral vertex-transitive graphs are cospectral. Moreover, in the light of the comment in the proof of Theorem 2.1, these vertex-deleted subgraphs also have cospectral complements. □

The following theorem relating the eigenvalues of G and a subgraph of G is also worth noting.

THEOREM 2.7. If v is a vertex of G , and $G_v(\lambda) = \Pi(\lambda - \mu_i)$ and $G(\lambda) = \Pi(\lambda - \lambda_i)$, then $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n$, for a suitable labelling of the eigenvalues.

Proof: This result is a consequence of a well known result in matrix theory, see, for example [4], p. 221. The theorem is noted in [74] and considered further in [47]. □

3: CHARACTERISATION OF GRAPHS BY THEIR SPECTRA

In this section we consider the problem of characterising a graph by its spectrum. Although this project for a general graph is doomed to failure by the examples of Section 1 and Section 4, there are a number of types of graphs for which characterisations are known. The results below fall into two categories. One of these says that a graph of a certain type with a particular spectrum belongs to a given set of graphs, while the other says that a graph of a certain type with a particular spectrum can only be a specified graph.

By and large, the results of this section deal with regular graphs. However, the reason for progress being made in these directions is due more to the

fact that the graphs considered have only a small number of distinct eigenvalues, rather than to the regularity, although naturally this latter is of some assistance.

We note that many of the results below have already been surveyed in the papers by Cvetković [16] and Wilson [74], but we include them again here for the sake of completeness. The first few concern line graphs.

THEOREM 3.1. The line graph of the complete graph on n vertices has as distinct eigenvalues, $\{-2, n, 2n - 2\}$. Except for $n = 8$, if G is a regular connected graph on n vertices with these eigenvalues, then $G \cong L(K_n)$.

If $n = 8$, then there are three exceptions.

Proof: See [7], [8], [12], [14], [35], [36]. The exceptions are given in [8]. \square

THEOREM 3.2. The line graph of the complete bipartite graph $K_{n,n}$ has as distinct eigenvalues, $\{-2, n - 2, 2n - 2\}$. Except for $n = 4$, if G is a regular connected graph on $2n$ vertices with these eigenvalues, then $G \cong L(K_{n,n})$. If $n = 4$, then there is one exception.

Proof: See [69]. \square

THEOREM 3.3. The line graph of the complete bipartite graph $K_{m,n}$ has as distinct eigenvalues, $\{-2, m - 2, n - 2, m + n - 2\}$.

If G has mn vertices and the eigenvalues listed above, then $G \cong L(K_{m,n})$ if and only if G has an m -clique.

Proof: See [21]. A number of other results concerning $L(K_{m,n})$ and its characterisation are given in this paper. \square

Now a bipartite graph on $2v$ vertices can be derived from a (v, k, λ) symmetric balanced incomplete block design, where every vertex has degree k and any two vertices in the same part are adjacent to exactly λ vertices in the other part. Call this graph B .

THEOREM 3.4. Let G be a regular connected graph on $2v$ vertices. The distinct

eigenvalues of G are given by $\pm k, \pm\sqrt{k - \lambda}$ if and only if G is isomorphic to the graph obtained from some symmetric b.i.b.d. with parameters (v, k, λ) .

Proof: See [41]. □

Continuing on the line graph theme, we have

THEOREM 3.5. $L(B)$ has as distinct eigenvalues $\{-2, 2k - 2, k - 2 \pm \sqrt{k - \lambda}\}$.

Except for $v = 4, k = 3, \lambda = 2$, if G is a regular connected graph on vk vertices with these eigenvalues, then G is isomorphic to the line graph of a symmetric b.i.b.d. with parameters (v, k, λ) . If $v = 4, k = 3, \lambda = 2$, there is one exception.

Proof: See [41]. □

Let π be a projective plane with $n + 1$ points on a line, i.e. π is of order n . Let $H(\pi)$ be the bipartite graph whose vertices are the $2(n^2 + n + 1)$ points and lines of π , where two vertices are adjacent if and only if one of the vertices is a point, the other is a line, and the point is on the line.

THEOREM 3.6. The line graph of $H(\pi)$, for any π of order n , has as distinct eigenvalues $\{-2, 2n, n - 1 \pm \sqrt{n}\}$. If G is a regular connected graph on $(n + 1)(n^2 + n + 1)$ vertices with these eigenvalues, then $G \cong L(H(\pi))$, for some π of order n .

Proof: This was proved directly in [37], but is properly a corollary to Theorem 3.5. □

On the way to proving Theorem 3.6, Hoffman also proved the following characterisation.

THEOREM 3.7. A regular connected graph G on $2(n^2 + n + 1)$ vertices has as distinct eigenvalues $\{\pm(n + 1), \pm\sqrt{n}\}$, if and only if $G = H(\pi)$ for some projective plane of order n . □

We now repeat the above process for finite affine planes. If Π is a finite affine plane with n points on a line, then we define the graph $H(\Pi)$ whose vertices are the points and lines of Π , with two vertices adjacent if and only if

one is a point, the other a line and the point is on the line.

THEOREM 3.8. For a particular Π , $L(H(\Pi))$ has as distinct eigenvalues $\{-2, 2n - 1, n - 2, \frac{1}{2}[2n - 3 \pm \sqrt{(4n + 1)}]\}$. If G is a regular connected graph on $n^2(n + 1)$ vertices with these eigenvalues, then $G \cong L(H(\Pi))$, for some Π of order n .

Proof: See [42]. □

A graph can also be made from a (v, b, r, k, λ) -balanced incomplete block design. This has $b + v$ vertices and two vertices are adjacent if and only if one corresponds to a block and the other corresponds to an element in that block. Call this graph N .

THEOREM 3.9. $L(N)$ has as distinct eigenvalues $\{-2, r + k - 2, \frac{1}{2}(r + k - 4) \pm D^{\frac{1}{2}}, k - 2\}$, where $D = \frac{1}{4}(r - k)^2 + r - \lambda$. If $r + k > 18$, $\lambda = 1$, and if G is a regular connected graph on vr vertices with these eigenvalues, then $G \cong L(N)$, for some N .

Proof: See [24]. □

In a similar vein we have

THEOREM 3.10. The line graph of a Steiner triple system is identified as such by its spectrum if $r > 15$.

Proof: This is noted in [20] as a private communication from the author to himself. □

Steiner graphs can be obtained from Steiner triple systems in a natural way by considering the blocks as vertices and saying two vertices are adjacent if the blocks have a common element (see [66]). It can then be shown, [2], that

THEOREM 3.11. For s sufficiently large, any strong graph on $\frac{1}{6}s(s - 1)$ vertices, with eigenvalues $\{-3, \frac{1}{2}(s - 9), \frac{3}{2}(s - 3)\}$ is isomorphic to some Steiner graph of order s . □

(A strong graph is a graph on n vertices which is not K_n or \bar{K}_n , and whose adjacency matrix $A = A(G)$ satisfies the following relation

$$[J - 2A - (\rho_1 + 1)I][J - 2A - (\rho_2 + 1)I] = (n - 1 + \rho_1\rho_2)J,$$

where J is the matrix all of whose entries are 1 and ρ_1, ρ_2 are suitable real numbers with $\rho_1 \neq \rho_2$.)

In Theorems 3.1, 3.2, 3.3, 3.5, 3.6, 3.8, and 3.9, which all deal with line graphs, it can be seen that in each case, -2 is an eigenvalue. Further, -2 is the smallest eigenvalue. We now consider generalisations of these observations.

THEOREM 3.12. (a) The minimum eigenvalue of a line graph is greater than, or equal to, -2 .

(b) If G is connected, the minimum eigenvalue of $L(G)$ is -2 if and only if either

$$|E(G)| - |V(G)| + 1 > 0 \quad \text{and } G \text{ is bipartite, or}$$

$$|E(G)| - |V(G)| > 0 \quad \text{and } G \text{ is not bipartite.}$$

(c) The minimum eigenvalue of $L(G)$ is -2 unless every connected component of G is a tree or has one cycle of odd length and no other cycles.

(d) If the diameter of G is D , then the minimum eigenvalue of $L(G)$ lies between -2 and $-2 \cos(\pi/(D + 1))$, and these bounds are best possible.

(e) If G is a regular graph of degree r , with n vertices, then

$$L(G)(\lambda) = (\lambda + 2)^{\frac{1}{2}(r-2)n} G(\lambda + 2 - r).$$

(f) Let G be a bipartite graph with n_i mutually non-adjacent vertices of degree r_i , $i = 1, 2$, and $n_1 \geq n_2$, then

$$L(G)(\lambda) = (\lambda + 2)^\beta \left\{ \left(-\frac{\alpha_1}{\alpha_2} \right)^{n_1 - n_2} G(\sqrt{\alpha_1\alpha_2}) G(-\sqrt{\alpha_1\alpha_2}) \right\}^{\frac{1}{2}},$$

where $\alpha_i = \lambda - r_i + 2$, $i = 1, 2$, and $\beta = n_1 r_1 - n_1 - n_2$.

(g) Let G be a regular connected graph of degree ≥ 17 and with smallest eigenvalue -2 , then G is either a line graph or the complement of the regular graph of degree 1. The number 17 is the best possible.

(h) If $G = L(H)$ and the minimum degree of H , $d(H)$, is ≥ 4 , then the minimum eigenvalue of G is -2 . Further, the number of vertices adjacent to both u_1 and u_2 , $\Delta(u_1, u_2)$, is such that for u_1, u_2 non-adjacent, $\Delta(u_1, u_2) < \deg_G u_i - 2$, $i = 1, 2$, where $u_1, u_2 \in VG$.

(i) If for a graph G , (α) $d(G) > 43$, (β) the minimum eigenvalue is -2 , and (γ) for non-adjacent vertices u_1, u_2 , we have $\Delta(u_1, u_2) < \deg_G u_i - 2$, $i = 1, 2$, then G is a line graph.

Proof: The proof of (a) can be found in [39], as is part of (b). The proof of (b), (c), (d) is in [22]. In [59], (e) is proved, and (f) is proved in [16]. No proof of (g) as yet seems to have appeared in print. It is referred to originally in [38] where it is attributed to Hoffman and Ray-Chaudhuri, and then later in [16] and [24] at least. Also in [38], an example due to Seidel is cited (but not given), to show that 17 is best possible. In [54], (h) and (i) are proved. It is expected that the number 43 in (α) is not best possible. \square

The following results are along similar lines to the work above, in that they deal with the number -2 .

THEOREM 3.13. If T is a tree on n vertices, $L(\lambda)$ is the characteristic polynomial of the line graph of T , and p is a prime, then $L(-2) \equiv 0 \pmod{p}$ if and only if $|VT| \equiv 0 \pmod{p}$.

Proof: See [22]. \square

THEOREM 3.14. The only strongly regular graphs with smallest eigenvalue -2 , are the lattice graphs, the triangular graphs, the pseudolattice graphs, the pseudotriangular graphs, the graphs of Petersen, Clebsch and Schläfli, and the complements of the ladder graphs.

Proof: See [65]. The graphs mentioned in this theorem are described in [65] and elsewhere. It should be pointed out that Seidel works with $(0, -1, 1)$ matrices in this paper, and hence the value 3 in its title. These results can be converted into results for $(0, 1)$ matrices. Other results on $(0, -1, 1)$ matrices may be

found in [31], [64], [66], [67]. □

We now see that there are graphs other than line graphs which are characterised by their spectra.

THEOREM 3.15. The graphs on a prime number of vertices, whose automorphism groups are transitive, are identified within this class of graphs by their spectra.

Proof: The eigenvalues of such graphs are given in [72] along with the proof of this result. They are easily obtained since the adjacency matrices of the graphs in question are circulant matrices. It should be noted that, in general, graphs whose adjacency matrices are circulants are not characterised by their spectra. An example of such graphs on 20 vertices is given in the Appendix. □

A cubic lattice graph with characteristic n ($n > 1$) is a graph whose vertices are all the n^3 ordered triplets on n symbols, with two triplets adjacent if and only if they differ in exactly one coordinate. These graphs are characterised as follows, where $\Delta(x, y)$ is the number of vertices adjacent to both x and y .

THEOREM 3.16. Except for $n = 4$, G is the cubic lattice graph with characteristic n , if and only if its eigenvalues are $\lambda_f = 3n - 3 - fn$, with multiplicity $p_f = \binom{n}{f}(n-1)^f$, $f = 0, 1, 2, 3$ and $\Delta(x, y) > 1$ for all non-adjacent x, y .

Proof: See [15] after [48] and [1]. □

A tetrahedral graph is defined to be a graph G , whose vertices are identified with the $\binom{n}{3}$ unordered triples on n symbols, two vertices being adjacent if and only if the corresponding triples have 2 symbols in common.

THEOREM 3.17. If G is a tetrahedral graph, then (i) $|VG| = \binom{n}{3}$, (ii) G is regular and connected, (iii) the number of vertices at distance 2 from a given vertex v is $\frac{3}{2}(n-3)(n-4)$ for all $v \in VG$, (iv) the distinct eigenvalues of G are $\{-3, 2n-9, n-7, 3n-9\}$. For $n > 16$ any graph possessing properties (i)-(iv) is tetrahedral.

Proof: See [6]. □

In [33], Harary and Schwenk pose the problem of determining all graphs whose spectrum consisted entirely of integers. They called these graphs integral graphs.

THEOREM 3.18. The set I_r of all regular connected integral graphs with a fixed degree r , is finite.

Proof: See [17]. □

The problem suggested by Theorem 3.18 then is to completely determine the set I_r . For $r \leq 2$, these are P_2 , C_3 , C_4 and C_6 (see [33]). What if $r = 3$?

THEOREM 3.19. There are thirteen connected cubic integral graphs.

Proof: See [17] and [62]. □

It can also be shown that Cayley graphs of \mathbb{Z}_2^n always have integral spectra.

At this stage little more seems to be known about integral graphs.

In a similar vein, Doob has tried to determine which graphs have a small number of eigenvalues. Some of this work relates back to earlier theorems concerning line graphs.

THEOREM 3.20. (a) G has one eigenvalue if and only if $G = \bar{K}_n$.

(b) G has two distinct eigenvalues $\alpha_1 > \alpha_2$ if and only if each component of G is K_{α_1+1} and $\alpha_2 = -1$.

(c) G has eigenvalues $r, 0, \alpha_2$ if and only if G is the complement of the union of complete graphs on $-\alpha_2$ vertices. (r is the degree of G .)

(d) G has eigenvalues $\pm\alpha, 0$ if and only if $G = K_{m,n}$ and $mn = \alpha^2$.

(e) If G is regular, then it has eigenvalues $\pm r, \pm 1$, if and only if

$G = K_{r+1, r+1}$ minus a 1-factor.

Proof: See [20]. □

THEOREM 3.21. If H is the graph of a b.i.b.d. and $G \cong L(H)$, then

(i) G has three eigenvalues if and only if the b.i.b.d. is symmetric and

trivial,

(ii) G has four eigenvalues if and only if the b.i.b.d. is symmetric or trivial, but not both,

(iii) G has five eigenvalues if and only if the b.i.b.d. is neither symmetric nor trivial.

Proof: See [20]. □

THEOREM 3.22. If G is a graph with four distinct eigenvalues, the smallest of which is -2 , and $G \cong L(H)$, then

(i) H is strongly regular,

(ii) H is the graph of a symmetric b.i.b.d.,

or (iii) $H \cong K_{m,n}$ with $m > n \geq 2$.

Proof: See [21]. In fact if G has four distinct eigenvalues, the smallest of which is -2 , then $G \cong L(H)$ for some H , except in a finite number of cases. □

4: COSPECTRAL GRAPHS

In this section we return to a consideration of those graphs which have a cospectral mate. The existence of cospectral graphs was recognised in the paper of Collatz and Sinogowitz [11]. Some of these graphs were rediscovered in [27] and [3] and no doubt elsewhere. In [32] the smallest (in terms of the number of vertices) cospectral graphs and trees were noted. Also in this paper, the smallest cospectral digraphs were listed. We note in passing that more work on cospectral digraphs is done in [46], [53].

In [30], the number of cospectral graphs on 5, 6, 7, 8, 9 vertices are given, while in [67] the eigenvalues of certain strongly regular graphs are listed, for the $(0, -1, 1)$ adjacency matrix.

In a general sense, it is doubtful whether very much can be said about cospectral graphs. It is possible to find cospectral graphs; cospectral connected graphs; cospectral trees; cospectral forests; cospectral regular graphs; cospectral vertex-transitive graphs; cospectral circulant graphs; cospectral regular graphs -

one of which is vertex-transitive, the other which is not; cospectral strongly regular graphs - one whose group is of order 1 and is cospectral to its complement, the other which is transitive and self-complementary; cospectral non-regular graphs with cospectral complements; cospectral trees with cospectral complements; non self-complementary graphs which are cospectral to their complements; cospectral graphs - one of which is self-complementary and one of which is not; cospectral trees with cospectral line graphs. An example of each of the above types of graphs is given in the Appendix. Where possible the smallest such pair is given. Further information on some of these graphs can be found in [30]. Two cospectral graphs with different chromatic number may be found in [40].

It is natural to ask, "How many cospectral graphs are there?" It should be no surprise that there are a non-finite number.

THEOREM 4.1. Given any positive integer k , there exists n such that at least k cospectral graphs exist on n vertices. Further, n may be chosen so that k of these cospectral graphs may be connected and regular.

Proof: Due to Hoffman, published in [53]. □

THEOREM 4.2. There are infinitely many pairs of non-isomorphic cospectral trees.

Proof: By construction in [53]. □

In fact, cospectral trees are more the rule than the exception.

THEOREM 4.3. If p_n is the probability that a randomly chosen tree on n vertices is cospectral to another tree on n vertices, then $p_n \rightarrow 1$ as $n \rightarrow \infty$.

Proof: See [61]. □

So in the above sense, almost all trees have a cospectral mate. But more can be said.

THEOREM 4.4. Let q_n be the property that an arbitrary tree S on n vertices, has a cospectral mate T which is also a tree and \bar{S}, \bar{T} are also cospectral. If

$$r_n = \frac{\text{number of trees with property } q_n}{\text{number of trees with } n \text{ vertices}},$$

then $r_n \rightarrow 1$ as $n \rightarrow \infty$.

The same result holds if in addition, we require S and T to have cospectral line graphs whose complements are also cospectral.

Proof: See [30] for the first part of the Theorem and [50] for the second part. \square

THEOREM 4.5. Given any graph G on n vertices, there exist at least $\binom{2n-2}{n-2}$ non-isomorphic pairs of cospectral graphs on $3n$ vertices such that each member of each of the pairs contains three disjoint induced subgraphs isomorphic to G , and is connected if G is.

Proof: See [29]. \square

COROLLARY 4.6. Asymptotically, there are at least $4^{n-1}/\sqrt{\pi n}$ pairs of cospectral graphs on $3n$ vertices. \square

Constructions for obtaining infinite families of cospectral pairs of graphs are given in [29], [30], [53].

Finally, we note that

THEOREM 4.7. Every graph can be embedded in each graph of a pair of cospectral regular graphs. In fact the degree and diameter of these cospectral graphs may be arbitrarily large.

Proof: See [23]. \square

5: INFORMATION FROM SPECTRA

The information that can be obtained from spectra basically breaks down into two kinds. In the first kind we obtain information about various properties of the graph itself, such as the chromatic number, while the second kind gives us information about something outside graph theory (and even outside mathematics),

such as molecular structure.

We consider what information is obtainable about the graph from the spectrum of the graph.

Suppose that U_r is a graph on r vertices whose components are solely edges and cycles. Then we have

THEOREM 5.1. If G is any graph with $G(\lambda) = \sum_{i=0}^n a_r \lambda^{n-r}$, then $a_r = \sum_{U_r \subset G} (-1)^{p(U_r)} 2^{c(U_r)}$, where $p(U_r)$ is the number of components of U_r and $c(U_r)$ is the number of components of U_r which are cycles.

Proof: See [57]. □

We now see that Theorem 1.1 is an immediate consequence of this more general result, as is Theorem 1.2 and the corollary below. An independent proof is given in [53].

COROLLARY 5.2. If G is a tree, then $|a_{2k}|$ is the number of matchings of order k in G . □

Several other results of Sachs, which are listed in [60], and follow more or less directly from standard matrix results, are given below.

THEOREM 5.3. (a) Let G be a graph with spectral radius ρ , eigenvalues λ_i , $|VG| = n$ and maximum degree δ .

(i) G is regular if and only if $\sum \lambda_i^2 = n\rho$.

(ii) G is regular if and only if $\rho = \delta$.

(iii) The chromatic number of G is less than or equal to $\rho + 1$.

(b) If G is regular, then the spectrum of G determines

(i) the length of the shortest odd cycle in G and the number of such cycles,

(ii) the girth, t , of G ,

(iii) the number of cycles of length h , where $h \leq 2t - 1$. □

It is also possible to determine the number of spanning trees of a graph.

THEOREM 5.4. Let G be a graph on n vertices.

(a) If G is regular of degree r , with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, r$, then the number of spanning trees of G is precisely

$$\frac{1}{n} \prod_{i=1}^{n-1} (r - \lambda_i) = \frac{1}{n} G'(r).$$

(b) If G is arbitrary, G^p is the graph obtained from G by adding sufficient loops to make the row sums of $A(G)$ equal to $n - 1$ and $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, n - 1$ are the eigenvalues of G^p , then the number of spanning trees of G is equal to

$$\frac{1}{n} \prod_{i=1}^{n-1} (n - 1 - \lambda_i).$$

Proof: See [60], [73], [74]. □

We mention in passing, that the number of spanning trees can equally well be determined by using the matrix $M(G)$, where $m_{ij} = -1$ if $v_i \sim v_j$ in G , $m_{ii} = \deg v_i$ and $m_{ij} = 0$ otherwise. Fiedler [25] also uses this matrix to give a definition of algebraic connectivity of graphs.

Some information about the automorphism group, $\Gamma(G)$, of a graph G can also be derived from its spectrum.

THEOREM 5.5. (a) If G has all eigenvalues of multiplicity 1, then every element of $\Gamma(G)$ is of order two, and so $\Gamma(G)$ is elementary abelian.

(b) If $G(\lambda)$ is irreducible over Z , then $|\Gamma(G)| = 1$.

(c) If $A = A(G)$ and the minimal and characteristic polynomials of A are identical over $GF(2)$, then $g \in \Gamma(G)$ can be expressed in the form

$$g = \mu(A) \sum_{i=0}^{n-m-1} b_i A^i + I,$$

for some $b_i \in GF(2)$, where $\mu(\lambda)$ is the minimal polynomial of A^2 and $m = \deg \mu(\lambda) = \lfloor \frac{1}{2}n \rfloor$.

Proof: See [51] for (a). The proofs of (b) and (c) are in [52]. It is worth commenting that (b) is actually proved in a more general setting, and is a generalisation of some remarks of Collatz and Sinogowitz [11]. The converse of (b) is not true. □

If we restrict our attention to the bipartite graph G , then we can derive some information from $\eta(G)$, the multiplicity of the eigenvalue 0.

THEOREM 5.6. (a) The maximal number of mutually non-adjacent edges of a tree G with n vertices is $\frac{1}{2}(n - \eta(G))$.

(b) If G is a bipartite connected graph and $\eta(G) = 0$, then G has a 1-factor.

Proof: Part (a) is given in [18] and follows from a result in [57] and from Corollary 5.2, while (b) can be found in [49]. □

It is of particular interest that [49] is not a graph theoretical paper. In fact, both chemists and physicists have taken interest in what amounts to the spectrum of a graph for a long period, indeed, they were interested in the topic even before graph theoreticians. It seems likely that Collatz and Sinogowitz came on the topic via physical motivations. A discussion of the relevance of the particular number $\eta(G)$, to chemistry, is given in [18]. If $\eta(G) > 0$, then the molecule corresponding to G cannot have the total electron spin being equal to zero. This implies molecular instability.

A number of references to chemical applications are available in [18]. Other papers which are of interest in this area can be found listed in [16]. The papers [43] and [70] are also in this area.

One question in physics which bears on the spectrum of a graph is whether or not one can "hear the shape of a drum". Collatz and Sinogowitz were probably motivated by this question. This problem is discussed in [3], [27], [44], for example.

APPENDIX: COSPECTRAL GRAPHS

In this appendix we give a number of examples of families of cospectral graphs. The graphs given in (1) to (7), (10) and (11) come from [30], while (8), (9) and (14) have been found by the authors.

For (1) to (11) we make the claim that there are no smaller families (i.e. on less vertices) with the properties stated.

(1) Cospectral graphs



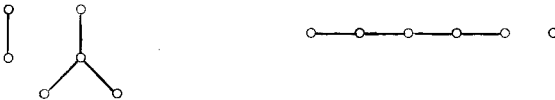
(2) Cospectral connected graphs



(3) Cospectral graphs with cospectral complements



(4) Cospectral forests



(5) Cospectral trees



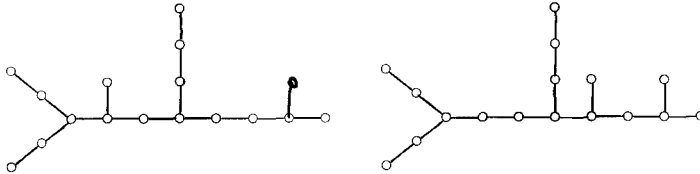
(6) Cospectral trees with cospectral complements



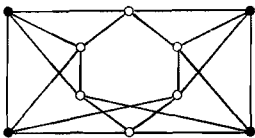
(7) Cospectral trees with cospectral line graphs



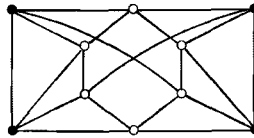
(8) Cospectral trees with cospectral complements, cospectral line graphs, cospectral complements of line graphs, cospectral line graphs of complements and cospectral distance matrices



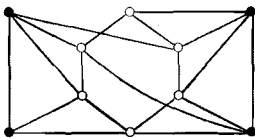
(9) Cospectral regular graphs: we present the two pairs of cospectral graphs on ten vertices. Each graph can be obtained from its cospectral mate by switching about the black points ((a) is cospectral to (b), (c) to (d)). Switching is defined in [67].



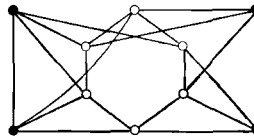
(a)



(b)

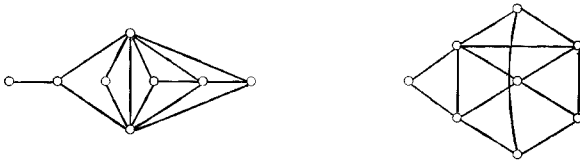


(c)

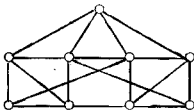


(d)

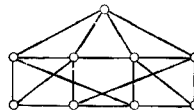
(10) Two examples of graphs cospectral but not isomorphic to their own complements



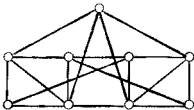
(11) A family of four cospectral graphs of which (a) and (b) are complements of each other, while (c) and (d) are self-complementary. Furthermore, the line graphs of (a) and (b) are cospectral, with cospectral complements.



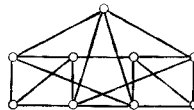
(a)



(b)

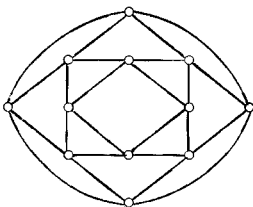


(c)

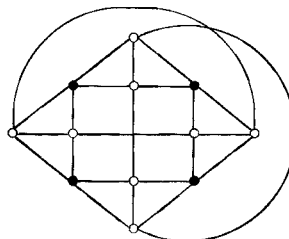


(d)

(12) Two cospectral regular graphs, the first transitive, the second not. These come from [41]. We remark that (a) is the line graph of the cube. (These graphs may not be the smallest such pair.)



(a)



(b)

(13) There are two cospectral strongly regular graphs on sixteen vertices, namely the line graph of $K_4 \times K_4$, and Shrikhande's graph (see [69]). These are the

smallest cospectral strongly regular graphs [67]. They are also both vertex-transitive. We know of no smaller pair of cospectral vertex-transitive graphs.

(14) We define a circulant graph $C(S)$ on n vertices with connection set S as follows: Let S be a subset of the integers (mod n) such that if $x \in S$, then $-x \in S$. Then $C(S)$ is the graph with vertices $0, 1, \dots, n-1$ and with vertex i adjacent to vertex j if and only if $i-j \in S$.

The circulant graphs on twenty vertices defined by the connection sets $S_1 = \{\pm 2, \pm 3, \pm 4, \pm 7\}$ and $S_2 = \{\pm 3, \pm 6, \pm 7, \pm 8\}$ are non-isomorphic and cospectral. There are no cospectral circulants on less than twenty vertices.

(15) Let G be the graph on twenty-five vertices obtained by regarding the vertices as elements of the field of order twenty-five, and taking the vertices to be adjacent if their difference is a square in the multiplicative group of the field.

Let H be the graph obtained from the graph denoted St 2.1 in [67] by switching about the neighbourhood of vertex three in that graph, and deleting the isolated vertex which results. Let \bar{H} be the complement of H .

Then G is self-complementary and vertex-transitive, H (and so \bar{H}) is an identity graph and G, H and \bar{H} are all cospectral.

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