THE VALUE OF THE RAMSEY NUMBER $R(3,8)$

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Abstract.

The Ramsey number $R(3,8)$ can be defined as the least number n such that every graph on n vertices contains either a triangle or an independent set of size 8. With the help of a substantial amount of computation, we prove that $R(3,8)=28$.

1. Introduction.

For integers $s,t \geq 1$, define the *Ramsey number* $R(s,t)$ to be the least number $n = n(s,t)$ such that every simple graph on n vertices contains either a clique of order s, or an independent set of order t. The existence of $R(s,t)$ is a well-known consequence of Ramsey's Theorem. The book [3] should be consulted for an extensive survey to 1980.

The previous best bounds on the number $R(3,8)$ were obtained by Grinstead and Roberts [4], who showed that $28 \leq R(3,8) \leq 29$. Further information on the structure of a minimal graph was obtained by Radziszowski and Kreher [6, 7]. It is possible that the results in the latter two papers might have facilitated the present work, but we were not aware of them at the time.

In this paper we describe a computation which demonstrated that every triangle-free graph on 28 vertices contains an independent set of order 8. In view of Grinstead and Robert's result, we conclude that $R(3,8)=28$.

2. Basic Theory.

For integers t,n,δ,e, let $\mathcal{R}(t,n,\delta,e)$ denote the set of all simple triangle-free graphs of order n with e edges, minimum degree δ, and no independent sets of order t. We will drop trailing arguments if we wish them to be free. Thus, $\mathcal{R}(t,n,\delta)=\bigcup_e \mathcal{R}(t,n,\delta,e)$ and $\mathcal{R}(t,n)=\bigcup_\delta \mathcal{R}(t,n,\delta)$.

If G is a graph, then VG denotes its vertex-set. If $v \in VG$, then $N_G(v)$ denotes the set of neighbours of v in G, and $\bar{N}_G(v)$ denotes $VG - N_G(v) - \{v\}$. If $W \subseteq VG$, then

$G[W]$ denotes the subgraph of G induced by W. For $v \in VG$, $\deg_G(v)$ is the degree in G of vertex v.

Our basic tool for the construction of $\mathcal{R}(t,n)$ is the following.

Lemma 1. *For $t \geq 2$, let $G \in \mathcal{R}(t,n,\delta,e)$ and let $v \in VG$ have degree δ. Then*
(a) $n - R(3,t-1) \leq \delta \leq \min(t-1, n/2)$, *and*
(b) $G[\bar{N}_G(v)] \in \mathcal{R}(t-1, n-\delta-1, \delta', e')$ *for some δ', and some e' satisfying*

$$\delta(n-\delta-1) - \delta(t-2) \leq 2e' \leq (t-1)(n-\delta-1) - \delta(\delta-1).$$

Proof. If $\delta \geq t$, then the neighbourhood of v would be an independent set of size t. Also, no triangle-free graph on n vertices has more than $n^2/4$ edges, which shows that $\delta \leq n/2$. Alternatively, if $\delta < n - R(3,t-1)$, then the existence of $G[\bar{N}_G(v)]$ would violate the definition of $R(3,t-1)$. This proves (a).

Let D be the sum of the degrees in G of the vertices in $\bar{N}_G(v)$. Then, clearly $\delta(n-\delta-1) \leq D \leq (t-1)(n-\delta-1)$. Similarly, if E is the number of edges in G between $N_G(v)$ and $\bar{N}_G(v)$, then $\delta(\delta-1) \leq E \leq \delta(t-2)$. Claim (b) now follows on noting that $2e' = D - E$. ∎

Let $H \in \mathcal{R}(t-1, n-\delta-1)$, and let G be a graph of order n and minimum degree $\delta \leq t-1$ such that $G[\bar{N}_G(v)] = H$ for some $v \in VG$ of degree δ. In this circumstance, we say that G is an *extension* of H. Let $N_G(v) = \{v_1, v_2, \ldots, v_\delta\}$ and, for $1 \leq i \leq \delta$, define $X_i = N_G(v_i) \cap \bar{N}_G(v)$. Note that $|X_i| = \deg_G(v_i) - 1$ for $1 \leq i \leq \delta$.

Lemma 2. *Let G be an extension of H, with $H = G[\bar{N}_G(v)] \in \mathcal{R}(t-1, n-\delta-1)$. Then $G \in \mathcal{R}(t,n)$ if and only if all of the following conditions hold.*
(a) *$N_G(v)$ is an independent set of G.*
(b) *Each X_i is an independent set of H.*
(c) *For each $I \subseteq \{1,2,\ldots,\delta\}$, no independent set in H of size $t - |I|$ is contained in $VH - \bigcup_{i \in I} X_i$.*

Proof. Since conditions (a) and (b) together are equivalent to requiring G to be triangle-free, it will suffice to consider the existence of independent sets of size t. Since $H \in \mathcal{R}(t-1, n-\delta-1)$, there can be no such sets including v. On the other hand, such a set not including v exists if and only if condition (c) fails to hold. ∎

Since any particular $G \in \mathcal{R}(t,n,\delta)$ may have many vertices of degree δ, it might be constructible from many different $H \in \mathcal{R}(t-1, n-\delta-1)$. To reduce the number of times each G was constructed, we employed several different techniques.

Consider a function $\theta(G)$ defined for any graph G and satisfying the following properties:
(1) $\theta(G)$ is an orbit of the action of the automorphism group $\mathrm{Aut}(G)$ on VG.

(2) The vertices in $\theta(G)$ have minimum degree in G.

(3) For any permutation γ of VG, $\theta(G^\gamma) = \theta(G)^\gamma$, where exponentiation by γ denotes taking the image under the action of γ.

The first author's program *nauty* [5] computes a permutation $\kappa = \kappa(G)$ of VG with the property that $(G^\gamma)^\kappa = G^\kappa$ for any permutation γ of VG. (Here, "=" means equality, not just isomorphism.) Such a permutation is called a *canonical labelling* of G, and G^κ is said to be *canonically labelled. nauty* can also compute the orbits of $\mathrm{Aut}(G)$. Suppose that $VG = \{1, 2, \ldots, n\}$. If we define $\theta(G)$ to be the orbit containing the vertex of minimum degree which appears first in the sequence $1^{\kappa^{-1}}, 2^{\kappa^{-1}}, \ldots, n^{\kappa^{-1}}$, then θ is easily seen to satisfy requirements (1)–(3) above.

For $t \le 7$, we constrained G to be only constructed from $H = G[\bar{N}_G(v)]$ for some $v \in \theta(G)$. The requirements on θ imply that isomorphic G can only arise amongst extensions of the same H, which greatly simplifies isomorph rejection. (In fact, they can only arise as a result of automorphisms of H, but that fact is not simple to use here.)

For $t = 8$, we were less concerned with isomorphism rejection as we expected few or no graphs to be found. Hence we did not use $\theta(G)$ as described above, but instead accepted any graph produced. This strategy permitted an additional technique to be used in one subcase which would have otherwise been too difficult. This subcase is described in more detail in the next section. In this case we constructed G only from some $H \in \mathcal{R}(t-1, n-\delta-1, \delta')$ with δ' as small as possible, with the help of the following lemma.

Lemma 3. *Let v and w be distinct vertices of a graph $G \in \mathcal{R}(t, n)$. Then the minimum degree of $G[\bar{N}_G(v)]$ is at most $t - 1 - |G(v) \cap G(w)|$.*

Proof. The claim is obvious if v and w are adjacent. If they are not adjacent, it is easily seen that w has such a degree in $G[\bar{N}_G(v)]$. ∎

3. The computational method.

Given the elementary nature of the underlying theory, the success of the computation must be attributed mostly to the details of the implementation. Consequently, we will describe this in some detail.

Subsets of $\{1, 2, \ldots, n\}$ were represented as the pattern of bits in a 32-bit machine word. This allows set operations such as union, intersection and containment to be performed in one or two machine instructions.

Suppose now that we have some $H \in \mathcal{R}(t-1, n-\delta-1)$ and we wish to find the extensions $G \in \mathcal{R}(t, n, \delta)$ of H. Let $S_1, S_2, \ldots, S_N$ be a list of all the independent sets of H with cardinality between $\delta - 1$ and $t - 2$, inclusive.

For $w \in VH$ and $X_1, X_2, \ldots, X_i \subseteq VH$, define

$$d_i(w) = d_i(w; X_1, \ldots, X_i) = \deg_H(w) + |\{X_j \mid w \in X_j, 1 \le j \le i\}|.$$

Now consider the following recursive procedure.

procedure $makeX(k,\ (X_1, \ldots, X_{k-1}),\ (Y_1, \ldots, Y_K))$

-- k and K are integers. Each X_i and each Y_i is a subset of VH.

if $k > \delta$ **then** $process((X_1, X_2, \ldots, X_\delta))$

else

Construct the list $(Z_1, \ldots, Z_L)$ of all elements Z of $(Y_1, \ldots, Y_K)$ such that

(i) For each $w \in VH$, if $d_{k-1}(w) < k-1$, then $w \in Z$.

(ii) For each $w \in VH$, if $d_{k-1}(w) = t-1$, then $w \notin Z$.

(iii) H has no independent set of size $t - 1 - |I|$ disjoint from $Z \cup \bigcup_{i \in I} X_i$ for any $I \subseteq \{1, 2, \ldots, k-1\}$.

for i **from 1 to** L **do** $makeX(k+1,\ (X_1, \ldots, X_{k-1}, Z_i),\ (Z_i, Z_{i+1}, \ldots, Z_L))$

endif

end $makeX$

Lemma 4. *Suppose procedure makeX is invoked with arguments* $(0, (\ ), (S_1, \ldots, S_N))$. *Then procedure process will be invoked exactly once for each sequence* $X_1 = S_{i_1}, X_2 = S_{i_2}, \ldots, X_\delta = S_{i_\delta}$ *such that* $1 \le i_1 \le i_2 \le \cdots \le i_\delta \le N$ *and the conditions of Lemma 2 are met, and for no other sequences.*

Proof. Procedure *makeX* has the general form of a standard backtrack procedure for this problem. The arguments to a general recursive call represent the index k of the set X_k which will be determined at this level, the values $X_1, \ldots, X_{k-1}$ determined so far, and a list of possibilities for X_k. Conditions (i) and (ii), respectively, ensure that the final graph G will have minimum degree δ and maximum degree at most $t-1$. Condition (iii) ensures that requirement (c) of Lemma 2 holds. ∎

By using the set representation described earlier, tests (i) and (ii) can easily be implemented in unit time per set Z. The most expensive part of the procedure is test (iii), and two different methods were used to implement it. Suppose, for definiteness, that we have a set $S \subseteq VH$ and we wish to determine if H has an independent set of size q disjoint from S. The first method was simply to scan a precomputed list of all the independent sets of size q. Only a few machine instructions per independent set are needed, but the number of independent sets can be large. To understand the second method, recognise that the subsets of VH can be identified with the integers $0, 1, \ldots, M$, where $M = 2^{|VH|} - 1$, by interpretting the binary representation of an integer as the characteristic vector of a set. We can construct in advance a vector of bits $(b_0, b_1, \ldots, b_M)$, such that $b_i = 1$ if and only if the set represented by i is disjoint from some independent set of size q. This vector clearly allows test (iii) to be performed with unit cost. Construction of the vector itself is expensive, but we found a technique based on Gray codes which was usably efficient. After much experiment, we found that the best overall performance was gained by using the second method only for large q, typically for $q \geq t-3$, and the first method for smaller q. The usual speed-up was a factor of 5–10 over using the first method alone.

For $t \leq 7$, the procedure *process* computed the function $\theta(G)$ defined in the previous section. If $v \notin \theta(G)$, then G was immediately rejected; otherwise it was written to a file in canonically labelled form. The canonical labelling was performed by the program *nauty* [5]. Isomorph rejection was then done using the system sorting utility.

The sets $\mathcal{R}(t, n, \delta)$ produced in this way are those shown in Table 1. In each case, it follows from Lemma 1 that each such set for $4 \leq t \leq 7$ can be obtained by extending members of other sets also in the table. For the case of $\mathcal{R}(7, 20)$, we did not generate the full set of graphs because our primary use for these graphs was to extend them to $\mathcal{R}(8, 28, 7)$. For $\mathcal{R}(7, 20, 2)$ and $\mathcal{R}(7, 20, 3)$, we generated the full subset with 49 edges, and the sizes of these subsets are given in Table 1. In the case of $\mathcal{R}(7, 20, 4)$, we restricted our generation to those graphs with 49 edges having no pair of distinct vertices with four or more common neighbours. This subset is sufficient, as will be shown below. This computation was the single most difficult step. Direct application of procedure *makeX* to $\mathcal{R}(6, 15)$ produced almost 695 million graphs, amongst which were 2820645 nonisomorphic members of the restricted subset just defined.

In the process of extending $\mathcal{R}(7, n)$ to $\mathcal{R}(8, 28)$, we avoided use of θ but instead accepted any graph G generated by procedure *makeX*. The steps in this computation can be described as follows, noting that $\mathcal{R}(8, 28) = \bigcup_{\delta=5}^{7} \mathcal{R}(8, 28, \delta)$ by Lemma 1.

(a) $\mathcal{R}(8, 28, 5)$ was found to be empty by applying *makeX* to the 191 members of $\mathcal{R}(7, 22)$.

(b) $\mathcal{R}(8, 28, 6)$ was found to be empty by applying *makeX* to each of the graphs in $\bigcup_{\delta'=3}^{5} \mathcal{R}(7, 21, \delta')$. These values of δ' are sufficient by Lemma 1.

(c) $\mathcal{R}(8, 28, 7)$ can be found by extending all the members of $\bigcup_{\delta'=2}^{4} \mathcal{R}(7, 20, \delta', 49)$. This was done by simple application of *makeX* for $\delta' = 2$ and $\delta' = 3$. (In fact, the case $\delta' = 2$ can easily be proven impossible, as has been demonstrated to us by Staszek Radziszowski.) For $\delta' = 4$, this approach appeared impossibly difficult, so we modified the algorithm to use Lemma 3. Since we had already computed all extensions from $\mathcal{R}(7, 20, \delta')$ for $\delta' < 4$, we could avoid constructing any graphs known to contain two vertices with four common neighbours. This simple test was made in three ways. Firstly, any H failing this test could be ignored (as indicated above). Secondly, no X_i can contain 4 vertices adjacent to any $w \in VH$, nor both of any pair $w, x \in VH$ which had three common

t	n	δ		t	n	δ		t	n	δ	
3	1	0	1	4	7	2	6	6	15	2	4236
	2	0	1		8	2	2			3	53343
		1	1			3	1			4	7145
	3	0	1	5	9	0	3		16	2	23
		1	1			1	105			3	1084
	4	1	2			2	172			4	1458
		2	1			3	10			5	11
	5	2	1			4	0		17	3	0
4	4	0	2		10	1	28			4	7
		1	3			2	240	7	20	2	39544*
		2	1			3	44			3	1698842*
	5	0	3			4	1			4	2820645*
		1	4		11	2	61		21	3	5674
		2	2			3	43			4	598971
	6	0	1			4	1			5	513460
		1	9		12	3	10		22	4	3
		2	4			4	2			5	178
		3	1		13	4	1			6	10
	7	1	3	6	15	1	8	*restricted, see text			

Table 1. Various values of $|\mathcal{R}(t,n,\delta)|$.

neighbours in H. Thirdly, $X_i \cap X_j$ must have cardinality at most two for any $i \neq j$. With these changes, this case was completed fairly easily. No graphs in $\mathcal{R}(8,28)$ were found.

4. Conclusions and discussion.

In the previous sections we have described a computation which demonstrated that $\mathcal{R}(8,28) = \emptyset$. Since it is already known that $R(3,8) \geq 28$, we have the following.

Theorem. $R(3,8) = 28$. ∎

This computation was carried out over a period of months on a network of SUN workstations. The total amount of computation used was a little below 10^{14} machine instructions.

As a partial check of our computational method, we verified the counts of $\mathcal{R}(t,n)$ by the number of edges, as given in [6]. Our results agree with theirs wherever they overlap,

t	$R(3,t)$	t	$R(3,t)$	t	$R(3,t)$
3	6	8	28	13	58–69
4	9	9	36	14	66–78
5	14	10	40–43	15	73–89
6	18	11	46–51		
7	23	12	51–60		

Table 2. Known values and bounds on $R(3,t)$.

namely for all cases with $t \leq 6$ and for $\mathcal{R}(7,22)$. With the help of a small amount of additional computation, we also verified the accepted values of $R(3,t)$ for $t \leq 7$.

In Table 2, we summarise the known values and bounds on $R(3,t)$ for $t \leq 15$. See [3] for $t \leq 7$ and [4] for $t = 9$. The upper bounds for $t \geq 10$ come from [7]. The lower bound for $t = 10$ is from [1], that for $t = 12$ from [2], that for $t = 14$ from Exoo (personal communication), and that for 15 from [8]. The lower and upper bounds for $t = 15$ are due to Wang and Wang [9] and to Radziszowski (personal communication), respectively. Other values in the table are repeated from [4].

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