Fast generation of planar graphs
(expanded version)

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Abstract

The program plantri is the fastest isomorph-free generator of many classes of planar graphs, including triangulations, quadrangulations, and convex polytopes. Many applications in the natural sciences as well as in mathematics have appeared. This paper describes plantri's principles of operation, the basis for its efficiency, and the recursive algorithms behind many of its capabilities. In addition, we give many counts of isomorphism classes of planar graphs compiled using plantri. These include triangulations, quadrangulations, convex polytopes, several classes of cubic and quartic graphs, and triangulations of disks.

This paper is an expanded version of the paper "Fast generation of planar graphs" to appeared in *MATCH*. The difference is that this version has substantially more tables.

1 Introduction

The program plantri can rapidly generate many classes of graphs embedded in the plane. Since it was first made available, plantri has been used as a tool for research of many types. We give only a partial list, beginning with masters theses [37, 40] and PhD theses [32, 42]. Applications have appeared in chemistry [8, 21, 28, 39, 44], in crystallography [19], in physics [25], and in various areas of mathematics [1, 2, 7, 20, 22, 24, 41].

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The purpose of this paper is to describe the facilities of `plantri` and to explain in general terms the method of operation. The program is written in `C` and available free [13].

Throughout the paper, we will only consider finite connected graphs.

A *planar graph* is a graph that can be drawn on the sphere (equivalently, the plane) without edge crossings. Since several distinct drawings (*embeddings*) may be possible, it is useful to define a *plane graph* to be a planar graph together with a crossing-free drawing. The combinatorial structure of a plane graph is described by giving the cyclic order of the edges incident with each vertex; for convenience we use clockwise order. The *mirror image* of a plane graph is obtained by reversing the cyclic order at each vertex; this corresponds to reflecting the plane about a line.

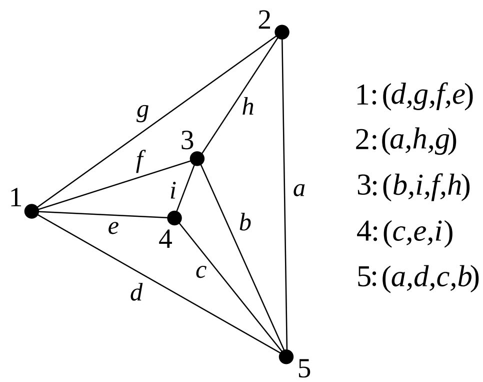

Figure 1: A plane graph with the associated cyclic orders

Two edges of a graph are *parallel* if they have the same endpoints. A *loop* is an edge whose endpoints are the same vertex. If there are neither parallel edges nor loops, a graph is called *simple*. *Triangulations* and *quadrangulations* are plane graphs in which each face is bordered by three edges, or four edges, respectively (see Figure 2).

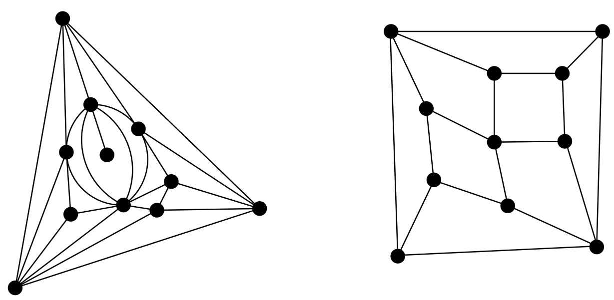

Figure 2: A non-simple triangulation and a simple quadrangulation

The *dual graph* of a plane graph G is a plane graph obtained from G by exchanging the functions of vertices and faces. In the left part of Figure 3, the graph with black vertices and solid edges is dual to the graph with white vertices and dashed edges, and

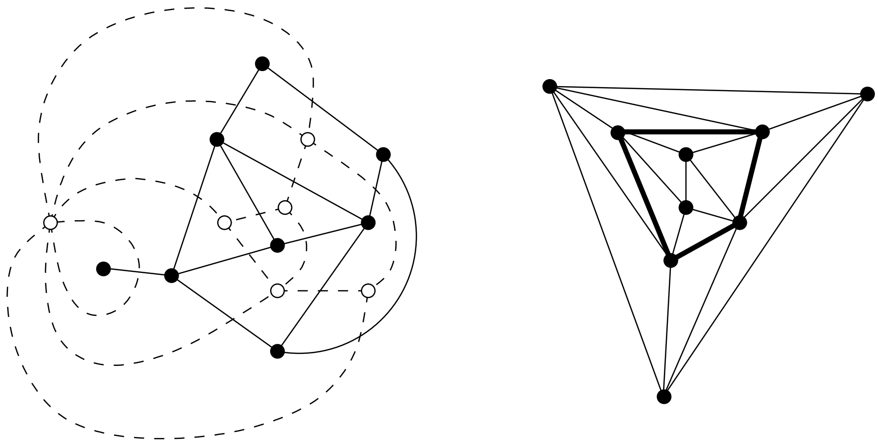

Figure 3: The dual graph, and a separating 4-cycle

vice-versa. In particular, the dual graph of a triangulation is a cubic graph and the dual graph of a quadrangulation is a quartic graph.

A *separating cycle* in a plane graph is a cycle that contains at least one vertex in its interior and at least one vertex in its exterior. The length of the smallest separating cycle in a triangulation is the same as the (vertex) connectivity, and is known to equal the *cyclic connectivity* of the cubic dual graph (see Figure 3).

1.1 Notions of isomorphism

Several types of isomorphism are important for plane graphs. Recall that we are assuming connectivity.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An *abstract isomorphism* from G_1 to G_2 is a pair of bijections $\phi : V_1 \to V_2$ and $\psi : E_1 \to E_2$ which preserve the vertex-edge incidence relationship. This is the usual notion of graph isomorphism.

If $G_1 = (V_1, E_1, C_1)$ and $G_2 = (V_2, E_2, C_2)$ are plane graphs, where C_i gives the embedding of G_i (as a specification of the clockwise order of the edges at each vertex), we can ask for isomorphisms that respect the embeddings in some fashion. An *orientation-preserving isomorphism (OP-isomorphism)* from G_1 to G_2 is an abstract isomorphism (ϕ, ψ) from (V_1, E_1) to (V_2, E_2) which preserves the embedding: if $(e_1, e_2, \ldots, e_k)$ is the cyclic order of the edges of G_1 incident with $v \in V_1$, then $(\psi(e_1), \psi(e_2), \ldots, \psi(e_k))$ is the cyclic order of the edges of G_2 incident with $\phi(v) \in V_2$. On the other hand, an *orientation-reversing isomorphism (OR-isomorphism)* reverses the cyclic order at each vertex: if $(e_1, e_2, \ldots, e_k)$ is the cyclic order of the edges of G_1 incident with $v \in V_1$, then $(\psi(e_k), \psi(e_{k-1}), \ldots, \psi(e_1))$ is the cyclic order of the edges of G_2 incident with $\phi(v) \in V_2$.

By an *isomorphism* we will always mean either an OP-isomorphism or an OR-isomorphism. Isomorphisms and OP-isomorphisms (but not OR-isomorphisms) are equivalence relations, so we have *isomorphism classes* and *OP-isomorphism classes*, respectively. Sim-

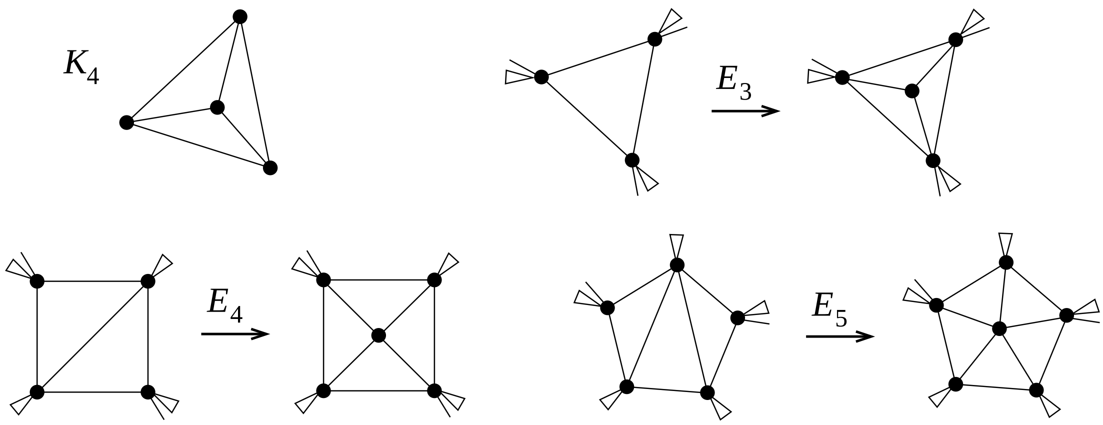

Figure 4: Generating simple plane triangulations

ilarly, the *automorphism group* $\mathrm{Aut}(G)$ of a plane graph is the group of all isomorphisms from G to itself, and the *OP-automorphism group* $\mathrm{Aut}_{\mathrm{OP}}(G)$ is the group of all OP-isomorphisms from G to itself. In some cases, $\mathrm{Aut}_{\mathrm{OP}}(G) = \mathrm{Aut}(G)$; otherwise, $\mathrm{Aut}_{\mathrm{OP}}(G)$ is a subgroup of index 2 of $\mathrm{Aut}(G)$ whose other coset consists of the OR-automorphisms.

1.2 Recursive generation

Let $\mathcal{C}$ be a class of plane graphs, closed under isomorphism. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be closed under isomorphism, and let $F_1, \ldots, F_k : \mathcal{C} \to 2^{\mathcal{C}}$ be a list of functions. We say that $(\mathcal{C}_0; F_1, \ldots, F_k)$ *generates* $\mathcal{C}$ if for each $G \in \mathcal{C}$ there is a sequence $G_0, G_1, \ldots, G_m = G$ such that $G_0 \in \mathcal{C}_0$ and for $1 \le i \le m$ we have $G_i \in F_j(G_{i-1})$ for some j. (This implies that $G_1, \ldots, G_m \in \mathcal{C}$.)

In our examples, each F_j consists of replacing some small subgraph by another, usually larger, subgraph under specified conditions. We will call $F_1, \ldots, F_k$ *expansions*, and the inverse operations *reductions*. If an expansion has a name, say F, we can call it an F-expansion, and its inverse an F-reduction.

For a definite example, we take $\mathcal{C}$ to be the class of simple plane triangulations of order at least 4. This class $\mathcal{C}$ is generated by $(\{K_4\}; E_3, E_4, E_5)$, where the tetrahedron K_4 and the expansions E_3, E_4, E_5 are shown in Figure 4 [26]. The proof consists of taking any $G \in \mathcal{C} - \{K_4\}$ and noting that some reduction can be applied. If there is a vertex of degree 3, its neighbours have degree at least 4, so an E_3-reduction can be applied. If there is no vertex of degree 3, but there is a vertex of degree 4, we can apply an E_4-reduction by removing that vertex and inserting an edge. One of the two ways to insert the edge might create a parallel edge (which this class of graphs does not allow), but the Jordan Curve Theorem assures that in such a case the other way is valid. In case G has minimum degree 5, which is the remaining possibility by Euler's formula, an E_5-reduction can be applied, by similar reasoning.

We note here the conventions we use in figures such as Figure 4. Vertices shown as distinct must be distinct. A half-edge attached to a vertex indicates that an edge must appear in that position, while a small open triangle represents zero or more edges in that position. Otherwise, vertices which are displayed cannot have additional incident edges. Also, there is an implicit requirement that expansions and reductions remain within the graph class under consideration; this often implies degree or connectivity constraints.

The method of isomorph rejection used by `plantri` is the *canonical construction path* method of McKay [36] (also called the method of canonical augmentation [34]). Another general description with emphasis on chemical applications appears in [9]. This method involves application of two rules. The first rule is that only one expansion of G from each equivalence class of expansions under $\mathrm{Aut}(G)$ is performed. The second rule requires the concept of canonical labelling. Select a unique labelled plane graph in each isomorphism class of plane graphs and call it the *canonical representative* of the isomorphism class. (We discuss below how such a canonical representative can be computed.) Given a canonical representative G^* of an isomorphism class in $\mathcal{C}-\mathcal{C}_0$, choose some reduction that applies to it and call this the *best* reduction of G^*. Now, suppose we apply an expansion F to G to obtain G'. Let f be an isomorphism from G^* to G', where G^* is the canonical member of the isomorphism class of G'. The second rule is that we must "accept" G' if f maps the best reduction of G^* to a reduction of G' that is equivalent under $\mathrm{Aut}(G')$ to the reduction inverse to F; otherwise we "reject" G'. Only accepted graphs are subject to further expansions. According to the theory of this method [36], application of both rules together means that exactly one member of each isomorphism class in $\mathcal{C}$ is accepted.

Let us see how this works for our triangulations example, ignoring for the moment expansions E_4 and E_5. Applying expansion E_3 to a triangulation G requires us to choose a face and insert a new vertex into it. The first rule says that we should do this with only one face in each equivalence class of faces under $\mathrm{Aut}(G)$. Next, suppose we perform such an expansion to make a larger triangulation G' by adding the new vertex v. We then compute an isomorphism f to G' from the canonical representative G^* of the isomorphism class of G'. A fair choice of the "best" reduction of G^* would be that which removes the least-labelled vertex w of degree 3. If so, the second rule tells us to accept G' if and only if $f(w)$ is equivalent to v in $\mathrm{Aut}(G')$.

1.3 Implementation issues

The literature contains a few algorithms [27, 33] that compute automorphisms or isomorphisms of planar graphs in linear time. Their applicability is too limited for our purposes, but the main reason `plantri` doesn't use them is they are so complicated that they could not possibly be competitive for very small graphs compared to the simple but highly tuned heuristics we use.

Let G be a connected plane graph with n vertices, and let v, e be a vertex of G and an edge incident with it. We will suppose that each vertex w has a *colour* $c(w) > n$. We also assign *labels* $\ell(w) \in \{1, 2, \ldots, n\}$, consecutively to vertices as they are discovered during a breadth-first scan starting at v (thus $\ell(v) = 1$). The "first" edge of v is defined to be e, while for other vertices it is the edge along which that vertex is first visited. To make the breadth-first search deterministic, the edges incident with each vertex are examined in clockwise order, starting with the first edge at that vertex.

The results of this scan is encoded in a string

$$\sigma_{v,e} = \bigl(c(v), r_1(1), \ldots, r_{d_1}(1), 0, r_1(2), \ldots, r_{d_2}(2), 0, \ldots, r_1(n), \ldots, r_{d_n}(n), 0\bigr),$$

where d_i is the degree of the vertex with label i, and $r_j(i)$ refers to the j-th edge in clockwise order about that vertex. If the other end of this edge is a vertex w, then $r_j(i) = c(w)$ if this is how w is first discovered, and $r_j(i) = \ell(w)$ otherwise.

The string $\sigma_{v,e}$ depends only on v, e and the graph structure. Moreover, the graph structure can be recovered from the string if the graph is simple. This remains true in the presence of parallel edges, since the uncertainty over which edge-ends are views of the same edge can be resolved from the topology. It is also true for triangulations with loops, but for the general case of loops it is not true.

There is a colour-preserving OP-automorphism mapping (v, e) onto (v', e') if and only if $\sigma_{v,e} = \sigma_{v',e'}$, so we can find $\mathrm{Aut}_{\mathrm{OP}}(G)$ by computing $\sigma_{v,e}$ for every vertex-edge pair. We can also find a canonical labelling by choosing the string $\sigma_{v,e}$ which is lexicographically least. To find OR-automorphisms, we perform similar scans using anticlockwise ordering instead of clockwise ordering.

This method is evidently of quadratic time, but in practice we can make it much faster on average. Instead of starting at every pair (v, e), we need start only at some combinatorially-defined subset of such pairs. For example, we could choose v to be a vertex of maximum degree, and, subject to that, e to be an edge incident with v that is incident with a face of largest size. Moreover, we can abort a scan as soon as it becomes apparent that it can give neither an automorphism nor a canonical labelling. This speedup is improved by using a nontrivial invariant, such as the degree, for the colour of each vertex. Our precise choices here were tuned experimentally for each of the graph classes that `plantri` generates. Since the great majority of graphs in all our graph classes have trivial automorphism groups, and in any case $|\mathrm{Aut}(G)| \le 4|E(G)|$, we do not attempt to use a sophisticated data structure to hold the group but simply list the action of every automorphism on each of the edges.

A similar heuristic can speed up the implementation of the second rule for isomorph rejection, defined in the previous subsection. In the terminology used there, the "best reduction" of G^* can be chosen from amongst those maximizing some combinatorial invariant. For example, a best reduction in our triangulation example might be chosen to be an E_3-reduction if one is available, and, if so, such that the sum of the degrees of the three surrounding vertices is as large as possible. For many of our graph classes, we found cheap combinatorial invariants which most of the time identified a unique reduction, in which case there is no need to find the isomorphism from G^* to the current graph.

The graph itself is stored as a set of directed edges, one for each orientation of each edge. Each directed edge knows its starting and ending vertex, its successor and predecessor in clockwise order about its starting vertex, and the oppositely directed edge. Expansions and reductions are performed by modifying the graph in place, not by copying the whole graph. This is essential for the performance.

The generators produce one graph from each isomorphism class. In case the user has requested representatives of OP-isomorphism classes instead, the output graphs are tested for the presence of an OR-automorphism. If there is none, the mirror image of the graph is output as well. There is also the option of computing the dual graph; this is done by writing the dual without explicitly constructing it.

To enable very large families of graphs to be generated, perhaps using many computers in parallel, `plantri` has a facility for selecting only a numbered portion of the family. For example, one can ask that the entire graph class be divided into 1000 parts but that only part 376 be generated. This is achieved with almost 100% efficiency. A line is drawn across the generation tree at an experimentally determined level, then all of the tree above that level is generated while assigning consecutive numbers to the subtrees that hang below that level. To obtain portion 376/1000, just those subtrees whose number is congruent to 376 modulo 1000 are developed.

1.4 Testing

A number of different approaches were taken to assure the correctness of the implementation. In some cases, earlier enumerations were available for comparison, such that of Dillencourt [23] and Aldred et al. [2], but they don't extend to very high order. Other tests consisted of checking that the outputs lie in the required graph classes, and that they are not isomorphic (using `nauty` [35]). We also generated very large collections of graphs by heuristic methods and checked that they were all included in `plantri`'s output. In some cases, we could compare the outputs of two distinct generation algorithms for the same classes.

Perhaps the best check was to compare the output to existing theoretical enumerations. Very few such enumerations exist for isomorphism classes of plane graphs, but there are several that enumerate classes of plane graphs that are rooted at a *flag*. A flag is a triple (v, e, f) where v is a vertex, e is an edge incident with v, and f is a face incident with e. Relevant examples include [6, 16, 30, 31, 43]. To compare these against `plantri`, we computed $\mathrm{Aut}(G)$ for each output graph G; the number of flag-rooted graphs isomorphic to G equals the number of orbits of flags.

1.5 Efficiency

It is easy to show that the running time per output graph is polynomial in the size of the graphs, but the precise complexity has not been determined as it depends in complex fashion on the average properties of unlabelled plane graphs. In practice, the running time per graph is approximately constant within each of the classes of plane graph generated by `plantri`. This cannot remain true in the limit of large size (even if the linear time required for outputting the objects is discounted), but we observe it up to the size beyond which exhaustive generation is impractical (say, more than 10^{15} objects).

In the following, the efficiency is indicated according to the approximate number of objects generated per second by a Pentium IV processor running at 1GHz.

In all cases the memory requirements are trivial. None of the examples we give in the Appendix require more than a few megabytes.

2 Classes of triangulations

Plane triangulations of order at least 4 are simple if and only if they are 3-connected. The generation of this class was described in Section 1.2 (310,000/second). The dual class consists of the 3-connected cubic plane graphs.

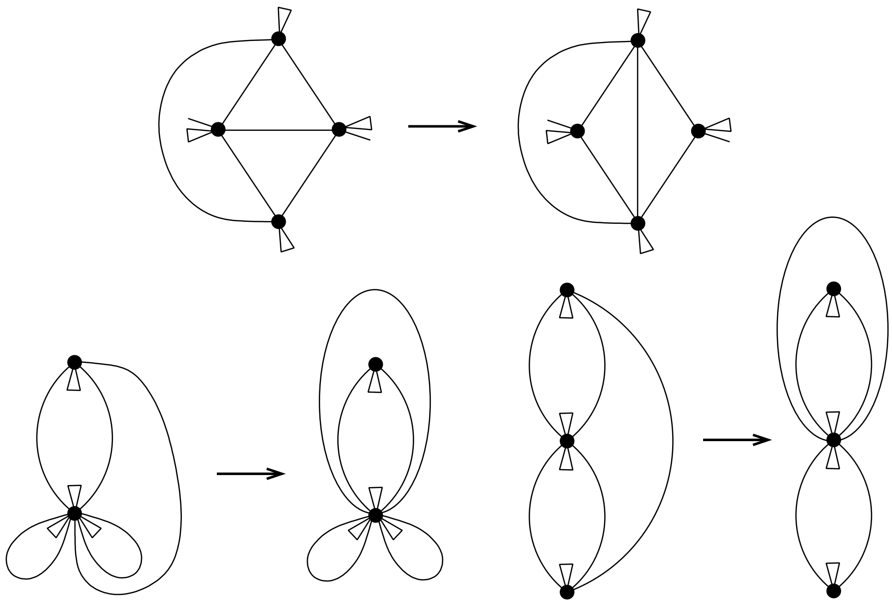

Figure 5: Creating double edges and loops in a triangulation

Allowing parallel edges as well, but excluding vertices of degree 2, we obtain the 2-connected triangulations with minimum degree at least 3, whose dual class consists of the 2-connected simple cubic plane graphs. This class can be generated by starting with simple triangulations and then performing the first operation shown in Figure 5. The operation must increase the number of double edges, so it is not possible to return to a previous configuration (110,000/second).

The next option is to allow loops. Loops in a triangulation of minimum degree at least 3 can have three different types, depending on whether the inside and outside faces are bounded by two additional loops or by a pair of parallel ordinary edges. However, by considering an innermost loop in a plane drawing, we see that a triangulation with loops must have a loop with one of the two types created by the operations shown in the lower half of Figure 5. These suffice to generate all possibilities starting with the 2-connected triangulations of minimum degree at least 3 (140,000/second). As an option, the generation can be limited to exclude triangulations having two faces with two edges (which must be parallel edges) in common. This gives us two useful dual classes: connected cubic plane simple graphs, and connected cubic plane multigraphs without faces of size less than 3.

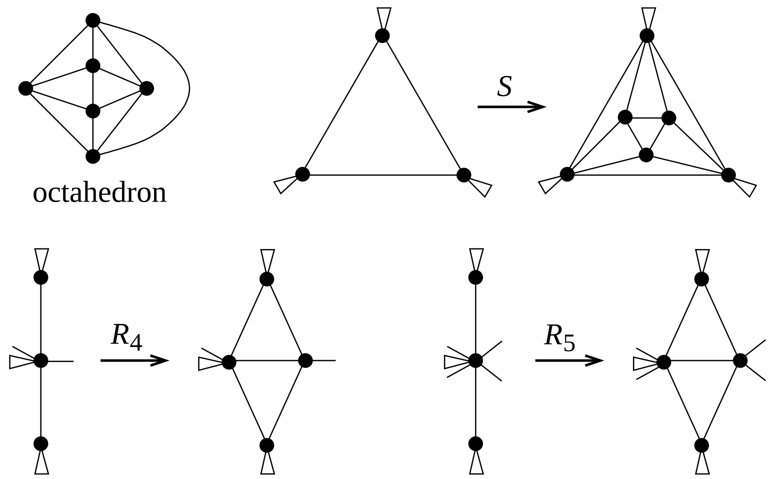

Figure 6: Generating triangulations with minimum degree at least 4

Define a *Cκ-δ-triangulation* to be a plane triangulation with minimum degree at least δ and connectivity at least κ.

A method for generating C3-4-triangulations was discovered by Batagelj [4]. It starts with the octahedron and applies three expansions R_4, R_5 and S that are the same as in Figure 6 except that the central vertex of R_5 is only required to have one left-pointing edge. Our stronger requirement, as shown in the figure, illustrates a technique we used repeatedly to improve efficiency. When we define the "best" reduction of an expanded graph, as in Section 1.2, we chose an R_4-reduction if there is any. This means that the result of an R_5 or S-expansion will not be accepted if it has an R_4-reduction. Experimentally, most graphs in this class have R_4-reductions, so this is a key observation. If graph G has 3 or more R_4-reductions, which is the most common case, then the result of applying an R_5-expansion is sure to have an R_4-reduction, so there is no point in making it. Finally, if the central vertex of an R_5-expansion has degree 4, then the expanded graph has an R_4-reduction; hence our requirement that that vertex have degree at least 6 (250,000/second).

We do not use a separate generation method for C4-4-triangulations, but maintain a list of all the separating 3-cycles which is updated as each expansion is done. At the last step before output, only expansions which eliminate all remaining separating 3-cycles are applied (160,000/second).

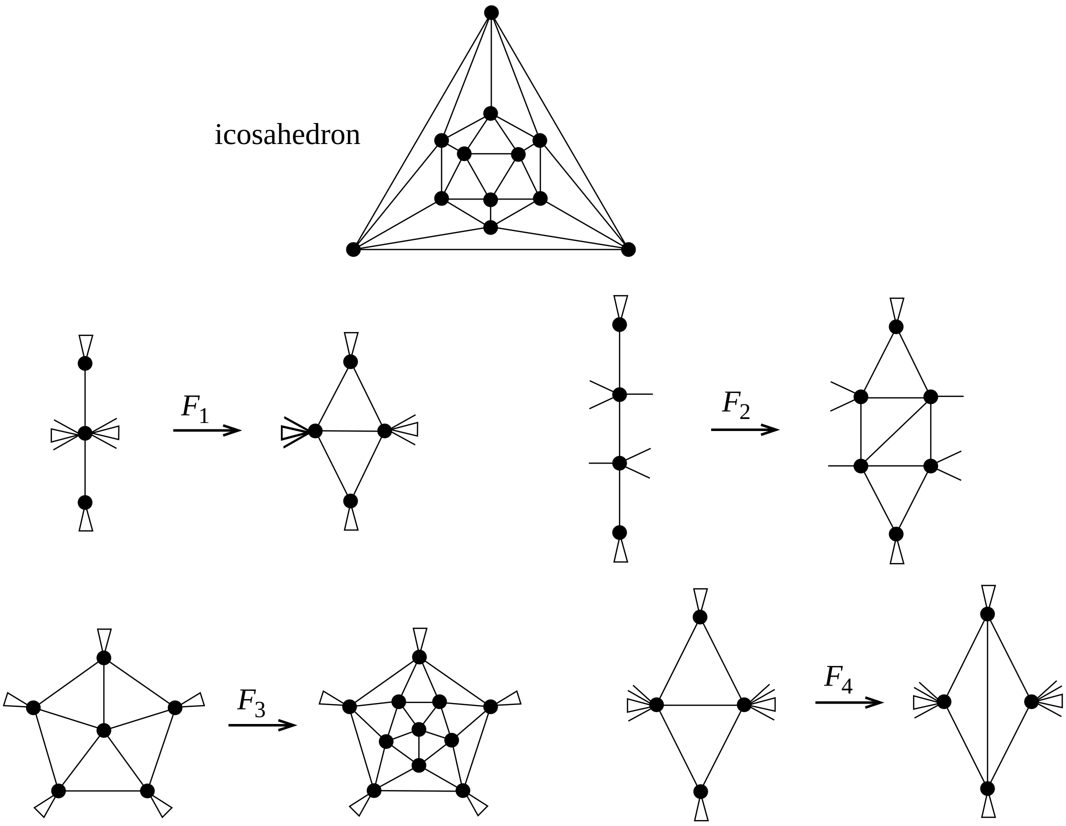

Figure 7: Generating triangulations with minimum degree 5

In the case of minimum degree 5, we first apply the method of Barnette [3] and Butler [17] to generate the C5-5-triangulations. This starts with the icosahedron and applies the expansions F_1, F_2 and F_3; see Figure 7 (50,000/second). Then we repeatedly apply operation F_4 in such a way that one additional separating 4-cycle, but no separating 3-cycle, is created each time. This suffices to generate the remaining C4-5-triangulations (43,000/second). Then F_4 is applied in similar manner to create one separating 3-cycle at a time to generate the remaining C3-5-triangulations (40,000/second). The details and proofs can be found in [14].

The simple eulerian plane triangulations (those with all vertices of even degree) are dual to the 3-connected bipartite cubic plane graphs. A recursive construction was given by Batagelj [4], but `plantri` uses a variation which is slightly more efficient. The starting triangulations are the *even double wheels* which consist of a cycle of even length and

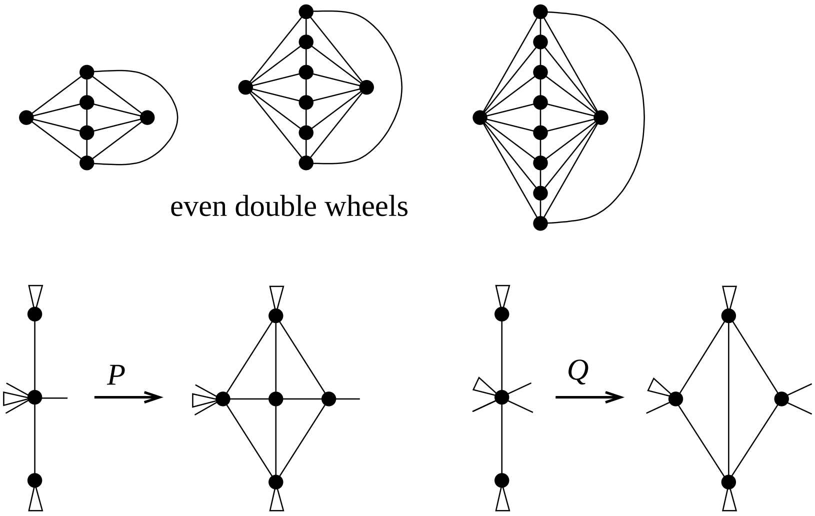

Figure 8: Generating eulerian triangulations

vertices in the interior and exterior adjacent to all of the cycle. The smallest three even double wheels are shown in Figure 8, as are the two expansions P and Q that complete the generation of this class. The version of Batagelj used only the smallest even double wheel (also known as the octahedron) and allowed the central vertex of a P-expansion to have degree 4. However, if a P-reduction with left vertex of degree 4 can be applied, there will also be a P-reduction with left vertex of degree greater than 4—unless the graph is an even double wheel. Our variation improved the generation efficiency because graphs on average have fewer reductions (110,000/second). Keeping track of separating 3-cycles allows the 4-connected subset to be generated (29,000/second).

3 Classes of quadrangulations

Algorithms for generating several classes of plane quadrangulations are given in [12]. The necessary expansions are shown in Figure 9. A *pseudo-double wheel* consists of a cycle of even length, a vertex in the interior adjacent to the odd-numbered vertices of the cycle, and a vertex in the exterior adjacent to the even-numbered vertices of the cycle. Let $\mathcal{D}$ be the class of all pseudo-double wheels. Also consider the square C_4.

The simple quadrangulations are generated by $(\{C_4\}; P_0, P_1)$ (270,000/second). If the minimum degree must be at least 3, the simple quadrangulations are generated by $(\mathcal{D}; P_1, P_3)$ (51,000/second). The same generator, restricted to remain within the class of 3-connected quadrangulations, generates that class (49,000/second). If both 3-connectivity and the absence of separating 4-cycles are required, a generator is $(\mathcal{D}; P_1)$ (46,000/second).

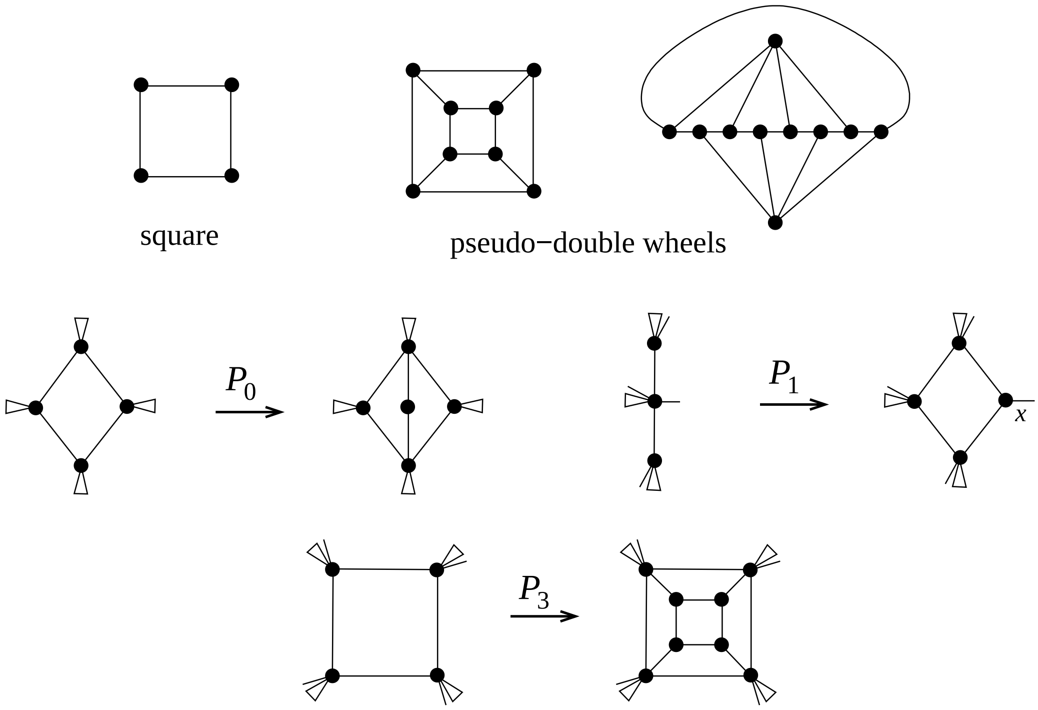

Figure 9: Generating quadrangulations

4 General simple plane graphs

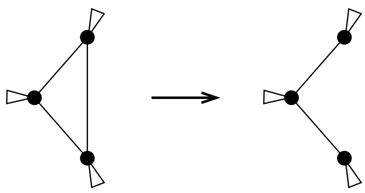

Figure 10: Generating plane graphs from triangulations

Let $F = v_0, v_1, \ldots, v_{m-1}$ be a face of length $m \geq 4$ in a connected simple plane graph. It is easily seen that for some i, the three vertices v_i, v_{i+1}, v_{i+2} (subscripts modulo m) are distinct and v_i is not adjacent to v_{i+2}. Therefore, an edge can be added from v_i to v_{i+2}, dividing f in a face of size 3 and a face of size $m - 1$, with the result being also a simple plane graph. Addition of an edge in this manner cannot decrease the minimum degree or the connectivity, and cannot increase the maximum face size. Repeated edge addition results in a simple triangulation. Also recall that simple triangulations are 3-connected.

Considering this process in reverse, given a lower bound δ on the minimum degree, a lower bound κ on the connectivity, a lower bound ε on the number of edges, and an upper bound ϕ on the maximum face size, we can generate all the plane graphs satisfying those four bounds by starting with triangulations and removing one edge at a time as in Figure 10. The starting triangulations must have minimum degree at least $\max\{\delta, 3\}$ and connectivity at least $\max\{\kappa, 3\}$. This is implemented in `plantri` except for the cases $\kappa = 4, 5$. Typical efficiencies are 200,000/second for $\kappa = \delta = 3$ and 220,000/second for $\kappa = \delta = 1$. The 3-connected simple plane graphs are also known as convex polytopal graphs, since they correspond to the skeletons of the convex polytopes in 3 dimensions.

5 Triangulations of a disk

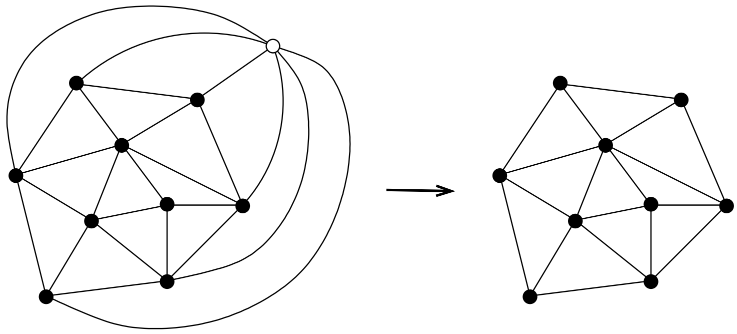

Figure 11: Making a disk triangulation from a triangulation

By a *triangulation of a disk* we mean a plane graph with one face distinguished as the "outer face" and all other faces bounded by 3 edges. The outer face may have any length but must be a simple polygon.

Here we only consider simple triangulations of a disk. In this case, it can be seen that vertices have degree at least 2 on the outer face and at least 3 otherwise. Also, the connectivity is at least 2 and the only possible 2-cuts consist of two vertices on the outer face that are connected by a chord.

The generation method used by `plantri` for generating a triangulation of a disk of length Δ with n vertices altogether is shown in Figure 11. We first generate a triangulation with $n + 1$ vertices, then delete a vertex of degree Δ to create the outer face. It can be specified whether the minimum degree is at least 2 or at least 3 and whether chords (i.e., 2-cuts) are allowed.

If all outer face sizes are required, this process is very efficient as each triangulation on average produces many outputs. For example, 3-connected disk triangulations are found at 540,000/second and 3-connected disk triangulations at one million per second. In case only a large outer face is required the efficiency is lower, but an improvement is achieved by filtering out many of the simple triangulations that cannot produce a vertex of sufficient degree, using the fact that the expansions in Figure 4 can only increase the maximum degree by 1.

6 Variations and applications

The program is usually used from the command line, with switches that specify the type of graph, its parameters, and output options. Several output formats are available, including some suitable for feeding to another program and some that are human-readable. Complete instructions appear in the manual that is packaged with the program [13].

`Plantri` can also be used in conjunction with the visualisation program `CaGe` [10]. This allows most of the graph classes generated by `plantri` to be visually examined in 2D or 3D form. For some of the functionalities of `plantri`, `CaGe` offers an interface that makes it unnecessary to type in the options; for others, the command line parameters must be given.

The structure of `plantri` allows external pieces of code to be attached at various points to generate subclasses more efficiently than filtering the output. Significant examples include code to limit triangulations to a specified set of vertex degrees [15] and similarly for quadrangulations [29].

More substantial variations on `plantri` include a program to generate plane triangulations with maximum degree 6 [2] (using in addition the efficient fullerene generator of Brinkmann and Dress [11]). Sulanke has made a program based on `plantri` to generate triangulations of higher-genus manifolds.

At the moment `plantri` does not provide for random generation. However, we note that a variety of random generation can be obtained with a simple modification of the algorithm [36].

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Appendix

In the following pages we give some of the counts of plane graphs that have been compiled using `plantri`.

The variables n_v, n_e and n_f are the numbers of vertices, edges and faces, respectively. For any symbol μ that enumerates isomorphism classes, μ' enumerates OP-isomorphism classes. The minimum degree is δ, the minimum face size is ϕ, the connectivity is κ, and the cyclic connectivity is λ. As with the whole paper, all graphs are connected.

- Tables 1–2. Plane triangulations with minimum degree at least 3; plane cubic multigraphs with minimum face size at least 3.

  $t_3(n_v)$: triangulations, $\kappa = 3$; simple cubic graphs, $\kappa = 3$
  $t_2(n_v)$: triangulations, $\kappa \geq 2$, $\delta \geq 3$; simple cubic graphs, $\kappa \geq 2$
  $t_{1+}(n_v)$: triangulations, $\delta \geq 3$ no two faces sharing two edges ; simple cubic graphs
  $t_1(n_v)$: triangulations $\delta \geq 3$; cubic multigraphs, $\phi \geq 3$

- Tables 3–5. Plane triangulations with lower bounds on degree and connectivity; plane simple cubic graphs with lower bounds on face size and cyclic connectivity.

  $t_{k,d}(n_v)$: triangulations, $\kappa \geq k$, $\delta \geq d$; simple cubic graphs, $\lambda \geq k$, $\phi \geq d$

- Table 6. Simple plane eulerian triangulations with lower bound on connectivity; plane simple cubic bipartite graphs with lower bound on cyclic connectivity.

  $e_k(n_v)$: simple eulerian triangulations, $\kappa \geq k$; cubic bipartite graphs, $\lambda \geq k$

- Tables 7–8. Plane quadrangulations with degree and connectivity conditions; simple quartic graphs with degree and connectivity conditions.

  $q_1(n_v)$: simple quadrangulations ; 4-edge-connected quartic multigraphs
  $q_2(n_v)$: simple quadrangulations, $\delta \geq 3$; 4-edge-connected simple quartic graphs
  $q_3(n_v)$: quadrangulations, $\kappa \geq 3$; simple quartic graphs, $\kappa \geq 3$
  $q_4(n_v)$: quadrangulations, $\kappa \geq 3$, no separating 4-cycles ; cubic graphs, $\kappa \geq 3$, cyclic edge connectivity at least 6

- Tables 9–18. 3-connected plane graphs with lower bound on degree; 3-connected plane graphs with lower bound on face size.

  $p_d(n_v, n_e)$: simple graph, $\delta \geq d$

  Note that for $d = 3$ this class is self-dual and $p_3(n_v, n_e) = p_3(n_f, n_e)$; these are the skeletons of convex polytopes.

- Tables 19–26. Simple plane graphs with lower bound on connectivity. Simplicity implies that the dual multigraph has at most one edge in common between two faces. Connectivity 2 implies in addition that the dual has only faces which are simple cycles.

  $c_k(n_v, n_e)$: simple graph, $\kappa \geq k$

- Table 27. Simple triangulations of a disk with lower bounds on degree and connectivity. The dual graphs are plane multigraphs with a distinguished vertex, with all vertices except possibly the distinguished vertex having degree 3.

  $d_1(n_v)$: disk triangulation, $\kappa \geq 3$
  $d_2(n_v)$: disk triangulation, $\delta \geq 3$
  $d_3(n_v)$: disk triangulation, $\delta \geq 2$
  $d_4(n_v)$: disk triangulation, $\Delta = n_v$
  The last class are also known as the 2-connected outerplanar graphs.

n_v	n_e	n_f	$t_3(n_v)$	$t_2(n_v)$	$t_{1+}(n_v)$	$t_1(n_v)$
4	6	4	1	1	1	1
5	9	6	1	1	1	1
6	12	8	2	3	3	3
7	15	10	5	8	9	9
8	18	12	14	32	37	38
9	21	14	50	131	172	178
10	24	16	233	723	993	1041
11	27	18	1249	4360	6308	6652
12	30	20	7595	29632	44145	46738
13	33	22	49566	213168	327051	347050
14	36	24	339722	1606633	2530761	2691419
15	39	26	2406841	12473723	20179785	21509955
16	42	28	17490241	99141919	164672106	175969274
17	45	30	129664753	802392930	1368137926	1465921468
18	48	32	977526957	6593377305	11536196188	12395111621
19	51	34	7475907149	54883010885	98494508358	106126249031
20	54	36	57896349553	462038444588	850073936750	918520748281
21	57	38	453382272049	3928893849911	7406965136219	8025676381104
22	60	40	3585853662949			
23	63	42	28615703421545			

Table 1: Isomorphism classes of plane triangulations

n_v	n_e	n_f	$t'_3(n_v)$	$t'_2(n_v)$	$t'_{1+}(n_v)$	$t'_1(n_v)$
4	6	4	1	1	1	1
5	9	6	1	1	1	1
6	12	8	2	3	3	3
7	15	10	6	9	10	10
8	18	12	17	37	42	43
9	21	14	73	183	230	236
10	24	16	389	1156	1523	1577
11	27	18	2274	7713	10737	11188
12	30	20	14502	55436	80319	84194
13	33	22	97033	412193	620134	653271
14	36	24	672781	3158392	4913112	5198809
15	39	26	4792530	24736138	39705720	42184083
16	42	28	34911786	197448348	326420796	348088277
17	45	30	259106122	1601481238	2723097802	2913967487
18	48	32	1954315346	13173471151	23012381739	24706425434
19	51	34	14949368524	109712447949	196713776094	211856940558
20	54	36	115784496932	923858502128	1698875856077	1835160731391
21	57	38	906736988527	7856893675780	14808015829668	16042357404748
22	60	40	7171613842488			

Table 2: OP-Isomorphism classes of plane triangulations

n_v	n_e	n_f	$t_{3-4}(n_v)$	$t_{4-4}(n_v)$	$t'_{3-4}(n_v)$	$t'_{4-4}(n_v)$
6	12	8	1	1	1	1
7	15	10	1	1	1	1
8	18	12	2	2	2	2
9	21	14	5	4	5	4
10	24	16	12	10	14	12
11	27	18	34	25	45	32
12	30	20	130	87	194	128
13	33	22	525	313	891	519
14	36	24	2472	1357	4499	2430
15	39	26	12400	6244	23603	11765
16	42	28	65619	30926	127887	59915
17	45	30	357504	158428	705770	311744
18	48	32	1992985	836749	3959653	1659633
19	51	34	11284042	4504607	22494163	8971845
20	54	36	64719885	24649284	129227103	49195863
21	57	38	375126827	136610879	749646288	272940855
22	60	40	2194439398	765598927	4387116659	1530417953
23	63	42	12941995397	4332047595	25878895923	8661936137
24	66	44	76890024027	24724362117	153765144588	49442678322
25	69	46	459873914230	142205424580	919704309272	284393946501
26	72	48	2767364341936	823687567019	5534600480206	1647327455726
27	75	50	16747182732792	4801749063379		

Table 3: Isomorphism classes of plane triangulations with minimum degree at least 4

n_v	n_e	n_f	$t_{3-5}(n_v)$	$t_{4-5}(n_v)$	$t_{5-5}(n_v)$
12	30	20	1	1	1
13	33	22	0	0	0
14	36	24	1	1	1
15	39	26	1	1	1
16	42	28	3	3	3
17	45	30	4	4	4
18	48	32	12	12	12
19	51	34	23	23	23
20	54	36	73	73	71
21	57	38	192	191	187
22	60	40	651	649	627
23	63	42	2070	2054	1970
24	66	44	7290	7209	6833
25	69	46	25381	24963	23384
26	72	48	91441	89376	82625
27	75	50	329824	320133	292164
28	78	52	1204737	1160752	1045329
29	81	54	4412031	4218225	3750277
30	84	56	16248772	15414908	13532724
31	87	58	59995535	56474453	48977625
32	90	60	222231424	207586410	177919099
33	93	62	825028656	764855802	648145255
34	96	64	3069993552	2825168619	2368046117
35	99	66	11446245342	10458049611	8674199554
36	102	68	42758608761	38795658003	31854078139
37	105	70	160012226334	144203518881	117252592450
38	108	72	599822851579	537031911877	432576302286
39	111	74	2252137171764	2003618333624	1599320144703
40	114	76	8469193859271	7488436558647	5925181102878

Table 4: Isomorphism classes of plane triangulations with minimum degree 5

n_v	n_e	n_f	$t'_{3-5}(n_v)$	$t'_{4-5}(n_v)$	$t'_{5-5}(n_v)$
12	30	20	1	1	1
13	33	22	0	0	0
14	36	24	1	1	1
15	39	26	1	1	1
16	42	28	4	4	4
17	45	30	4	4	4
18	48	32	17	17	17
19	51	34	33	33	33
20	54	36	117	117	115
21	57	38	331	330	325
22	60	40	1180	1177	1144
23	63	42	3899	3874	3736
24	66	44	14052	13910	13225
25	69	46	49667	48878	45904
26	72	48	180502	176538	163456
27	75	50	654674	635653	580704
28	78	52	2398527	2311572	2083116
29	81	54	8800984	8415829	7485349
30	84	56	32447008	30785420	27033550
31	87	58	119883207	112855620	97890740
32	90	60	444226539	414972649	355702718
33	93	62	1649550311	1529287903	1296014495
34	96	64	6138874486	5649427132	4735513531
35	99	66	22890091062	20914166059	17347212127
36	102	68	85511947468	77587152924	63705666521
37	105	70	320013030067	288398164702	234500056176
38	108	72	1199620598580	1074044692104	865141832437
39	111	74	4504219709753	4007195731866	3198618016486
40	114	76	16938267502048	14976784750710	11850315368675

Table 5: OP-isomorphism classes of plane triangulations with minimum degree 5

n_v	n_e	n_f	$e_3(n_v)$	$e_4(n_v)$	$e'_3(n_v)$	$e'_4(n_v)$
6	12	8	1	1	1	1
7	15	10	0	0	0	0
8	18	12	1	1	1	1
9	21	14	1	0	1	0
10	24	16	2	2	2	2
11	27	18	2	1	2	1
12	30	20	8	5	9	6
13	33	22	8	3	11	3
14	36	24	32	18	41	22
15	39	26	57	19	89	25
16	42	28	185	79	296	112
17	45	30	466	134	829	214
18	48	32	1543	501	2772	817
19	51	34	4583	1147	8746	2058
20	54	36	15374	3976	29461	7188
21	57	38	50116	11055	98342	21036
22	60	40	171168	37231	336881	71185
23	63	42	582603	114560	1156559	224103
24	66	44	2024119	384053	4024297	753561
25	69	46	7057472	1244056	14075250	2464355
26	72	48	24873248	4193857	49638364	8321649
27	75	50	88111772	13977946	176037177	27841706
28	78	52	314301078	47522279	628107157	94737950
29	81	54	1126716000	161222224	2252541666	321889797
30	84	56	4060375677	553033544	8118442511	1104620101
31	87	58	14697571234	1899744032	29390845869	3796766424
32	90	60	53432834170	6571595339	106854715443	13136256710
33	93	62	195015189626	22793047258	390009407529	45572625554
34	96	64	714404259151	79449718217	1428755867040	158865787212
35	99	66	2626130395699	277760027418	5252157292165	555452882736
36	102	68	9685071313079	974836112457		

Table 6: Isomorphism classes of eulerian plane triangulations

n_v	n_e	n_f	$q_1(n_v)$	$q_2(n_v)$	$q_3(n_v)$	$q_4(n_v)$
4	4	2	1			
5	6	3	1			
6	8	4	2			
7	10	5	3			
8	12	6	9	1	1	1
9	14	7	18	0	0	0
10	16	8	62	1	1	1
11	18	9	198	1	1	1
12	20	10	803	3	3	2
13	22	11	3378	3	3	2
14	23	12	15882	12	11	9
15	26	13	77185	19	18	11
16	28	14	393075	64	58	37
17	30	15	2049974	155	139	79
18	32	16	10938182	510	451	249
19	34	17	59312272	1514	1326	671
20	36	18	326258544	5146	4461	2182
21	38	19	1815910231	16966	14554	6692
22	40	20	10213424233	58782	49957	22131
23	42	21	57974895671	203269	171159	72405
24	44	22	331820721234	716607	598102	243806
25	46	23	1913429250439	2536201	2098675	822788
26	48	24	11109119321058	9062402	7437910	2815119
27	50	25	64901418126997	32533568	26490072	9679205
28	52	26		117498072	94944685	33551192
29	54	27		426212952	341867921	116900081
30	56	28		1553048548	1236864842	409675567
31	58	29		5681011890	8984888982	1442454215
32	60	30		20858998805	16387852863	5102542680
33	62	31		76850220654	59985464681	18124571838
34	64	32		284057538480	220320405895	64634480340
35	66	33		1053134292253	811796327750	231334873091
36	68	34		3915683667721	3000183106119	830828150081
37	70	35				2993489821771

Table 7: Isomorphism classes of plane quadrangulations

n_v	n_e	n_f	$q'_1(n_v)$	$q'_2(n_v)$	$q'_3(n_v)$	$q'_4(n_v)$
4	4	2	1			
5	6	3	1			
6	8	4	2			
7	10	5	3			
8	12	6	10	1	1	1
9	14	7	21	0	0	0
10	16	8	83	1	1	1
11	18	9	298	1	1	1
12	20	10	1339	4	4	3
13	22	11	6049	3	3	2
14	23	12	29765	16	15	12
15	26	13	148842	26	25	16
16	28	14	770267	99	92	59
17	30	15	4054539	256	234	133
18	32	16	21743705	895	803	445
19	34	17	118237471	2789	2469	1248
20	36	18	651370528	9740	8512	4162
21	38	19	3628421181	32799	28290	13014
22	40	20	20416662314	115024	98148	43474
23	42	21	115919209155	401180	338673	143304
24	44	22	663548898942	1421170	1188338	484444
25	46	23	3826577783917	5046539	4180854	1639388
26	48	24	22217382001865	18066772	14840031	5617205
27	50	25	129800215435088	64940825	52904562	19332596
28	52	26		234712099	189724510	67048051
29	54	27		851801048	683384218	233691112
30	56	28		3104690139	2472961423	819121608
31	58	29		11358900851	4493270976	2884443024
32	60	30		41710948878	32772085447	10204104900
33	62	31		153684688127	119963084542	36247138920
34	64	32		568079430741	440623586740	129264732757
35	66	33		2106188450292	1623555117611	462661038926
36	68	34		7831185534651	6000283550482	1661637913984
37	70	35				5986941546017

Table 8: OP-isomorphism classes of plane quadrangulations

n_v	n_e	n_f	$p_3(n_v, n_e)$	$p'_3(n_v, n_e)$	n_v	n_e	n_f	$p_3(n_v, n_e)$	$p'_3(n_v, n_e)$
4	6	4	1	1	9	18	11	768	1441
4	total		1	1	9	19	12	558	1032
5	8	5	1	1	9	20	13	219	386
5	9	6	1	1	9	21	14	50	73
5	total		2	2	9	total		2606	4798
6	9	5	1	1	10	15	7	5	6
6	10	6	2	3	10	16	8	76	128
6	11	7	2	2	10	17	9	633	1188
6	12	8	2	2	10	18	10	2635	5096
6	total		7	8	10	19	11	6134	11982
7	11	6	2	2	10	20	12	8822	17265
7	12	7	8	11	10	21	13	7916	15466
7	13	8	11	16	10	22	14	4442	8582
7	14	9	8	10	10	23	15	1404	2652
7	15	10	5	6	10	24	16	233	389
7	total		34	45	10	total		32300	62754
8	12	6	2	2	11	17	8	38	60
8	13	7	11	16	11	18	9	768	1441
8	14	8	42	69	11	19	10	6134	11982
8	15	9	74	127	11	20	11	25626	50586
8	16	10	76	128	11	21	12	64439	127765
8	17	11	38	60	11	22	13	104213	206880
8	18	12	14	17	11	23	14	112082	222472
8	total		257	419	11	24	15	79773	158057
9	14	7	8	10	11	25	16	36528	71980
9	15	8	74	127	11	26	17	9714	18914
9	16	9	296	541	11	27	18	1249	2274
9	17	10	633	1188	11	total		440564	872411

Table 9: 3-connected planar graphs

n_v	n_e	n_f	$p_3(n_v, n_e)$	$p'_3(n_v, n_e)$
12	18	8	14	17
12	19	9	558	1032
12	20	10	8822	17265
12	21	11	64439	127765
12	22	12	268394	534292
12	23	13	709302	1414264
12	24	14	1263032	2519753
12	25	15	1556952	3106586
12	26	16	1338853	2670345
12	27	17	789749	1573849
12	28	18	306470	609084
12	29	19	70454	139264
12	30	20	7595	14502
12	total		6384634	12728018
13	20	9	219	386
13	21	10	7916	15466
13	22	11	104213	206880
13	23	12	709302	1414264
13	24	13	2937495	5865150
13	25	14	8085725	16154030
13	26	15	15535572	31044880
13	27	16	21395274	42757876
13	28	17	21317178	42599870
13	29	18	15287112	30543400
13	30	19	7706577	15391064
13	31	20	2599554	5185408
13	32	21	527235	1048947
13	33	22	49566	97033
13	total		96262938	192324654

n_v	n_e	n_f	$p_3(n_v, n_e)$	$p'_3(n_v, n_e)$
14	21	9	50	73
14	22	10	4442	8582
14	23	11	112082	222472
14	24	12	1263032	2519753
14	25	13	8085725	16154030
14	26	14	33310550	66582243
14	27	15	94713809	189357113
14	28	16	193794051	387478495
14	29	17	292182191	584222152
14	30	18	328192346	656222622
14	31	19	274542869	548932992
14	32	20	168992630	337857349
14	33	21	74424566	148765545
14	34	22	22229616	44410011
14	35	23	4037671	8057026
14	36	24	339722	672781
14	total		1496225352	2991463239
15	23	10	1404	2652
15	24	11	79773	158057
15	25	12	1556952	3106586
15	26	13	15535572	31044880
15	27	14	94713809	189357113
15	28	15	388431688	776705379
15	29	16	1134914458	2269538208
15	30	17	2447709924	4894956314
15	31	18	3981512855	7962395520
15	32	19	4939809506	9878872040
15	33	20	4686995652	9373223282
15	34	21	3380569040	6760462428
15	35	22	1823658612	3646797274
15	36	23	713331098	1426340694
15	37	24	191283058	382390200
15	38	25	31477887	62893270
15	39	26	2406841	4792530
15	total		23833988129	47663036427

Table 10: 3-connected planar graphs (continued)

n_v	n_e	n_f	$p_3(n_v, n_e)$	$p'_3(n_v, n_e)$
16	24	10	233	389
16	25	11	36528	71980
16	26	12	1338853	2670345
16	27	13	21395274	42757876
16	28	14	193794051	387478495
16	29	15	1134914458	2269538208
16	30	16	4637550072	9274453627
16	31	17	13865916560	27730625000
16	32	18	31277856206	62553740764
16	33	19	54271705726	108540645892
16	34	20	73247405678	146491362077
16	35	21	77220397213	154437090617
16	36	22	63443012728	126882463218
16	37	23	40232230880	80461570728
16	38	24	19322611431	38643116145
16	39	25	6799902944	13598589828
16	40	26	1654924768	3309214738
16	41	27	249026400	497840520
16	42	28	17490241	34911786
16	total		387591510244	775158142233
17	26	11	9714	18914
17	27	12	789749	1573849
17	28	13	21317178	42599870
17	29	14	292182191	584222152
17	30	15	2447709924	4894956314
17	31	16	13865916560	27730625000
17	32	17	56493493990	112984297173
17	33	18	172301697581	344598307174
17	34	19	404008232288	808008004874
17	35	20	741171341224	1482330409238
17	36	21	1075323264149	2150630733021
17	37	22	1240159791730	2480301615624
17	38	23	1136847700529	2273677366634
17	39	24	823788552428	1647561027749
17	40	25	466224664031	932436981890
17	41	26	201829738768	403651052112
17	42	27	64563924319	129123206316
17	43	28	14386939428	28771596080
17	44	29	1994599707	3988465676
17	45	30	129664753	259106122
17	total		6415851530241	12831576165782

n_v	n_e	n_f	$p_3(n_v, n_e)$
18	27	11	1249
18	28	12	306470
18	29	13	15287112
18	30	14	328192346
18	31	15	3981512855
18	32	16	31277856206
18	33	17	172301697581
18	34	18	700335433295
18	35	19	2173270387051
18	36	20	5270785332349
18	37	21	10150757285258
18	38	22	15683069986564
18	39	23	19547663107721
18	40	24	19682306885581
18	41	25	15962912975720
18	42	26	10348108651919
18	43	27	5288847843415
18	44	28	2084335836704
18	45	29	611239308239
18	46	30	125619037674
18	47	31	16147744792
18	48	32	977526957
18	total		107854282197058

Table 11: 3-connected planar graphs (continued)

n_v	n_e	n_f	$p_4(n_v, n_e)$	$p'_4(n_v, n_e)$
6	12	8	1	1
6	total		1	1
7	15	10	1	1
7	total		1	1
8	16	10	1	1
8	17	11	1	1
8	18	12	2	2
8	total		4	4
9	18	11	1	1
9	19	12	4	5
9	20	13	4	5
9	21	14	5	5
9	total		14	16
10	20	12	3	4
10	21	13	10	15
10	22	14	25	39
10	23	15	17	27
10	24	16	12	14
10	total		67	99
11	22	13	3	3
11	23	14	36	58
11	24	15	107	186
11	25	16	159	276
11	26	17	89	152
11	27	18	34	45
11	total		428	720
12	24	14	11	15
12	25	15	119	211
12	26	16	580	1080
12	27	17	1095	2087
12	28	18	1089	2035
12	29	19	491	909
12	30	20	130	194
12	total		3515	6531
13	26	15	18	25
13	27	16	456	843
13	28	17	2815	5444
13	29	18	7562	14765
13	30	19	10096	19791
13	31	20	7485	14520
13	32	21	2806	5398
13	33	22	525	891
13	total		31763	61677
14	28	16	58	92
14	29	17	1714	3280
14	30	18	14102	27691
14	31	19	47890	94823
14	32	20	85805	170029
14	33	21	87124	172780
14	34	22	51844	102171
14	35	23	16534	32422
14	36	24	2472	4499
14	total		307543	607787
15	30	17	139	234
15	31	18	6678	13024
15	32	19	67651	134140
15	33	20	288534	574277
15	34	21	651596	1298861
15	35	22	870969	1735951
15	36	23	712861	1420596
15	37	24	355286	705869
15	38	25	98587	195245
15	39	26	12400	23603
15	total		3064701	6101800

Table 12: 3-connected planar graphs with minimum degree at least 4

n_v	n_e	n_f	$p_4(n_v, n_e)$	$p'_4(n_v, n_e)$
16	32	18	451	803
16	33	19	26053	51346
16	34	20	321633	640157
16	35	21	1655945	3304524
16	36	22	4596362	9178285
16	37	23	7694805	15370991
16	38	24	8201794	16380369
16	39	25	5623132	11228632
16	40	26	2419038	4822966
16	41	27	594236	1182790
16	42	28	65619	127887
16	total		31199068	62288750
17	34	19	1326	2469
17	35	20	102303	202921
17	36	21	1495862	2984594
17	37	22	9162421	18304658
17	38	23	30452356	60865282
17	39	24	62068706	124072684
17	40	25	82398857	164721872
17	41	26	73098873	146113658
17	42	27	43159731	86258968
17	43	28	16358800	32670411
17	44	29	3607916	7198627
17	45	30	357504	705770
17	total		322264655	644101914
18	36	20	4461	8512
18	37	21	401839	799894
18	38	22	6871117	13724201
18	39	23	49151202	98250487
18	40	24	192208694	384298704
18	41	25	465884287	931575989
18	42	26	748153542	1496024456
18	43	27	822759274	1645223548
18	44	28	625673674	1251042907
18	45	29	324655428	649108896
18	46	30	110109217	220069087
18	47	31	22046012	44040684
18	48	32	1992985	3959653
18	total		3369911732	6738127018
19	38	21	14554	28290
19	39	22	1580624	3152716
19	40	23	31144629	62247109
19	41	24	257114746	514091455
19	42	25	1165392704	2330464143
19	43	26	3301434495	6602256719
19	44	27	6271404415	12541919157
19	45	28	8303354116	16605547117
19	46	29	7797302305	15593472680
19	47	30	5192671355	10384273735
19	48	31	2404902987	4809128447
19	49	32	738539257	1476613802
19	50	33	135456226	270757489
19	51	34	11284042	22494163
19	total		35611596455	71216447022

Table 13: 3-connected planar graphs with minimum degree at least 4 (continued)

n_v	n_e	n_f	$p_4(n_v, n_e)$	$p'_4(n_v, n_e)$	n_v	n_e	n_f	$p_4(n_v, n_e)$
20	40	22	49957	98148	21	42	23	171159
20	41	23	6216228	12413412	21	43	24	24454736
20	42	24	139772014	279440528	21	44	25	621638040
20	43	25	1316733885	2633118911	21	45	26	6623527353
20	44	26	6835771143	13670629344	21	46	27	38988464261
20	45	27	22314671743	44627575118	21	47	28	144986607904
20	46	28	49293619937	98584249961	21	48	29	367384415512
20	47	29	76876264695	153748652342	21	49	30	663656374611
20	48	30	86591104361	173177527208	21	50	31	877516979065
20	49	31	70984826506	141965438774	21	51	32	860271765669
20	50	32	42074212402	84144667518	21	52	33	626038224719
20	51	33	17604812790	35207381912	21	53	34	334467204910
20	52	34	4942097075	9882727713	21	54	35	127681045881
20	53	35	836535543	1672603778	21	55	36	33020478701
20	54	36	64719885	129227103	21	56	37	5190532666
20	total		379881408164	759735751770	21	57	38	375126827
					21	total		4086847012014

Table 14: 3-connected planar graphs with minimum degree at least 4 (continued)

n_v	n_e	n_f	$p_5(n_v, n_e)$	$p'_5(n_v, n_e)$	n_v	n_e	n_f	$p_5(n_v, n_e)$	$p'_5(n_v, n_e)$
12	30	20	1	1	21	56	37	540	1017
12	total		1	1	21	57	38	192	331
13	total		0	0	21	total		1587	2961
14	36	24	1	1	22	55	35	14	24
14	total		1	1	22	56	36	325	616
15	39	26	1	1	22	57	37	1550	3005
15	total		1	1	22	58	38	2955	5734
16	40	26	1	1	22	59	39	2162	4185
16	41	27	1	1	22	60	40	651	1180
16	42	28	3	4	22	total		7657	14744
16	total		5	6	23	58	37	196	365
17	43	28	1	1	23	59	38	2591	5058
17	44	29	3	3	23	60	39	9270	18274
17	45	30	4	4	23	61	40	13615	26814
17	total		8	8	23	62	41	8549	16797
18	45	29	1	2	23	63	42	2070	3899
18	46	30	7	12	23	total		36291	71207
18	47	31	10	15	24	60	38	96	173
18	48	32	12	17	24	61	39	2810	5497
18	total		30	46	24	62	40	20206	39974
19	48	31	3	4	24	63	41	52823	104898
19	49	32	24	40	24	64	42	63095	125146
19	50	33	35	58	24	65	43	34124	67568
19	51	34	23	33	24	66	44	7290	14052
19	total		85	135	24	total		180444	357308
20	50	32	6	9	25	63	40	1694	3307
20	51	33	37	63	25	64	41	28649	56820
20	52	34	136	244	25	65	42	138525	275764
20	53	35	140	253	25	66	43	284520	567010
20	54	36	73	117	25	67	44	284102	565701
20	total		392	686	25	68	45	135439	269342
21	53	34	26	45	25	69	46	25381	49667
21	54	35	231	433	25	total		898310	1787611
21	55	36	598	1135					

Table 15: 3-connected plane graphs with minimum degree at least 5

n_v	n_e	n_f	$p_5(n_v, n_e)$	$p'_5(n_v, n_e)$
26	65	41	518	990
26	66	42	27247	54028
26	67	43	251687	501717
26	68	44	884431	1764979
26	69	45	1474446	2943645
26	70	46	1265456	2524800
26	71	47	537493	1071577
26	72	48	91441	180502
26	total		4532719	9042238
27	68	43	14674	29075
27	69	44	315002	628215
27	70	45	1943074	3880657
27	71	46	5285560	10560455
27	72	47	7387374	14761187
27	73	48	5547143	11080030
27	74	49	2126514	4245308
27	75	50	329824	654674
27	total		22949165	45839601
28	70	44	3917	7689
28	71	45	262170	522777
28	72	46	3064076	6121002
28	73	47	13674643	27332100
28	74	48	30081720	60132817
28	75	49	36052160	72069944
28	76	50	24062148	48089612
28	77	51	8400155	16782891
28	78	52	1204737	2398527
28	total		116805726	233457359
29	73	46	129558	258217
29	74	47	3395462	6784218
29	75	48	25980495	51937427
29	76	49	89502100	178953032
29	77	50	164317521	328554612
29	78	51	172082986	344079630
29	79	52	103295735	206511268
29	80	53	33113060	66186792
29	81	54	4412031	8800984
29	total		596228948	1192066180
30	75	47	29821	59206
30	76	48	2540458	5075116
30	77	49	36153637	72280336
30	78	50	199284603	398489524
30	79	51	553245996	1106343494
30	80	52	868499404	1736780076
30	81	53	806515573	1612816382
30	82	54	439841613	879491006
30	83	55	130336575	260584336
30	84	56	16248772	32447008
30	total		3052696452	6104366484
31	78	49	1145111	2287156
31	79	50	36028132	72031083
31	80	51	333673154	667247944
31	81	52	1413054897	2825865636
31	82	53	3264576190	6528731430
31	83	54	4465329366	8930094363
31	84	55	3721853265	7443174579
31	85	56	1859260375	3718075225
31	86	57	512281901	1024362305
31	87	58	59995535	119883207
31	total		15667197926	31331752928

Table 16: 3-connected plane graphs with minimum degree at least 5 (continued)

n_v	n_e	n_f	$p_5(n_v, n_e)$	$p'_5(n_v, n_e)$
32	80	50	240430	479446
32	81	51	24468620	48918024
32	82	52	416399311	832689068
32	83	53	2767321897	5534305556
32	84	54	9412162103	18823569658
32	85	55	18541480725	37081796296
32	86	56	22433623830	44865765346
32	87	57	16951098902	33900894153
32	88	58	7811471882	15621888283
32	89	59	2011226628	4021998166
32	90	60	222231424	444226539
32	total		80591725752	161176530535
33	83	52	10152741	20295368
33	84	53	376951752	753810321
33	85	54	4149278837	8298153553
33	86	55	21111408725	42221707361
33	87	56	59571105445	119140021626
33	88	57	101993247858	203983308997
33	89	58	110506546904	221009334051
33	90	59	76335350545	152667508151
33	91	60	32643939837	65285438093
33	92	61	7888416533	15775800762
33	93	62	825028656	1649550311
33	total		415411427833	830804928594

Table 17: 3-connected plane graphs with minimum degree at least 5 (continued)

n_v	n_e	n_f	$p_5(n_v, n_e)$	$p'_5(n_v, n_e)$
34	85	53	1957382	3910515
34	86	54	234846981	469623164
34	87	55	4698066344	9395720509
34	88	56	36973254903	73945022947
34	89	57	150610142121	301216777356
34	90	58	361402022519	722797642328
34	91	59	546056821115	1092105078640
34	92	60	535227995999	1070446321676
34	93	61	340413582639	680819405952
34	94	62	135792191605	271578632193
34	95	63	30915951931	61829568488
34	96	64	3069993552	6138874486
34	total		2145396827091	4290746578254
35	88	55	90171828	180309786
35	89	56	3899705466	7799068373
35	90	57	50252955201	100504272959
35	91	58	301760294018	603515614576
35	92	59	1016954066033	2033897372915
35	93	60	2115688345019	4231358798972
35	94	61	2856474952904	5712927114015
35	95	62	2554640081343	5109255971021
35	96	63	1505165810142	3010312797687
35	97	64	562599608075	1125185937779
35	98	65	121088625406	242171956724
35	99	66	11446245342	22890091062
35	total		11100060860777	22199999305869

Table 18: 3-connected plane graphs with minimum degree at least 5 (continued)

n_v	n_e	n_f	$s_1(n_v, n_e)$	$s'_1(n_v, n_e)$
1	0	1	1	1
1	total		1	1
2	1	1	1	1
2	total		1	1
3	2	1	1	1
3	3	2	1	2
3	total		2	2
4	3	1	2	2
4	4	2	2	2
4	5	3	1	1
4	6	4	1	1
4	total		6	6
5	4	1	3	3
5	5	2	7	8
5	6	3	7	8
5	7	4	5	6
5	8	5	2	2
5	9	6	1	1
5	total		25	28
6	5	1	6	6
6	6	2	22	29
6	7	3	42	60
6	8	4	49	73
6	9	5	35	52
6	10	6	18	25
6	11	7	5	6
6	12	8	2	2
6	total		179	253
7	6	1	12	14
7	7	2	76	113
7	8	3	237	388
7	9	4	442	768
7	10	5	510	903
7	11	6	412	728
7	12	7	218	379
7	13	8	84	136
7	14	9	18	26
7	15	10	5	6
7	total		2014	3461
8	7	1	27	34
8	8	2	271	444
8	9	3	1293	2303
8	10	4	3539	6584
8	11	5	6205	11782
8	12	6	7482	14321
8	13	7	6318	12113
8	14	8	3833	7298
8	15	9	1623	3048
8	16	10	485	872
8	17	11	88	147
8	18	12	14	17
8	total		31178	58963
9	8	1	65	95
9	9	2	1001	1763
9	10	3	6757	12650
9	11	4	25842	49806
9	12	5	63254	123547
9	13	6	106985	210314
9	14	7	129782	255884
9	15	8	115988	228807
9	16	9	76582	150929
9	17	10	37421	73428
9	18	11	13111	25536
9	19	12	3228	6142
9	20	13	489	892
9	21	14	50	73
9	total		580555	1139866

Table 19: Simple plane graphs

n_v	n_e	n_f	$s_1(n_v, n_e)$	$s'_1(n_v, n_e)$
10	9	1	175	280
10	10	2	3765	6951
10	11	3	34289	66036
10	12	4	173890	341048
10	13	5	563715	1114697
10	14	6	1266019	2513423
10	15	7	2064899	4107464
10	16	8	2520682	5018648
10	17	9	2340428	4661292
10	18	10	1665254	3315602
10	19	11	904432	1799396
10	20	12	370667	735850
10	21	13	111177	219906
10	22	14	23376	45710
10	23	15	3071	5876
10	24	16	233	389
10	total		12046072	23952568
11	10	1	490	854
11	11	2	14381	27395
11	12	3	169146	331103
11	13	4	1095253	2167814
11	14	5	4522819	8994907
11	15	6	12962663	25838666
11	16	7	27156110	54191372
11	17	8	43021440	85900472
11	18	9	52653941	105164088
11	19	10	50438521	100749234
11	20	11	38019564	75940910
11	21	12	22531494	44995125
11	22	13	10404904	20770210
11	23	14	3680668	7340180
11	24	15	965204	1921576
11	25	16	178166	352832
11	26	17	20667	40490
11	27	18	1249	2274
11	total		267836680	534729502
12	11	1	1473	2694
12	12	2	55450	107672
12	13	3	814935	1610019
12	14	4	6540667	13007783
12	15	5	33414914	66638772
12	16	6	118601261	236825891
12	17	7	308937020	617272349
12	18	8	612495575	1224182030
12	19	9	947188002	1893440560
12	20	10	1161385024	2321819219
12	21	11	1140422860	2279997573
12	22	12	901120070	1801560786
12	23	13	572806006	1145135868
12	24	14	291326699	582350001
12	25	15	117141432	234118427
12	26	16	36490499	72898910
12	27	17	8509444	16986695
12	28	18	1403778	2795556
12	29	19	146381	290020
12	30	20	7595	14502
12	total		6258809085	12511055327

Table 20: Simple plane graphs (continued)

n_v	n_e	n_f	$s_1(n_v, n_e)$	$s'_1(n_v, n_e)$
13	12	1	4588	8714
13	13	2	214880	422330
13	14	3	3847045	7640733
13	15	4	37396327	74563644
13	16	5	231042429	461400518
13	17	6	992347643	1983154498
13	18	7	3135908805	6269025509
13	19	8	7574882641	15145530340
13	20	9	14361574531	28717622644
13	21	10	21771980560	43537636164
13	22	11	26728140900	53449879149
13	23	12	26786453756	53567092452
13	24	13	22003820967	44002935573
13	25	14	14820237437	29637017484
13	26	15	8154755112	16307265965
13	27	16	3636115210	7270899128
13	28	17	1295548971	2590420902
13	29	18	360765580	721213960
13	30	19	75769154	151420444
13	31	20	11315138	22588920
13	32	21	1072760	2136488
13	33	22	49566	97033
13	total		151983244000	303919972592

Table 21: Simple plane graphs (continued)

n_v	n_e	n_f	$s_1(n_v, n_e)$	$s'_1(n_v, n_e)$
14	13	1	14782	28640
14	14	2	835663	1654180
14	15	3	17850770	35560340
14	16	4	206278148	411862507
14	17	5	1513397328	3024437423
14	18	6	7721917289	15437825408
14	19	7	29037045381	58061851096
14	20	8	83706257879	167391740331
14	21	9	190208046338	380385840201
14	22	10	347588206583	695137815312
14	23	11	518230987068	1036418333036
14	24	12	636786027736	1273527733979
14	25	13	649139024125	1298237616992
14	26	14	550913928145	1101794307544
14	27	15	389487996579	778950963059
14	28	16	228840399800	457663707370
14	29	17	111096539594	222182705492
14	30	18	44127749090	88249715922
14	31	19	14121413352	28240051024
14	32	20	3556481698	7111733177
14	33	21	679613347	1358792263
14	34	22	92771734	185398789
14	35	23	8071728	16113254
14	36	24	339722	672781
14	total		3807081193879	7613826460120

Table 22: Simple plane graphs (continued)

n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$	n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$
3	3	2	1	1	8	14	8	2144	4055
3	total		1	1	8	15	9	1246	2332
4	4	2	1	1	8	16	10	447	804
4	5	3	1	1	8	17	11	88	147
4	6	4	1	1	8	18	12	14	17
4	total		3	3	8	total		7593	14162
5	5	2	1	1	9	9	2	1	1
5	6	3	2	2	9	10	3	7	7
5	7	4	4	5	9	11	4	104	161
5	8	5	2	2	9	12	5	915	1664
5	9	6	1	1	9	13	6	5046	9659
5	total		10	11	9	14	7	16009	31252
6	6	2	1	1	9	15	8	30183	59244
6	7	3	3	3	9	16	9	33719	66289
6	8	4	13	17	9	17	10	23749	46521
6	9	5	21	31	9	18	11	10585	20604
6	10	6	16	22	9	19	12	3017	5743
6	11	7	5	6	9	20	13	489	892
6	12	8	2	2	9	21	14	50	73
6	total		61	82	9	total		123874	242110
7	7	2	1	1	10	10	2	1	1
7	8	3	4	4	10	11	3	9	9
7	9	4	29	42	10	12	4	181	286
7	10	5	94	157	10	13	5	2239	4151
7	11	6	183	318	10	14	6	17876	34700
7	12	7	154	265	10	15	7	85550	168757
7	13	8	76	123	10	16	8	254831	505410
7	14	9	18	26	10	17	9	478913	952044
7	15	10	5	6	10	18	10	581324	1156127
7	total		564	942	10	19	11	468388	931227
8	8	2	1	1	10	20	12	255156	506318
8	9	3	6	6	10	21	13	93028	183980
8	10	4	59	87	10	22	14	22077	43180
8	11	5	328	576	10	23	15	3071	5876
8	12	6	1146	2128	10	24	16	233	389
8	13	7	2114	4009	10	total		2262877	4492455

Table 23: Simple 2-connected plane graphs

n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$	n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$
11	11	2	1	1	12	24	14	138993896	277822669
11	12	3	11	11	12	25	15	72858380	145607615
11	13	4	283	460	12	26	16	27705872	55347935
11	14	5	4920	9266	12	27	17	7444800	14861459
11	15	6	53908	105587	12	28	18	1344104	2676888
11	16	7	360828	715594	12	29	19	146381	290020
11	17	8	1545170	3077004	12	30	20	7595	14502
11	18	9	4342371	8662224	12	total		904777809	1808322585
11	19	10	8185754	16339528	13	13	2	1	1
11	20	11	10537211	21038806	13	14	3	15	15
11	21	12	9462526	18891305	13	15	4	645	1085
11	22	13	5996409	11967922	13	16	5	19120	36684
11	23	14	2680961	5345964	13	17	6	352415	696524
11	24	15	828434	1649252	13	18	7	4029097	8026593
11	25	16	169576	335867	13	19	8	30252253	60409193
11	26	17	20667	40490	13	20	9	155343066	310459801
11	27	18	1249	2274	13	21	10	561633452	1122815588
11	total		44190279	88181555	13	22	11	1458502789	2916267568
12	12	2	1	1	13	23	12	2762714021	5524390423
12	13	3	13	13	13	24	13	3867958565	7734703651
12	14	4	440	725	13	25	14	4049812367	8098392055
12	15	5	10030	19079	13	26	15	3196009270	6390977323
12	16	6	144513	284523	13	27	16	1905204103	3809636821
12	17	7	1286139	2557736	13	28	17	853252330	1706037382
12	18	8	7445568	14853386	13	29	18	282757123	565262540
12	19	9	29007422	57938504	13	30	19	67281212	134458388
12	20	10	78002990	155876134	13	31	20	10885047	21730964
12	21	11	147357964	294537519	13	32	21	1072760	2136488
12	22	12	198748443	397287301	13	33	22	49566	97033
12	23	13	194273258	388346576	13	total		19207129217	38406536120

Table 24: Simple 2-connected plane graphs (continued)

n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$
14	14	2	1	1
14	15	3	18	18
14	16	4	933	1585
14	17	5	34651	66880
14	18	6	797273	1579648
14	19	7	11397821	22730489
14	20	8	107688633	215151463
14	21	9	703236220	1405855204
14	22	10	3278258667	6555101486
14	23	11	11157309442	22311929605
14	24	12	28183908019	56363414831
14	25	13	53528076491	107050064823
14	26	14	77287026330	154566703701
14	27	15	85631816052	171256090392
14	28	16	73282811954	146558815643
14	29	17	48559480162	97113759448
14	30	18	24846491977	49689459876
14	31	19	9724984643	19447991228
14	32	20	2858351093	5715708409
14	33	21	610686053	1220985282
14	34	22	89583129	179028906
14	35	23	8071728	16113254
14	36	24	339722	672781
14	total		419870351012	839691224953

Table 25: Simple 2-connected plane graphs (continued)

n_v	n_e	n_f	$s_2(n_v, n_e)$	$s'_2(n_v, n_e)$
15	15	2	1	1
15	16	3	20	20
15	17	4	1296	2234
15	18	5	60022	116449
15	19	6	1692814	3360546
15	20	7	29659179	59192537
15	21	8	344617558	688740212
15	22	9	2787166095	5572787546
15	23	10	16247029396	32490034160
15	24	11	69967888504	139927031083
15	25	12	226648323613	453280266975
15	26	13	559821248634	1119616299740
15	27	14	1065997244505	2131957799072
15	28	15	1579447768761	3158851211581
15	29	16	1835018112997	3669988946517
15	30	17	1680911336451	3361779089184
15	31	18	1217057936678	2434080203072
15	32	19	695669679375	1391314290972
15	33	20	312070865975	624126015244
15	34	21	108552153253	217096040999
15	35	22	28680789703	57357685690
15	36	23	5562076007	11122712222
15	37	24	746644170	1492805661
15	38	25	61990477	123878966
15	39	26	2406841	4792530
15	total		9405626692325	18810933303213

Table 26: Simple 2-connected plane graphs (continued)

n_v	$d_1(n_v)$	$d_2(n_v)$	$d_3(n_v)$	$d_4(n_v)$
3	0	0	1	1
4	1	1	2	1
5	2	2	4	1
6	7	8	16	3
7	27	31	63	4
8	132	159	328	12
9	773	936	1933	27
10	5017	6148	12633	82
11	34861	42891	87466	228
12	253676	313088	633015	733
13	1903584	2351945	4717745	2282
14	14616442	18063992	35980100	7528
15	114254053	141141422	279418926	24834
16	906266345	1118604721	2202903618	83898
17	7277665889	8972884862	17590599410	285357
18	59066524810	72732678436	142025760202	983244
19	483864411124	595001005461	1157868883224	3412420
20	3996427278475	4907386823804	9520828261067	11944614
21	33250623548406	40771329386840	78888071847324	42080170

n_v	$d'_1(n_v)$	$d'_2(n_v)$	$d'_3(n_v)$	$d'_4(n_v)$
3	0	0	1	1
4	1	1	2	1
5	2	2	4	1
6	8	9	20	4
7	37	42	93	6
8	213	255	554	19
9	1386	1675	3554	49
10	9524	11654	24256	150
11	68057	83688	171676	442
12	501858	619177	1255194	1424
13	3788747	4680413	9399396	4522
14	29170667	36048019	71837656	14924
15	228295618	282009376	558420702	49536
16	1811802818	2236264516	4404369524	167367
17	14552804492	17942491936	35176227916	570285
18	118124257451	145453903206	284034186632	1965058
19	967698049455	1189961845145	2315677128324	6823410
20	7992746427963	9814632026718	19041443086870	23884366
21	66500865364037	81542157841982	157775390173844	84155478

Table 27: Triangulations of a disk