Construction of planar triangulations with minimum degree 5

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Abstract

In this article we describe a method of constructing all simple triangulations of the sphere with minimum degree 5; equivalently, 3-connected planar cubic graphs with girth 5. We also present the results of a computer program based on this algorithm, including counts of convex polytopes of minimum degree 5.

Introduction

A set of operations is said to *generate* a class of graphs from a set of starting graphs in the class if every graph in the class can be constructed (up to isomorphism, however defined) by a sequence of these operations from one of the starting graphs and the class is closed under the construction operations.

There are two main reasons why methods to construct an infinite class from a finite set of starting graphs are of interest: on one hand they provide a basis for inductive proofs, and on the other they can be used to develop efficient algorithms for the constructive enumeration of the structures. Classes of polyhedra were among the first graph classes for which construction methods were published (see [7] and [11]) and also among the first classes for which a computer was used for their enumeration (see [9]). Today, the most extensive tables for various classes of polyhedra are given by Dillencourt in [6].

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We will restrict our attention to a subclass of all polyhedra. Isomorphisms must preserve the embedding, but since we will deal only with 3-connected graphs there is a one-to-one correspondence between embedding-preserving isomorphisms and abstract graph isomorphisms. D. Barnette [2] and J. W. Butler [5] independently described a method for constructing all planar cyclically 5-connected cubic graphs. In the language of the dual graph this class is the set of all 5-connected planar triangulations. We call such triangulations *C5-5-triangulations*. More generally, *Ck-5-triangulations* are the k-connected planar triangulations with minimum degree 5.

A *separating* k-cycle in a graph embedded on the plane is a k-cycle such that both the interior and the exterior contain one or more vertices. For a simple planar triangulation, 3-cuts correspond to separating 3-cycles, while 4-cuts correspond to separating 4-cycles. Thus a planar triangulation with minimum degree 5 is a C3-5-triangulation always, a C4-5-triangulation if there are no separating 3-cycles, and a C5-5-triangulation if there are no separating 3-cycles or separating 4-cycles.

Barnette and Butler's method starts with the icosahedron graph and uses the operations given in the following figure. In all our figures, edges and half edges drawn are always required to be present, while black triangles correspond to any number—zero or nonzero—of outgoing edges. No edges are incident with the depicted vertices except those indicated by the depicted edges or black triangles.

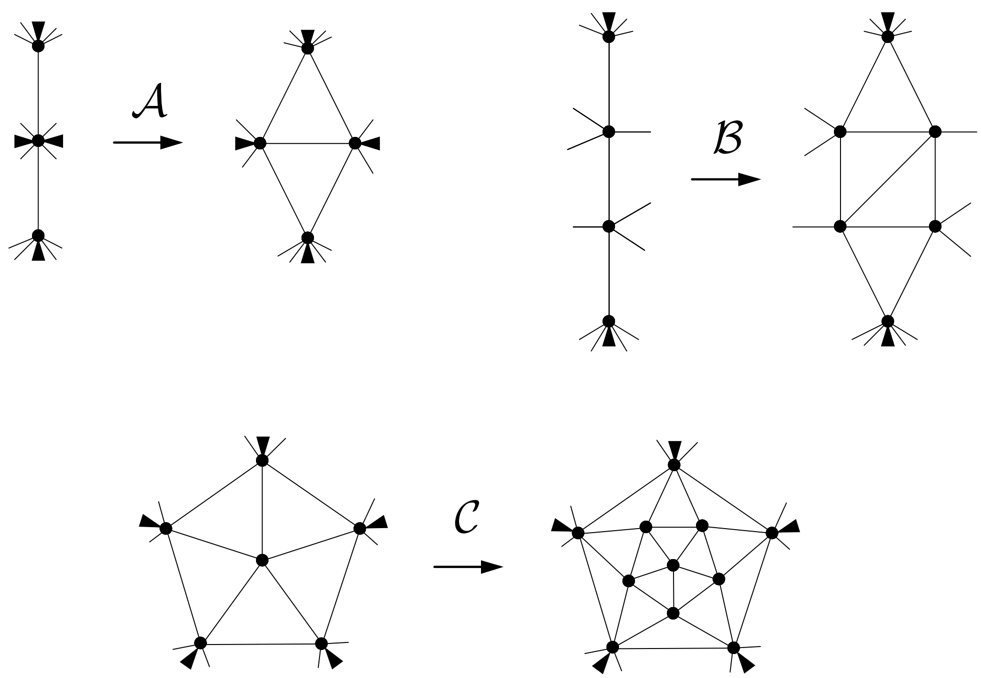

Figure 1: Barnette and Butler's operations

Theorem 1 *(Barnette [2], Butler [5]) All C5-5-triangulations can be generated from the icosahedron graph by using operations $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.*

Batagelj [3] has described a method for constructing all *C3-5-triangulations.*

He uses the operations $\mathcal{A}$ and $\mathcal{B}$ also used by Barnette and Butler and in addition a switching operation $\mathcal{D}$ as depicted in Figure 2. This operation assumes that the top and bottom vertices do not share an edge.

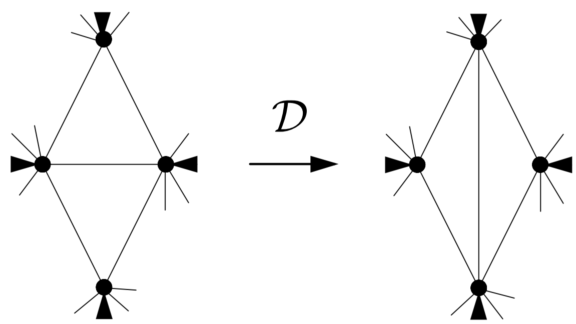

Figure 2: Switching operation

Theorem 2 *(Batagelj [3]) All C3-5-triangulations can be generated from the icosahedron graph by using operations $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$.*

Unfortunately Batagelj's proof contains an error, as he acknowledges (private communication), but nevertheless his theorem is correct as we will prove. However, we will focus on an approach that uses all four operations $\mathcal{A}$–$\mathcal{D}$ and thereby also allows construction of the intermediate class of C4-5-triangulations. In fact it enables a computer program to efficiently restrict its output to C4-5-triangulations or C5-5-triangulations only in addition to generating all C3-5-triangulations.

For $k \in \{4, 5\}$ let us denote a $\mathcal{D}$ operation such that the central edge does not belong to a separating cycle of length $k-1$ or less after the operation as a $\mathcal{D}_k$ operation.

Theorem 3

(a) *All C3-5-triangulations on n vertices with at least one separating 3-cycle can be constructed from C3-5-triangulations of the same size with fewer separating 3-cycles by applying operation $\mathcal{D}$.*

(b) *All C4-5-triangulations on n vertices with at least one separating 4-cycle can be constructed from C4-5-triangulations of the same size with fewer separating 4-cycles by applying operation $\mathcal{D}_4$ or from C4-5-triangulations with fewer vertices by applying operation $\mathcal{A}$.*

Recall that C4-5-triangulations without separating 4-cycles are just C5-5-triangulations and C3-5-triangulations without separating 3-cycles are C4-5-triangulations. So a computer program can first list all C5-5-triangulations using Theorem 1, then construct all additional C4-5-triangulations using Theorem 3(a), then finally all construct all additional C3-5-triangulations using Theorem 3(b). Restricting the generation to a subclass (C4-5-triangulations or C5-5-triangulations) is simply a matter of stopping the generation process at the correct point.

We will infer from our proof that Theorem 2 is correct, and also show that the operations given by Batagelj are able to generate just the C5-5-triangulations or C4-5-triangulations.

Theorem 4

(a) *The set of all C5-5-triangulations can be generated from the icosahedron graph by operations* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{D}_5$*.*

(b) *The set of all C4-5-triangulations can be generated from the icosahedron graph by operations* $\mathcal{A}$*,* $\mathcal{B}$*, and* $\mathcal{D}_4$*.*

An importand subclass of C5-5-triangulations, with many practical applications, are those with maximum degree 6, best known via their duals, the *fullerenes*. A very efficient generator of fullerenes has been given by Brinkmann and Dress [4].

Proofs of the Theorems

For $k \in \{3, 4\}$ an *innermost* separating k-cycle is a separating k-cycle such that either the interior or exterior does not contain any edges of another separating k-cycle. It can be easily seen that if a separating 3-cycle exists there is an innermost one and if a separating 4-cycle exists and no separating 3-cycle exists, then there is an innermost separating 4-cycle.

We will always draw innermost separating k-cycles in such a way that the interior does not contain edges of another separating k-cycle.

Proof of Theorem 3: In order to prove the theorem, we consider an arbitrary graph satisfying the conditions of the theorem and show how to apply the inverse of operation $\mathcal{D}$ (in case (a)), or either $\mathcal{D}_4$ or $\mathcal{A}$ (in case (b)), to produce a parent in the specified class.

Proof of part (a):

Let G be a C3-5-triangulation with an innermost separating 3-cycle C. First note that at each vertex of C at least two edges must lead into the interior, since otherwise the endpoint v of the single edge would be adjacent to the two remaining vertices on C, forming three 3-cycles in the interior, which—due to C being innermost—must be faces. But in this case v can not have additional edges, so it would have degree 3 (a contradiction). So C includes three internal faces as in part (a) of Figure 3.

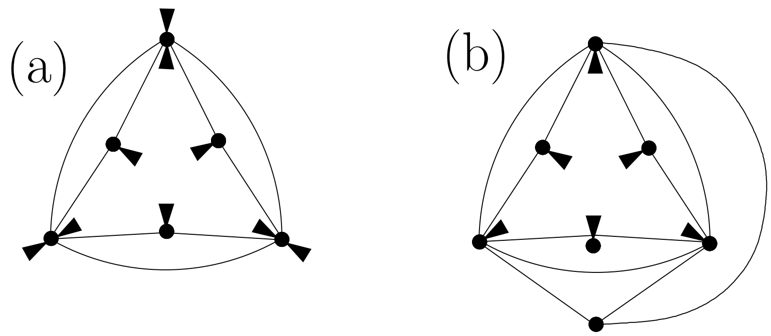

Figure 3: Possibilities for a separating 3-cycle

Since the exterior of C is not a face, each vertex has at least one edge sticking out. If two vertices on C had exactly one edge sticking out (w.l.o.g. the lower two in the picture), the situation of Figure 3(b) would occur—again introducing a vertex of valency 3 (a contradiction). So at least two vertices v, w on C must have at least two edges sticking out—giving a total degree of at least 6 for v, w and therefore the conditions for applying the inverse of operation $\mathcal{D}$ to (v, w) without violating the minimal valency are fulfilled. In the resulting graph G' the separating 3-cycle C has been destroyed, and the new edge cannot have created a new separating 3-cycle due to the minimality of C. So there is a smaller number of separating 3-cycles in G' and G can be constructed from G' by applying $\mathcal{D}$.

Proof of part (b):

Suppose we have no separating 3-cycles, but at least one separating 4-cycle. Let C be an innermost separating 4-cycle.

Again the property of being innermost implies that each vertex on an innermost separating 4-cycle C of a graph G has at least two edges sticking in. So the situation is as depicted in Figure 4(a). Vertices opposite on C cannot be adjacent, since this would either introduce a separating 3-cycle or the exterior would not contain vertices at all (contradicting C being a separating 4-cycle). This fact implies that each vertex on C must have at least one edge sticking out. Two consecutive vertices on C with each just one edge sticking out can be easily seen to imply either a vertex of degree 4 in the exterior or a separating 3-cycle—both contradictions. So we have at least two vertices v, w on C with at least two edges sticking outwards from each of them.

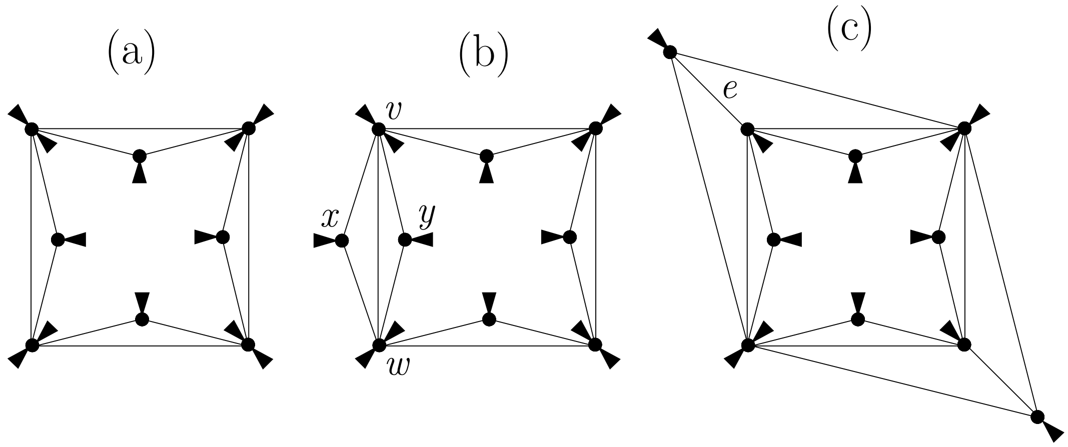

Figure 4: Possibilities for a separating 4-cycle

First suppose v and w are neighbours on C. The vertices neigbouring the edge (v, w), x on the outside of C and y on the inside of C (see Figure 4(b)), cannot be adjacent to either of the two remaining vertices on C, since this would imply a separating 3-cycle in the graph. Therefore, if we apply operation $\mathcal{D}$ to replace (v, w) by edge (x, y), the only possibility for (x, y) to lie on a new separating 4-cycle would be that the cycle passes through C at v or w—again implying a separating 3-cycle in the original graph. So in this case this $\mathcal{D}$ operation reduces the number of separating 4-cycles while obeying the degree constraints.

The only remaining case is that we have two vertices opposite to each other on C with each having exactly one edge sticking out and the others having at least two edges sticking out. So the situation is as in Figure 4(c).

In this case the inverse of operation $\mathcal{A}$ can be applied by contracting edge e with the result a graph of smaller order. A separating 3-cycle in the new graph that wasn't there before would have to cross the interior of C and can easily be seen not to exist by checking the various possibilities how this is possible. ■

In fact it can even be shown that in the last case the inverse of operation $\mathcal{A}$ need only be applied if the endpoint of e on the cycle C has valency 5. Otherwise we can again apply operation $\mathcal{D}$, but since it is not needed for the proof, we will not discuss it in detail here.

Proof of Theorem 4: Theorem 1 implies that every graph that can not be reduced by the inverse of operation $\mathcal{A}$ or $\mathcal{B}$ can be reduced by the inverse of operation $\mathcal{C}$, so it must contain the configuration on the right hand side of operation $\mathcal{C}$ in Figure 1. So for part (a) it is enough to show that a graph containing this configuration can be reduced by the inverse of operation $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{D}_5$.

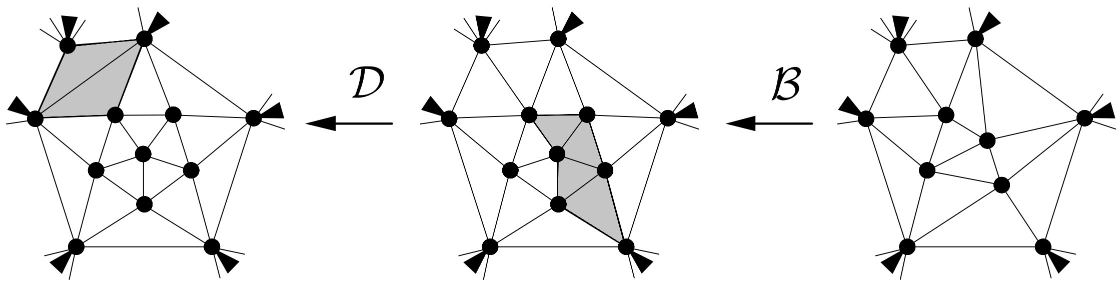

Figure 5: Replacing a $\mathcal{C}$ operation by $\mathcal{B}$ followed by $\mathcal{D}$

In Figure 5 it is shown that a reverse $\mathcal{D}_5$ (which is easily seen not to produce separating 4-cycles, so the resulting graph is in the same class) paves the ground for the inverse of operation $\mathcal{B}$ to be applied.

So every C5-5-triangulation containing this configuration can be constructed from a smaller C5-5-triangulation by applying a $\mathcal{B}$ operation followed by a $\mathcal{D}_5$ operation. This proves part (a). Of course these operations could also be combined to form a single new operation.

Part (b) now follows easily from (a) and part (b) of Theorem 3. ■

Proof of Theorem 2: Theorem 4 shows that all C4-5-triangulations (which include the C5-5-triangulations) can be generated using $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$. The remaining C3-5-triangulations, which are those having separating 3-cycles, can be made from the C4-5-triangulations using only $\mathcal{D}$, as is shown in Theorem 3(a). ■

Computer implementation

The aim of a computer program for the construction of triangulations with minimum degree 5 is to list exactly one member of every isomorphism class. Ideally, such a program should have modest space requirements even when a vast number of graphs are produced, and should be fast enough that generation will not be the bottleneck in most computations where all the outputs are tested for conformance to some non-trivial conjecture.

The first objective, and possibly also the second, is not met by the previously best implementation for the present class of graphs, namely that of Dillencourt [6].

In order to avoid the generation of isomorphic copies, we used the *canonical construction path* method described in [10]. This method considers a sequence of graphs known to include at least one from each isomorphism class, then rejects all but one in each class without explicit isomorphism testing. This is not the place to discuss the exact implementation of the method, but the reader is referred to the source code which can be obtained from `http://cs.anu.edu.au/~bdm/plantri`.

The following lemma is useful in speeding the overall computation, since it reduces the number of graphs which are generated only to be be rejected.

Lemma 5 *Let G be a C5-5-triangulation which can be constructed by a $\mathcal{B}$ operation from the C5-5-triangulation G' which can be constructed by an $\mathcal{A}$ operation. Then there is a C5-5-triangulation G'' from which G can be constructed by an $\mathcal{A}$ operation.*

The main impact of this lemma is that if we never apply a $\mathcal{B}$ operation immediately after an $\mathcal{A}$ operation, we still get a member of each isomorphism class.

Proof: There are two requirements an edge has to fulfill in order to be a possible center edge for an inverse $\mathcal{A}$ operation: It may not lie on a separating 5-cycle (otherwise there would be a separating 4-cycle after the reduction), and both the opposite vertices on the faces incident with the edge must have valency at least 6.

Clearly such an edge exists in G', since G' was formed using an $\mathcal{A}$ operation. We have to show that such an edge exists in G after G is formed from G' using a $\mathcal{B}$ operation.

First suppose that e, the edge in G' which is the central edge created by the $\mathcal{A}$ operation used to form G', is none of the 3 edges depicted vertically on the left hand side of the $\mathcal{B}$ operation in Figure 1. In this case, the opposite vertices on the faces incident with e still have degree at least 6 after the $\mathcal{B}$ operation, since $\mathcal{B}$ does not decrease any vertex degrees. Furthermore it can be seen that any possible separating 5-cycle in G through e would correspond to a separating 5-cycle or even a separating 4-cycle through e in G', which is not possible as G' is a C5-5-triangulation.

Suppose instead that e be one of the 3 initial edges of the $\mathcal{B}$ operation (those drawn vertically in Figure 1), w.l.o.g. the central one or the upper one. Figure 6 shows the $\mathcal{B}$ operation forming G from G' and part of its neighbourhood. A square surrounding a vertex on the right side shows that the vertex must have degree at least 6, either because the $\mathcal{B}$ operation forces it or because the preceding $\mathcal{A}$ operation forces it. Some edges are drawn bold or dashed for reference.

We see that the two opposite vertices on the faces incident with the bold edge have valency at least 6, so this edge is a candidate for an inverse $\mathcal{A}$ operation. So suppose this edge is on a separating 5-cycle. If this cycle uses one of the dashed edges, there would

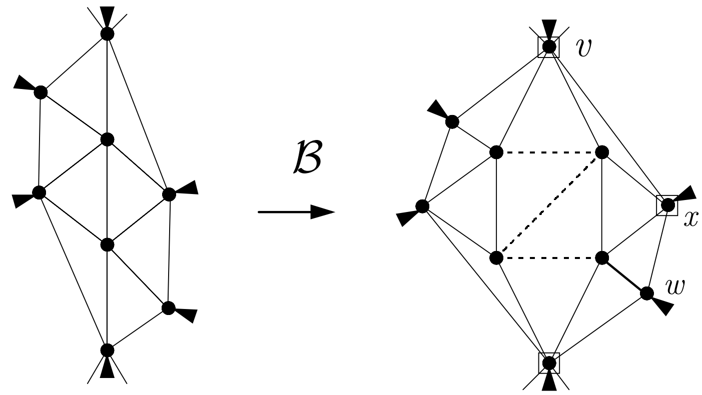

Figure 6: Following an $\mathcal{A}$ operation by a $\mathcal{B}$ operation

be a separating 4-cycle in G'. If the cycle does not use any of the dashed edges then the fact that without separating 4-cycles present every separating 5-cycle has to have edges sticking in and out at every vertex implies that it has to pass through vertex v. But then a shortcut through x would give a separating 4-cycle, since x can not be the only vertex inside the separating 5-cycle. None of these possibilities can happen, since G' is a C5-5-triangulation. Therefore, the bold edge is the center of a valid inverse $\mathcal{A}$ operation, proving the lemma. ■

Results

We now present some counts obtained by our program. Two types of equivalence classes are recognised. "Isomorphism classes" permit orientation-reversing (reflectional) isomorphisms, whereas "orientation-preserving (O-P) isomorphism classes" do not.

In addition, we give some counts of convex polytopes (equivalent to 3-connected planar graphs) with minimum degree 5. These can be generated by successively removing edges from C3-5-triangulations without violating the degree and connectivity conditions. In the tables, n, e and f are the numbers of vertices, edges and faces, respectively.

Some checks on the results are available. Aldred et al. [1] found the numbers of C3-5-trianguations and C4-5-triangulations up to 25 vertices, and C5-5-triangulations up to 27 vertices. An unpublished program of ours, using quite a different method, gave the same results up to 34 vertices.

Gao, Wanless and Wormald [8] theoretically determined the number of 5-connected planar triangulations which are rooted at a flag. By finding the automorphism group of each of the generated graphs, we have matched their values up to 38 vertices.

We can incidentally tidy up a loose end from [1]. The smallest nonhamiltonian cubic simple planar graphs of girth 5 with cyclic 3-cuts have 48 vertices. There are two such graphs formed by joining together the two fragments shown in Figure 7. Either join a–A, b–B, c–C, d–D, or join a–C, b–D, c–A, d–B.

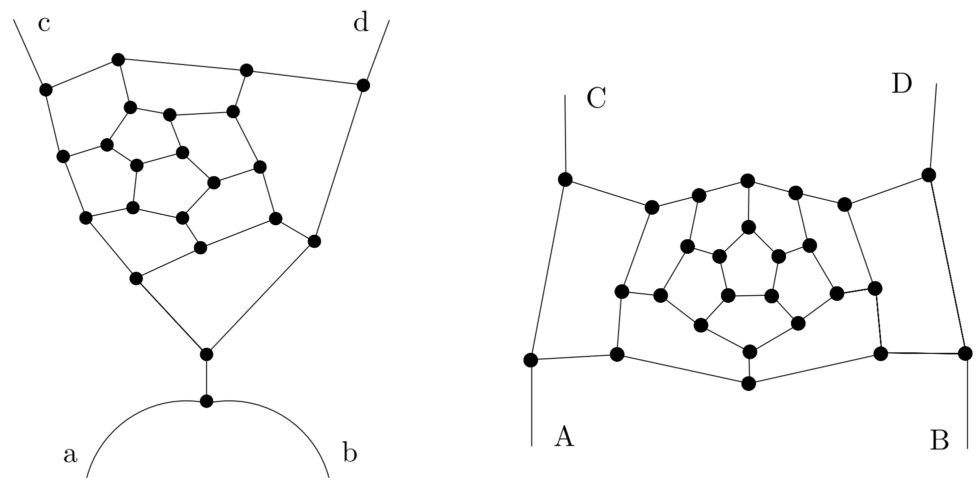

Figure 7: Nonhamiltonian planar cubic graphs of girth 5 with cyclic 3-cuts

Final note

The main theorem from Batagelj's paper [3] was independently proven in this article. Another proof is to use the method of our Lemma 3(a) to remove all separating 3-cycles at the beginning, after which the remainder of Batagelj's argument applies correctly. However, neither of these two ways to prove the theorem also gives a proof of the additional remark at the end of Batagelj's article that operation $\mathcal{D}$, which does not increase the number of vertices, can be replaced by two other operations which do. The difficulty is that both the correct proofs use $\mathcal{D}$ in ways that it was not used by the original incorrect proof. It would be interesting to know whether Batagelj's remark is nevertheless true.

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n	C3-5-triangulations	C4-5-triangulations	C5-5-triangulations
12	1	1	1
13	0	0	0
14	1	1	1
15	1	1	1
16	3	3	3
17	4	4	4
18	12	12	12
19	23	23	23
20	73	73	71
21	192	191	187
22	651	649	627
23	2070	2054	1970
24	7290	7209	6833
25	25381	24963	23384
26	91441	89376	82625
27	329824	320133	292164
28	1204737	1160752	1045329
29	4412031	4218225	3750277
30	16248772	15414908	13532724
31	59995535	56474453	48977625
32	222231424	207586410	177919099
33	825028656	764855802	648145255
34	3069993552	2825168619	2368046117
35	11446245342	10458049611	8674199554
36	42758608761	38795658003	31854078139
37	160012226334	144203518881	117252592450
38	599822851579	537031911877	432576302286
39	2252137171764	2003618333624	1599320144703
40	8469193859271	7488436558647	5925181102878

Table 1: Isomorphism classes of triangulations with minimum degree 5

n	C3-5-triangulations	C4-5-triangulations	C5-5-triangulations
12	1	1	1
13	0	0	0
14	1	1	1
15	1	1	1
16	4	4	4
17	4	4	4
18	17	17	17
19	33	33	33
20	117	117	115
21	331	330	325
22	1180	1177	1144
23	3899	3874	3736
24	14052	13910	13225
25	49667	48878	45904
26	180502	176538	163456
27	654674	635653	580704
28	2398527	2311572	2083116
29	8800984	8415829	7485349
30	32447008	30785420	27033550
31	119883207	112855620	97890740
32	444226539	414972649	355702718
33	1649550311	1529287903	1296014495
34	6138874486	5649427132	4735513531
35	22890091062	20914166059	17347212127
36	85511947468	77587152924	63705666521
37	320013030067	288398164702	234500056176
38	1199620598580	1074044692104	865141832437
39	4504219709753	4007195731866	3198618016486
40	16938267502048	14976784750710	11850315368675

Table 2: O-P isomorphism classes of triangulations with minimum degree 5

n	e	f	all classes	O-P classes
12	30	20	1	1
12	total		1	1
13	total		0	0
14	36	24	1	1
14	total		1	1
15	39	26	1	1
15	total		1	1
16	40	26	1	1
16	41	27	1	1
16	42	28	4	3
16	total		5	6
17	43	28	1	1
17	44	29	3	3
17	45	30	4	4
17	total		8	8
18	45	29	2	1
18	46	30	12	7
18	47	31	15	10
18	48	32	17	12
18	total		30	46
19	48	31	4	3
19	49	32	40	24
19	50	33	58	35
19	51	34	33	23
19	total		85	135
20	50	32	9	6
20	51	33	63	37
20	52	34	244	136
20	53	35	253	140
20	54	36	117	73
20	total		392	686
21	53	34	45	26
21	54	35	433	231
21	55	36	1135	598
21	56	37	1017	540
21	57	38	331	192
21	total		1587	2961

Table 3: Polytopes with minimum degree 5

n	e	f	all classes	O-P classes
22	55	35	24	14
22	56	36	616	325
22	57	37	3005	1550
22	58	38	5734	2955
22	59	39	4185	2162
22	60	40	1180	651
22	total		7657	14744
23	58	37	365	196
23	59	38	5058	2591
23	60	39	18274	9270
23	61	40	26814	13615
23	62	41	16797	8549
23	63	42	3899	2070
23	total		36291	71207
24	60	38	173	96
24	61	39	5497	2810
24	62	40	39974	20206
24	63	41	104898	52823
24	64	42	125146	63095
24	65	43	67568	34124
24	66	44	14052	7290
24	total		180444	357308
25	63	40	3307	1694
25	64	41	56820	28649
25	65	42	275764	138525
25	66	43	567010	284520
25	67	44	565701	284102
25	68	45	269342	135439
25	69	46	49667	25381
25	total		898310	1787611

Table 4: Polytopes with minimum degree 5 (continued)

n	e	f	all classes	O-P classes
26	65	41	990	518
26	66	42	54028	27247
26	67	43	501717	251687
26	68	44	1764979	884431
26	69	45	2943645	1474446
26	70	46	2524800	1265456
26	71	47	1071577	537493
26	72	48	180502	91441
26	total		4532719	9042238
27	68	43	29075	14674
27	69	44	628215	315002
27	70	45	3880657	1943074
27	71	46	10560455	5285560
27	72	47	14761187	7387374
27	73	48	11080030	5547143
27	74	49	4245308	2126514
27	75	50	654674	329824
27	total		22949165	45839601
28	70	44	7689	3917
28	71	45	522777	262170
28	72	46	6121002	3064076
28	73	47	27332100	13674643
28	74	48	60132817	30081720
28	75	49	72069944	36052160
28	76	50	48089612	24062148
28	77	51	16782891	8400155
28	78	52	2398527	1204737
28	total		116805726	233457359
29	73	46	258217	129558
29	74	47	6784218	3395462
29	75	48	51937427	25980495
29	76	49	178953032	89502100
29	77	50	328554612	164317521
29	78	51	344079630	172082986
29	79	52	206511268	103295735
29	80	53	66186792	33113060
29	81	54	8800984	4412031
29	total		596228948	1192066180

Table 5: Polytopes with minimum degree 5 (continued)

n	e	f	all classes	O-P classes
30	75	47	59206	29821
30	76	48	5075116	2540458
30	77	49	72280336	36153637
30	78	50	398489524	199284603
30	79	51	1106343494	553245996
30	80	52	1736780076	868499404
30	81	53	1612816382	806515573
30	82	54	879491006	439841613
30	83	55	260584336	130336575
30	84	56	32447008	16248772
30	total		3052696452	6104366484
31	78	49	2287156	1145111
31	79	50	72031083	36028132
31	80	51	667247944	333673154
31	81	52	2825865636	1413054897
31	82	53	6528731430	3264576190
31	83	54	8930094363	4465329366
31	84	55	7443174579	3721853265
31	85	56	3718075225	1859260375
31	86	57	1024362305	512281901
31	87	58	119883207	59995535
31	total		15667197926	31331752928
32	80	50	479446	240430
32	81	51	48918024	24468620
32	82	52	832689068	416399311
32	83	53	5534305556	2767321897
32	84	54	18823569658	9412162103
32	85	55	37081796296	18541480725
32	86	56	44865765346	22433623830
32	87	57	33900894153	16951098902
32	88	58	15621888283	7811471882
32	89	59	4021998166	2011226628
32	90	60	444226539	222231424
32	total		80591725752	161176530535

Table 6: Polytopes with minimum degree 5 (continued)

n	e	f	all classes	O-P classes
33	83	52	20295368	10152741
33	84	53	753810321	376951752
33	85	54	8298153553	4149278837
33	86	55	42221707361	21111408725
33	87	56	119140021626	59571105445
33	88	57	203983308997	101993247858
33	89	58	221009334051	110506546904
33	90	59	152667508151	76335350545
33	91	60	65285438093	32643939837
33	92	61	15775800762	7888416533
33	93	62	1649550311	825028656
33	total		415411427833	830804928594
34	85	53	3910515	1957382
34	86	54	469623164	234846981
34	87	55	9395720509	4698066344
34	88	56	73945022947	36973254903
34	89	57	301216777356	150610142121
34	90	58	722797642328	361402022519
34	91	59	1092105078640	546056821115
34	92	60	1070446321676	535227995999
34	93	61	680819405952	340413582639
34	94	62	271578632193	135792191605
34	95	63	61829568488	30915951931
34	96	64	6138874486	3069993552
34	total		2145396827091	4290746578254
35	88	55	180309786	90171828
35	89	56	7799068373	3899705466
35	90	57	100504272959	50252955201
35	91	58	603515614576	301760294018
35	92	59	2033897372915	1016954066033
35	93	60	4231358798972	2115688345019
35	94	61	5712927114015	2856474952904
35	95	62	5109255971021	2554640081343
35	96	63	3010312797687	1505165810142
35	97	64	1125185937779	562599608075
35	98	65	242171956724	121088625406
35	99	66	22890091062	11446245342
35	total		11100060860777	22199999305869

Table 7: Polytopes with minimum degree 5 (continued)