Small Hypohamiltonian Graphs

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Abstract

A graph G is said to be hypohamiltonian if G is not hamiltonian but for each $v \in V(G)$ the vertex deleted subgraph $G - v$ is hamiltonian. In this paper we show that there is no hypohamiltonian graph on 17 vertices and thereby complete the answer to the question, "for which values of n do there exist hypohamiltonian graphs on n vertices?". In addition we present an exhaustive list of hypohamiltonian graphs on fewer than 18 vertices and extend previously obtained results for cubic hypohamiltonian graphs.

1. Introduction

Definition. A graph G is said to be *hypohamiltonian* if it is not hamiltonian but for each $v \in V(G)$ the vertex deleted subgraph $G - v$ is hamiltonian.

Hypohamiltonian graphs first appeared in the literature in response to a problem of Sousselier [9]. The solution, due to Gaudin, Herz and Rossi [5] established that the Petersen graph is the smallest hypohamiltonian graph. Since that time Herz Duby and Vigué [6] have used exhaustive computer searches to show that there are no hypohamiltonian graphs on 11 or 12 vertices. A later exhaustive search by Collier and Schmeichel [3] showed that there is no hypohamiltonian graph on 14 vertices. For $n = 10, 13, 15, 16$ and for $n \geq 18$ it is known that there are hypohamiltonian graphs of order n (see [1], [2], [4], [6], [8], [10], [11]) leaving only the case of $n = 17$ to be determined. (A fuller rundown of the evolution of the problem may be found in Chapter 7 of [7].) In section 2 we present details of a computer search employed by the authors to establish the following result.

Theorem. *There is no hypohamiltonian graph on 17 vertices.*

The search techniques were also employed to search exhaustively for hypohamiltonian graphs on 13, 15, and 16 vertices thereby yielding a complete list of hypohamiltonian graphs on fewer than 18 vertices. Exhaustive searching of graphs on more than 17 vertices is too costly using the current methods but when applied to cubic graphs, with some girth restrictions, the procedure is reasonably quick and efficient. This application has been pursued and the results have been included in tabular form in Section 3. For other studies of cubic, planar and infinite hypohamiltonian graphs, the reader might consult [12], [13], [14].

2. The Computation Method

This is a summary of our computation method for generating small hypohamiltonian graphs.

Definition. A graph G is *hypocyclic* if $G - v$ is Hamiltonian for each $v \in VG$.

Thus, *hypohamiltonian* = *nonhamiltonian* + *hypocyclic* .

Definition. For a possibly empty graph G, define $p(G)$ to be the minimum number of vertex disjoint paths needed to cover VG.

Definition. Define an invariant $k(G)$ for possibly empty G as follows.

(1) If G is empty, $k(G) = 0$.

(2) Else, if G has no isolated vertices or edges,

$$k(G) = \max \{1, \lceil((\text{number of vertices of degree } 1)/2)\rceil\}.$$

(3) Else, $k(G)$ = number of isolated vertices and edges + k(remainder of G).

Lemma 1. $k(G) \le p(G)$.

Proof. The truth of the lemma is easily seen. ∎

We next describe some “obstructions” for hypocyclicity. In each case, an *obstruction* is a disjoint non-trivial partition $VG = W \cup X$.

Type-A Obstruction: $p(\langle W \rangle) \ge |X|$.

Type-B Obstruction: $k(\langle W\rangle) \geq |X|$.

Type-C Obstruction: W is an independent set. Furthermore, for some vertex $v \in X$, defining n_1 and n_2 to be the number of vertices of $X - v$ joined to one or more than one vertex of W (respectively), we have $2n_2 + n_1 < 2|W|$.

Lemma 2. *If G has an obstruction (W, X) of type-A, -B or -C, then G is not hypocyclic.*

Proof. Let v be in X, and consider a hamiltonian cycle C in $G - v$. The number of components of C restricted to W must be at least $p(\langle W\rangle)$, so the number of components in C restricted to $X - v$ must also be at least $p(\langle W\rangle)$. This handles type-A and, by Lemma 1, type-B obstructions. For type-C, consider the same cycle C. The number of edges of C between W and X must be $2|W|$, but X can supply at most $2n_2 + n_1$. ∎

Lemma 3([C-S 78]). *If a hypohamiltonian graph has a vertex v of degree 3, then v lies on no triangles.* ∎

Definition. Let G be a nonhamiltonian graph, with maximum degree D. Let *top*(G) denote any graph T obtainable by the following process:

(a) set $T := G$
(b) add to T every edge uv such that uv is not an edge of G, $G + uv$ is nonhamiltonian, u and v have degree $< D$ in G. Note that T need not be nonhamiltonian.
(c) Repeat this any number of times you please: Choose one vertex of degree 3 in T, and delete from T any edges joining two of its neighbours.

Lemma 4. *If H is a supergraph of G that is hypohamiltonian and has maximum degree D, then H is a subgraph of top(G).*

Proof. Clearly, the value of graph T after step (b) is a supergraph of every nonhamiltonian supergraph of G having maximum degree D. If, after that, vertex v has degree 3 in T we know that the neighbourhood of v in T must be the neighbourhood of v in any hypohamiltonian graph between G and T (since hypohamiltonian graphs cannot have vertices of degree 2). In that case Lemma 3 is violated unless we remove edges between the neighbours of v. ∎

Note that *top*(G) might not be a supergraph of G because step (c) might remove some edges that are in G. The lemma still holds, implying in such case that G has no hypohamiltonian supergraphs of maximum degree D.

Corollary 5. *If G is hamiltonian or $top(G)$ is not hypocyclic, there is no hypohamiltonian supergraph of G with the same maximum degree.* ∎

We can now describe the search for hypohamiltonian graphs of order n and maximum degree D.

Take a vertex, called the *hub*, and an $(n-1)$-cycle disjoint from it. We assume that the degree of the hub is D. There are only a small number of nonequivalent ways to join the hub to the cycle by D edges without creating a hamiltonian cycle. Given one of those ways, by the *feet* we mean the vertices of the cycle adjacent to the hub.

It is clear that the set S of graphs consisting of the $(n-1)$-cycle and D edges from the hub to the cycle has this property:

P: For any hypohamiltonian graph H of order n and maximum degree D, there is an isomorph of H which lies in the interval $[G, top(G)]$ for some G in S.

The basic idea is to iteratively replace members of G by other graphs while maintaining property P. By Corollary 5, we can delete from S any G which is hamiltonian, or for which $top(G)$ is not hypocyclic. The efficiency depends very strongly on the order in which the replacement is performed. We have found that a good general technique is to look for obstructions in G and add edges which destroy them. Also, the continual use of tests for hypocyclicity of $top(G)$ can be replaced except at the final stages by a much faster obstruction test.

We will describe this process in terms of a series of filters, where each filter takes out a member of S satisfying some predicate and replaces it by zero or more new graphs. This is repeated until no graphs in S satisfy the predicate. In all cases, potential members G of S are rejected if $top(G)$ is found to have an obstruction.

Phase One.

In this phase, each graph G in S is associated with a subset $W = W(G) \subseteq VG$. Initially, each G (cycle plus edges from the hub) has W equal to the set of vertices which are not feet, including the hub.

Predicate: $(W, VG - W)$ is a type-A obstruction.

Action: Consider each non-edge e joining two components of W, such that $G + e$ has maximum degree D and is not Hamiltonian. The replacement for (G, W) is the set of $(G + e, W')$, where W' is obtained from W by deleting every vertex which has degree 3 in $\langle W \rangle$.

The subgraph $\langle W \rangle$ is always a set of disjoint paths, so testing if $(W, VG - W)$ is a type-A obstruction is easy. Furthermore, $(W, VG - W)$ is destined to remain a type-A obstruction unless we decrease $p(\langle W \rangle)$, which requires an edge between two components of $\langle W \rangle$. Thus, property P is preserved.

Phase Two.

During this phase we intermingle three filters. Our program chooses a "best" application of one of the first two, if one of them applies, otherwise it attempts the third filter. In attempting to apply the third filter, we looked at all partitions $(W, VG - W)$ where W consists of an independent set and the vertices not adjacent to it.

Predicate: G has a vertex v of degree 2.
Action: Replace G by the set of $G + e$, where e is an edge in $top(G) - G$ which is incident with v.

Predicate: G has a vertex v of degree 3 incident with a triangle.
Action: Replace G by the set of $G + e$, where e is an edge in $top(G) - G$ which is incident with v.

Predicate: G has a type-B obstruction (W, X).
Action: Replace G by the set of $G + e$, where e is an edge in $top(G) - G$ that joins two vertices of W, at least one of these vertices having degree 0 or 1 in $\langle W \rangle$.

The preservation of property P under the first and second filters is obvious. To see this in the case of the third filter, note that only edges of the type mentioned can reduce $k(\langle W \rangle)$ and so eliminate this type-B obstruction.

Phase Three.

In the third phase, hypohamiltonian graphs are identified exhaustively.

Predicate: true
Action: If G is hypohamiltonian, output it. Replace G by the set of $G + e$, where e is an edge in $top(G) - G$.

Property P is violated only to the extent that hypohamiltonian graphs which are missing have been output. Eventually, the set S will be empty and we will have found isomorphs of all hypohamiltonian graphs. Clearly this process can only be efficient if the difference between G and $top(G)$ is small, but this was always true after Phase Two in our computations (rarely more than a few edges).

Summary.

Phase One:

$n =$	**12**	**13**	**14**	**15**	**16**	**17**
inputs	9	8	23	26	55	71
outputs	8	17	228	1721	20600	187798
cpu time	0.8s	6s	73s	16m	5h	211h

The output counts are after isomorph rejection separately for each input graph. This was performed using the second author's program "nauty".

Phase Two:

$n =$	**12**	**13**	**14**	**15**	**16**	**17**
outputs	0	1	0	3	17	126
cpu time	0.2s	2s	19s	7.5m	3h	272h

In this case, the output counts are for nonisomorphic outputs.

Phase Three:

$n =$	**12**	**13**	**14**	**15**	**16**	**17**
outputs	0	1	0	1	4	0
maxdim	-	0	-	2	5	12
cpu time	-	-	-	-	3s	145s

Again, complete isomorph rejection was done. The value "maxdim" is the maximum number of edges between G and $top(G)$.

In the case of $n = 17$ we did many parts of the computation twice using different procedures to find cycles (one kindly provided by Gordon Royle). In Figure 1, we include drawings of all hypohamiltonian graphs on fewer than 18 vertices.

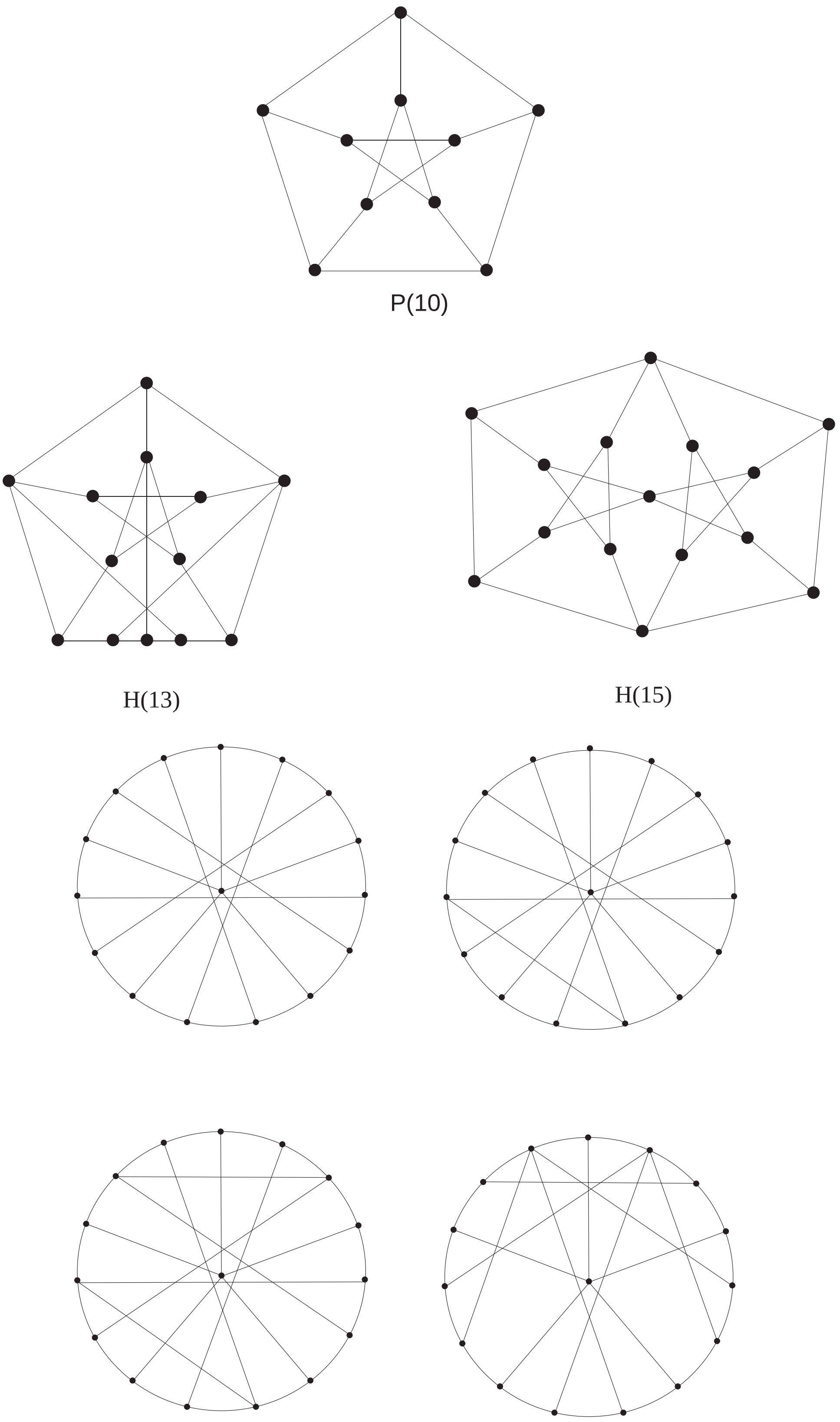

Figure 1: All the hypohamiltonian graphs on fewer than 18 vertices

3. Cubic hypohamiltonian graphs

The same method as above, beginning directly at Phase Two, is quite fast at finding the cubic hypohamiltonian graphs for, perhaps, $n \leq 20$. However it begins to become rather slow at greater sizes, so we used a more crude approach. We ran the program "minibaum3" of Gunnar Brinkmann and Carsten Saager to find all triangle-free cubic graphs with $n \leq 26$, and rejected those that were not hypohamiltonian. Note that, in the light of Lemma 3, all cubic hypohamiltonian graphs have girth at least 4, so this procedure is indeed exhaustive. The results were as follows.

Counts of connected cubic graphs

n	Girth searched	Total	Non-hamiltonian	Nonham. & 3-connected	Hypo-hamiltonian
10	≥ 4	6	1	1	1
12	≥ 4	22	0	0	0
14	≥ 4	110	2	1	0
16	≥ 4	792	8	3	0
18	≥ 4	7805	59	20	2
20	≥ 4	97546	425	129	1
22	≥ 4	1435720	3862	1166	3
24	≥ 4	23780814	41293	12652	1
26	≥ 4	432757568	518159	162969	100
28	≥ 5	656783890	239126	218556	34
30	≥ 6	122090544	1	1	1
32	≥ 7	30368	0	0	0
34	≥ 8	1	0	0	0
36	≥ 8	3	0	0	0
38	≥ 8	13	0	0	0

The girth distributions of the cubic hypohamiltonian graphs indicated in the table above are as follows. Of girth 4 we found 1 on 24 and 4 on 26 vertices. Of girth 6 we found 1 on 28 and 1 on 30 vertices. Of girth 7 we found 1 on 28 vertices. All of the remaining cubic hypohamiltonian graphs found in the search are of girth 5.

In addition to the two cubic hypohamiltonian graph on 18 vertices, we found

11 others but our search was not exhaustive and there may be others.

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