Asymptotic enumeration of graphs with a given upper bound on the maximum degree

Brendan D. McKay†
Department of Computer Science
Australian National University
Canberra ACT 0200 Australia

bdm@cs.anu.edu.au

Ian M. Wanless
Christ Church
Oxford OX1 1DP UK

wanless@maths.ox.ac.uk

Nicholas C. Wormald†
Department of Mathematics and Statistics
University of Melbourne
Parkville Vic 3052 Australia

nick@ms.unimelb.edu.au

Consider the class of graphs on n vertices which have maximum degree at most $\frac{1}{2}n - 1 + \tau$, where $\tau \geq -n^{1/2+\epsilon}$ for sufficiently small $\epsilon > 0$. We find an asymptotic formula for the number of such graphs and show that their number of edges has a normal distribution whose parameters we determine. We also show that expectations of random variables on the degree sequences of such graphs can often be estimated using a model based on truncated binomial distributions.

1. Introduction

Given the variety of solved problems on graph enumeration and random graphs, the following question is quite natural: what is the probability that a random graph on n vertices has maximum vertex degree at most $\frac{1}{2}n$? It is hard to trace the origin of this question, but Bollobás [6] attributes it to Sós about 1982. Alon [2] showed that the probability is at most $2^{-n/2}$, whilst the lower bound 2^{-n} comes easily from the FKG inequality. Riordan and Selby [12] found the exact value of the limit of the nth root of the probability, numerically $.6102304\cdots$ (and the corresponding limit for edge probabilities

† Research Supported by the Australian Research Council.

other than $\frac{1}{2}$). The question is of course equivalent to enumerating n-vertex graphs with all vertex degrees at most $\frac{1}{2}n$.

In this paper we solve a generalization of this question. Namely, we find an asymptotic formula for the number $G_{\le k}(n)$ of graphs on n vertices with maximum degree $\Delta \le k$, where $k = \frac{1}{2}n - 1 + \frac{1}{2}Tn^{1/2}$ with $T = O(n^{\epsilon})$ for some fixed $\epsilon > 0$. We will meet several situations where ϵ needs to be sufficiently small, so without further ado we assume that ϵ is small enough to meet all those needs simultaneously. We also note at this time that all asymptotics in this paper are for $n \to \infty$ and that constants implicit in our order notation are independent of any other variable.

Apart from the approximations mentioned above, the only prior work we are aware of is that of Bollobás [5], who determined $\mathbf{P}(\Delta \le k)$ in the case where it is constant or decreasing very slowly.

Our general approach is to use the asymptotic formula of McKay and Wormald [9] for the number of graphs with given degrees. In principle we just need to sum the formula over all relevant degree sequences, but such a multidimensional sum has considerable difficulties. Instead, we use the asymptotic formula to estimate the difference between the degree sequence of a random graph with degrees restricted to $\Delta \le k$, and a sequence of independent binomial variables subject to the same upper bound and restricted to even sum. The same approach was used by McKay and Wormald [11] to derive properties of the degree sequence of a random graph.

Theorem 1. *Let $k = \frac{1}{2}n - 1 + \frac{1}{2}Tn^{1/2}$ be an integer, where $T = O(n^{\epsilon})$ for a fixed, sufficiently small $\epsilon > 0$. Then the number of graphs on n vertices with maximum degree at most k is asymptotic to*

$$2^{\binom{n}{2}}\Bigl(\Phi(L_0+T)e^{-L_0^2/2}\Bigr)^n \frac{\exp\bigl(-\frac{1}{12}L_0(6L_0^3+11L_0^2T-15L_0+4L_0T^2-11T-T^3)\bigr)}{\sqrt{2L_0^2+L_0T+1}},$$

where $L_0 = L_0(T)$ is the unique solution of $\phi(L_0+T) = L_0\Phi(L_0+T)$. Here Φ, defined in terms of the normal density, is

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt \quad \text{where} \quad \phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

Note that the restriction of k to integer is essential, as the value is sensitive to changes in k as small as $n^{-1/2}$.

If $T \to \infty$ sufficiently quickly, most of the formula in Theorem 1 becomes insignificant. The leading asymptotic term was established by Bollobás [5] in the case that the result is not a vanishingly small proportion of the total number of graphs on n vertices, that is, $e^{-T^2/2}/T = O(n^{-1})$. Our result shows the following simplification under the weaker condition $e^{-T^2} = o(n^{-1})$, which permits T to grow more slowly.

Corollary 1. *Let $T \to \infty$ in Theorem 1 (without the $O(n^\epsilon)$ restriction). Then the number of graphs on n vertices with maximum degree at most k is*

$$2^{\binom{n}{2}} \Phi(T)^n \exp\big(O(ne^{-T^2}) + o(1)\big).$$

We will also give a simpler version of Theorem 1 in the case where the degree bound is quite close to $\frac{1}{2}$. Define $\ell = L_0(0)$, the unique solution of the equation $\phi(\ell) = \ell\Phi(\ell)$. The approximate value of ℓ is 0.506054469.

Corollary 2. *Let $k = \frac{1}{2}n - 1 + \tau$ be an integer, where $\tau = o(n^{1/4})$. Then the number of graphs on n vertices with maximum degree at most k is*

$$2^{\binom{n}{2}} \zeta_0 \zeta_1^n \exp\big(\zeta_2 \tau n^{1/2} + \zeta_3 \tau^2 + \zeta_4 \tau^3 n^{-1/2} + o(1)\big),$$

where

$$\zeta_0 = \frac{\exp(\frac{5}{4}\ell^2 - \frac{1}{2}\ell^4)}{\sqrt{1+2\ell^2}} = 1.083878\cdots \qquad \zeta_3 = -\frac{4\ell^2}{1+2\ell^2} = -0.677408\cdots$$

$$\zeta_1 = \frac{e^{-\ell^2}}{\ell\sqrt{2\pi}} = 0.610230\cdots \qquad \zeta_4 = \frac{4\ell(6\ell^2-1)}{3(1+2\ell^2)^3} = 0.104696\cdots$$

$$\zeta_2 = 2\ell = 1.012108\cdots$$

Moreover, many properties of the degree sequence of random graphs subject to a bound on the maximum degree can be computed using our method. Theorem 2 shows how the number of edges is distributed and Theorem 3 shows that, for a fixed number of edges, the asymptotic distribution is very close to one based on independent truncated binomials.

Theorem 2. *Let k, T, ϵ and L_0 be as in Theorem 1. In a random graph on n vertices with maximum degree at most k, the number of edges is distributed asymptotically normally with expectation $\frac{1}{2}\binom{n}{2} - \frac{1}{2}L_0 n^{3/2} + O(n^{1/2+5\epsilon})$ and variance $\frac{1}{8}n^2 \frac{1-2L_0^2-L_0T}{1+2L_0^2+L_0T}\big(1+o(1)\big)$.*

The error term in the expectation is insignificant compared with the standard deviation, which we shall show to be at least the order of $n^{1-\epsilon}$. Our proof of Theorem 2 will give the asymptotic probability that the number of edges has a given value, over a wide range of values.

Define $\mathrm{Bin}_{\le k}(n-1,p)$ to be the truncated binomial distribution: $\mathrm{Bin}(n-1,p)$ truncated at k. Then $R_p = R_p(n,k)$ is the probability space of vectors of n independent components each distributed as $\mathrm{Bin}_{\le k}(n-1,p)$. A property of the binomial distribution is that the slice of R_p containing vectors with a given sum is independent of p, provided $0 < p < 1$. For this reason the choice of $p = \frac{1}{2}$ in the next theorem is unimportant.

Theorem 3. *Let k, T, ϵ and L_0 be as in Theorem 1. Suppose $m = m(n)$ satisfies $n^2/(2\log n) \leq m \leq \lfloor \frac{1}{2}kn \rfloor$. Let F be a nonnegative function defined on integer sequences. Define $\mathbf{E}_1(F,n)$ to be the expectation of F over degree sequences of random graphs with n vertices, m edges, and maximum degree at most k. Similarly, define $\mathbf{E}_2(F,n)$ to be the expectation of F over $R_{1/2}$, conditional on having sum $2m$. Then $\mathbf{E}_1(F,n) \sim \mathbf{E}_2(F,n)$ as $n \to \infty$ provided that the maximum value of F over a sequence of n integers with sum $2m$ and maximum at most k is $O(n^t)\mathbf{E}_2(F,n)$ for some fixed t.*

By taking F to be the characteristic function of an event, Theorem 3 can also be used to estimate probabilities which are not too small.

2. Outline and definitions.

Our approach to estimating $G_{\leq k}(n)$ is to first find the contribution due to graphs with exactly m edges and then sum over the m which contribute significantly. Let $p = m/N$ where $N = \binom{n}{2}$ is the total number of possible edges. Let D_p be the space of degree sequences of random graphs on n vertices where each edge occurs independently with probability p, and let B_p be the space of n-tuples of independent random variables each with distribution $\text{Bin}(n-1,p)$. In [9], the probability of any sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ in the graph model D_p is shown to be asymptotic to the probability of $\mathbf{d}$ in the independent binomial model B_p multiplied by some function $g = g(\mathbf{d}, p)$ which, for present purposes, we can regard as a function of n and k only. This converts the problem to one of estimating the probability that $\mathbf{d}$ in B_p satisfies $\sum d_i = 2m$, conditional upon $d_i \leq k$ for each i, and then summing over m. We find this conditional probability in B_p by relating it to the same probability in B_r, where r is chosen so as to maximise the conditional probability of $\sum d_i = 2m$. Standard central limit methods then give us the answer. Incidentally, consideration of B_r also assists us in establishing the above claim about the behaviour of g.

We next introduce some more notation which will be used throughout, and make some of the statements in the above outline more precise.

For a graph G with vertices $v_1, v_2, \ldots, v_n$ let $\mathbf{d}(G) = \big(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)\big)$ denote the degree sequence of G. For any degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ we use $\Delta = \Delta(\mathbf{d})$, $\delta = \delta(\mathbf{d})$ and $\bar{d} = \bar{d}(\mathbf{d})$ to be respectively the maximum, minimum and mean of the d_j. The degree sequence $\mathbf{d}$ (or, more generally, any sequence $\mathbf{d} = \mathbf{d}(n)$ of sequences indexed by n) is *concentrated* if $|d_j - \bar{d}| \leq n^{1/2+\epsilon^*}$ uniformly over j as $n \to \infty$, where $\epsilon^* > 0$ is a constant implicit in [9]. We can assume that $\epsilon < \epsilon^*/4$. In particular, this allows us to apply the results of Theorem 3 of [9] to our concentrated sequences.

Define V to be the set of all sequences of n integers from the range $[0,k]$, where $k = \frac{1}{2}n - 1 + \frac{1}{2}Tn^{1/2}$. For any integer m, let V_m be the subset of V containing just those sequences with sum $2m$ and let V_m^{con} be the set of concentrated sequences in V_m.

In addition to the probability spaces D_p and B_p defined above, recall the definition of R_p from Section 1. The restriction of R_p to $V_m^{\rm con}$ is independent of p, and we denote it by $R_m^{\rm con}$.

The function g mentioned above is expressed in [9] as a function of γ_2 and p, where γ_2 is a scaled second moment of the components of the sequence $\mathbf{d}$ (defined precisely in (18) below). However, that formula is only valid for concentrated sequences. So we first show (in Section 4) that we can restrict attention to concentrated sequences, with negligible asymptotic loss. Then for each m we estimate the contribution of $V_m^{\rm con}$ to $G_{\leq k}(n)$ as follows. Calculate a number (probability) $r = r(m)$ such that $2m$ is the expected sum of the components of R_r, in order that $\mathbf{P}_{R_r}(V_m^{\rm con})$ is not too small. Thus, when we show that γ_2 is sharply concentrated in R_r, it will follow that it is also sharply concentrated in $R_m^{\rm con}$. In this way, we can find $\mathbf{P}_{D_p}(V_m^{\rm con})$ in terms of $\mathbf{P}_{R_r}(V_m^{\rm con})$ which is simpler to determine.

Since each graph with m edges has probability $p^m(1-p)^{N-m}$ in D_p, the contribution to $G_{\leq k}(n)$ of $V_m^{\rm con}$ is simply $\mathbf{P}_{D_p}(V_m^{\rm con})p^{-m}(1-p)^{m-N}$. Our analysis will show that we can restrict m to a certain interval I_1 of length $O(n^{1+\epsilon})$ in which $\mathbf{P}_{D_p}(V_m^{\rm con})p^{-m}(1-p)^{m-N}$ is asymptotically normally distributed with respect to m. Summing over m then gives Theorem 1.

In the next section some binomial approximations are given. In Section 4 we show that the vast bulk of the graphs we are counting have roughly the same number of edges and all degrees reasonably close to k. In Section 5 we study the integral equation which is central to our formulation of Theorem 1. The key parts of the analysis required for the main theorem are performed in Section 6. Section 7 is concerned with the random variable γ_2. We find its expected value and prove the sharp concentration result mentioned in the plan above. In Section 8 we will bring the various parts together to complete the proof of our main theorems.

3. Binomial Approximations

We need approximations to several functions involving binomial coefficients. These approximations will make use of the following Euler-Maclaurin summation.

Lemma 1. *Let $f(x)$ be a real-valued function such that $f^{(4)}(x)$ is absolutely integrable on $(0,\infty)$. Then for $m \geq 1$ we have*

$$\sum_{j=0}^{m} f(j) = \int_0^m f(x)\,dx + \frac{1}{2}\big(f(0)+f(m)\big) - \frac{1}{12}\big(f'(0)-f'(m)\big) + R(m),$$

where

$$|R(m)| \leq \frac{1}{384}\int_0^m |f^{(4)}(x)|\,dx.$$

Proof: See [14; page 36], for example. ⊙

For the following we assume that T and L are functions of n satisfying $T = O(n^\epsilon)$ and $L = O(n^\epsilon)$, with $\epsilon > 0$ as in Section 1. Define:

$$\begin{aligned} r &= \tfrac{1}{2} - \tfrac{1}{2}Ln^{-1/2} \\ k &= \tfrac{1}{2}n - 1 + \tfrac{1}{2}Tn^{1/2} \\ B &= \binom{n-1}{k} \\ S = S(r,k) &= \sum_{i=0}^{k} \binom{n-1}{i} r^i (1-r)^{n-1-i}. \end{aligned} \tag{1}$$

Our first two results are straightforward. By Stirling's formula we get

$$B = 2^n \phi(T)\Big(\frac{1}{\sqrt{n}} + \frac{T}{n} - \frac{T^4 - 6T^2 + 3}{12n^{3/2}} + O(n^{-3/2-\epsilon})\Big), \tag{2}$$

while from 26.5.1 of [1] we deduce that

$$\frac{dS}{dr} = -(n-1)\binom{n-2}{k} r^k (1-r)^{n-2-k}. \tag{3}$$

Next we put Lemma 1 to work.

Lemma 2.

$$S(r,k) = \Big(\Phi(L+T) + \frac{3L^3 + T^3 + L^2T - LT^2 - 3L + 5T}{12n}\phi(L+T)\Big)\big(1 + O(n^{-1-\epsilon})\big).$$

Proof: Write $S = s_0 \sum_{j=0}^{k} (s_j/s_0)$, where $s_j = \binom{n-1}{k-j} r^{k-j}(1-r)^{n-1-k+j}$. By (2) and Taylor's Theorem, we have

$$\begin{aligned} s_0 = \Big(&\frac{2}{n^{1/2}} + \frac{2(L+T)}{n} - \frac{3L^4 + T^4 + 4L^3T - 12L^2 - 6T^2 - 12LT + 3}{6n^{3/2}} \\ &+ O(n^{-3/2-\epsilon})\Big)\phi(L+T). \end{aligned} \tag{4}$$

To estimate the term s_j/s_0 we write

$$\frac{s_j}{s_0} = \prod_{i=0}^{j-1} \frac{s_{i+1}}{s_i}, \text{ and find } \frac{s_{i+1}}{s_i} = \frac{(n + T\sqrt{n} - 2i - 2)(\sqrt{n} + L)}{(n - T\sqrt{n} + 2i + 2)(\sqrt{n} - L)}.$$

Define $C = \lfloor n^{1/2+2\epsilon} \rfloor$. Then, for $0 \le j \le C$, expanding $\log(s_{i+1}/s_i)$ by Taylor's Theorem and summing gives

$$\frac{s_j}{s_0} = e^{A(j)}\big(1 + B(j) + \Delta(j)\big), \tag{5}$$

where

$$A(j) = \frac{2(L+T)j}{n^{1/2}} - \frac{2j(j+1)}{n},$$
$$B(j) = \frac{2j(L^3+T^3)}{3n^{3/2}} - \frac{2T^2j^2}{n^2} + \frac{8Tj^3}{3n^{5/2}} - \frac{4j^4}{3n^3},$$

and

$$\Delta(j) = O(n^{-1-\epsilon}) \text{ uniformly over } j.$$

It is easy to see that s_j is decreasing for $j \geq C$, and furthermore that $s_C/s_0 = O(e^{-n^\epsilon})$. Also, $B(j) \to 0$ as $n \to \infty$ for $0 \leq j \leq C$. Therefore we have

$$\sum_{j=0}^{k} \frac{s_j}{s_0} = \bigl(1 + O(n^{-1-\epsilon})\bigr) \sum_{j=0}^{C} f(j), \tag{6}$$

where $f(j) = e^{A(j)}\bigl(1 + B(j)\bigr)$.

For $0 \leq x \leq C$, $|f^{(4)}(x)| = f(x)O(n^{-1-\epsilon})$. This means that we can apply Lemma 1 to obtain

$$\sum_{j=0}^{C} f(j) = \Biggl(\Bigl(\tfrac{1}{2}n^{1/2} - \tfrac{1}{2}(L+T) + \frac{3L^4 + T^4 + 4L^3T + 6T^2 + 12LT + 3}{24n^{1/2}} \Bigr) \frac{\Phi(L+T)}{\phi(L+T)} + \frac{3L^3 + T^3 + L^2T - LT^2 - 3L + 5T}{24n^{1/2}} \Biggr) \bigl(1 + O(n^{-1-\epsilon})\bigr). \tag{7}$$

Note that $\Phi(x)/\phi(x) \to \infty$ as $x \to \infty$ and $\Phi(x)/\phi(x) = O(x^{-1})$ as $x \to -\infty$. Thus we can take the product of (4) and (7) to obtain the desired result. ⊙

We note one spin-off of the above proof for later use. By (6) and (7) we have

$$\frac{B}{S} r^k (1-r)^{n-1-k} = \frac{s_0}{S} = O(n^{-1/2+\epsilon}). \tag{8}$$

Finally, we require a good bound on the lower tail of a truncated binomial distribution.

Lemma 3. *Let p satisfy $0 < p < 1$. Let X be a random variable whose distribution is* $\mathrm{Bin}_{\leq K}(n-1, p)$. *Suppose $0 \leq t \leq K$. Then*

$$\mathbf{P}(X \leq K - t) \leq \frac{K(K-1)\cdots(K-t+1)\,(1-p)^t}{(n-K)(n-K+1)\cdots(n-K+t-1)\,p^t}.$$

Proof: Let $b(i) = \binom{n-1}{i} p^i (1-p)^{n-i-1}$ denote the probabilty of the value i in the non-truncated binomial distribution. Then $\mathbf{P}(X \leq K - t) = \sum_{i=0}^{K-t} b(i) / \sum_{i=0}^{K} b(i) \leq \sum_{i=t}^{K} b(i-t) / \sum_{i=t}^{K} b(i) \leq \max_{i=t}^{K} \bigl(b(i-t)/b(i)\bigr)$. It is easy to see that the maximum occurs for $i = K$ and equals the value given in the lemma. ⊙

4. Initial results on degree sequences

Our first task will be to show that we can safely restrict our attention to graphs with concentrated degree sequences and many edges. We do this by first showing that a tight lower bound on the number of edges implies one on the minimum degree good enough to imply concentration, and then that almost all graphs have that required number of edges.

Lemma 4. *Suppose that $2m \geq N - O(n^{3/2+\epsilon})$. Let G be chosen uniformly at random from the graphs on n vertices with m edges and maximum degree $\Delta \leq k$. Then the probability that G has minimum degree $\delta \leq \Delta - n^{1/2+2\epsilon}$ is $O(e^{-n^{\epsilon}})$ as $n \to \infty$.*

Proof: Let v be a given vertex in G and $\mathcal{Z}_v$ the set of vertices at distance at least 2 from v. We use a switching argument to bound the probability that $\deg(v)$ is low. The switching operation is to replace some edge (a, b) with an edge (a, v) or vice versa. Switchings must not create multiple edges or increase Δ beyond k. For any $d < \Delta$, define s^+ to be the number of switchings which increase $\deg(v)$ from d to $d+1$ and s^- to be the number of switchings which decrease $\deg(v)$ from $d+1$ to d. The ratio of these counts is the same as the ratio of the numbers of graphs in each of the two classes.

We get a lower bound for s^+ by noting that there are at least $2m - \deg(v)(\Delta+1)$ ways to choose the edge (a, b) with $a \in \mathcal{Z}_v$. For the reverse operation, there are $n - 1 - \deg(a)$ possible moves involving each edge (a, v), except that some of these choices may breach our bound on Δ. Hence $s^- \leq (d+1)(n-1-\bar{a})$ where $\bar{a}$ is the mean degree of the neighbours of v. The least possible value of $\bar{a}$ occurs when all vertices in $\mathcal{Z}_v$ have degree Δ, which tells us that $(d+1)\bar{a} \geq 2m - (n-d-2)\Delta - (d+1)$. Hence

$$\frac{s^+}{s^-} \geq \frac{2m - d(\Delta+1)}{(d+1)n - 2m + (n-d-2)\Delta}. \tag{9}$$

Note that the bound in (9) increases with m but has an inverse dependence on Δ and d. Also, from the definition of k and the assumption we are making about m in this lemma, $\Delta \leq \frac{1}{2}n + c_1 n^{1/2+\epsilon}$ and $2m \geq \Delta n - c_2 n^{3/2+\epsilon}$ for positive constants c_1 and c_2. So then, for all $d < \Delta - (2c_2+1)n^{1/2+\epsilon}$, (9) implies that

$$\frac{s^+}{s^-} \geq 1 + 4n^{-1/2+\epsilon} + O(n^{-1+2\epsilon}). \tag{10}$$

Let p_v be the probability that $\deg(v) \leq \Delta - n^{1/2+2\epsilon}$. As a consequence of (10),

$$p_v \leq \exp\Big(-\big(n^{1/2+2\epsilon} - O(n^{1/2+\epsilon})\big)\log\big(1 + 4n^{-1/2+\epsilon} + O(n^{-1+2\epsilon})\big)\Big) = O(e^{-n^{2\epsilon}}).$$

The probability that $\delta \leq \Delta - n^{1/2+2\epsilon}$ is trivially bounded above by np_v. ⊙

Lemma 5. *With probability $1 - O(e^{-n^\epsilon})$ as $n \to \infty$, graphs with maximum degree at most k have $m > \frac{1}{2}N - n^{3/2+2\epsilon}$ edges and concentrated degree sequences.*

Proof: Recall that $N = \binom{n}{2}$. The number of graphs with $m = \frac{1}{2}(N - \alpha n)$ edges is

$$\binom{N}{m} \le 2^N \exp\bigl(-\alpha^2 + O(\log n)\bigr).$$

Hence, even without a restriction on the maximum degree, the number of graphs with $m < \frac{1}{2}N - n^{3/2+2\epsilon}$ is

$$2^N O\bigl(\exp(-n^{1+4\epsilon})\bigr). \tag{11}$$

The bound in (11) can be compared to a simple lower bound on $G_{\le k}(n)$. The degree of a given vertex in a random graph on n vertices with edge probability $\frac{1}{2}$ has distribution $\mathrm{Bin}(n-1, \frac{1}{2})$. If the vertex degrees were independent then the probability of having $\Delta \le k = \frac{1}{2}n - 1 + \frac{1}{2}Tn^{1/2}$ would be $S(\frac{1}{2}, k)^n$, where S is given by Lemma 2. The degrees are not independent, but since the events $\deg(v_i) \le k$ for $i = 1, \ldots, n$ are positively correlated, Theorem 3.3 of Alon and Spencer [3, p.78] shows that $2^N S(\frac{1}{2}, k)^n$ is valid as a lower bound. For $T = O(n^\epsilon)$, Lemma 2 tells us that $2^N S(\frac{1}{2}, k)^n$ is at least $2^N \exp\bigl(-O(n^{1+2\epsilon})\bigr)$, which is much larger than (11). It follows that $m \ge \frac{1}{2}N - n^{3/2+2\epsilon}$ with probability at least $1 - O(e^{-n^\epsilon})$.

Uniformly over $m \ge N - n^{3/2+2\epsilon}$, Lemma 4 tells us that almost all graphs with $\Delta \le k$ have concentrated degree sequences. ⊙

We also need a similar lemma about R_r.

Lemma 6. *For $\frac{1}{2} - O(n^{-1/2+\epsilon}) \le r < 1$, sequences in R_r are concentrated with probability $1 - O(e^{-n^{3\epsilon}})$ as $n \to \infty$.*

Proof: Each component d_i of R_r is distributed as $\mathrm{Bin}_{\le k}(n-1, r)$. Applying Lemma 3 with $p = r$, $K = k$ and $t = \lfloor n^{1/2+2\epsilon} \rfloor$, we find that $\mathbf{P}(d_i \le k - n^{1/2+2\epsilon}) = O(e^{-n^{4\epsilon}/2})$. Therefore, with probability at least $1 - O(e^{-n^{3\epsilon}})$, all the components of R_r lie in the interval $[k - n^{1/2+2\epsilon}, k]$, which implies that they are concentrated. ⊙

5. The integral equation.

In this section we look at a particular integral equation which will crop up later in our calculations. We perform the analysis separately here to avoid disrupting the flow of our argument in the next section. The pivotal equation has already appeared in the statement of Theorem 1. It is

$$L = \frac{\phi(L+T)}{\Phi(L+T)}. \tag{12}$$

For a given T, we use L_0 to denote any value of L which satisfies (12). We aim to justify the assertion in Theorem 1 that L_0 is uniquely determined as a function of T, as well as studying other properties of L_0.

For a given T, any solution of (12) corresponds to a point (L_0+T, L_0) on the graph of the function $\phi(x)/\Phi(x)$. This point is also on the line of unit gradient through the point $(T,0)$. Since $\phi(x)/\Phi(x)$ is continuous, positive and strictly decreasing, such a point of intersection exists and is unique. Hence L_0 is determined uniquely. It follows from the corresponding properties of $\phi(x)/\Phi(x)$ that L_0 is a positive, continuous, strictly decreasing function of T.

Suppose $T \to -\infty$. Then by inspection, (12) has no solution unless $L+T \to -\infty$ too. In that case,

$$\frac{1}{L_0} = \frac{\Phi(L_0+T)}{\phi(L_0+T)} = -(L_0+T)^{-1} + (L_0+T)^{-3} - 3(L_0+T)^{-5} + O\big((L_0+T)^{-7}\big),$$

and hence

$$L_0 = -\tfrac{1}{2}T - T^{-1} + 6T^{-3} + O(T^{-5}). \tag{13}$$

Two other cases of interest are $T=0$, in which case $L_0(0) = 0.506054468989\cdots$, and $T \to \infty$, in which case $L_0 \sim \phi(T)$.

A quantity which will be of particular importance is $\xi = 2L_0^2 + L_0 T$. By differentiating (12) we find that

$$\frac{dL_0}{dT} = \frac{-\xi}{1+\xi}. \tag{14}$$

Now L_0 decreases with T and ξ is clearly positive for $T>0$, so by continuity ξ is always positive. We next show that it is below 1. If $\xi = 1$ then $L_0 + T = -L_0 + L_0^{-1}$ so consider $h = \phi(-L_0 + L_0^{-1})/L_0 - \Phi(-L_0 + L_0^{-1})$. It is elementary to establish that

$$\frac{dh}{dL_0} = \frac{\phi(L_0 - L_0^{-1})}{L_0^4} > 0 \quad \text{and} \quad \lim_{L_0 \to \infty} h(L_0) = 0.$$

We infer that there is no (finite) T for which $\xi = 1$. In the case when $T \to -\infty$ we observe that $\xi = 1 - 4T^{-2} + O(T^{-4})$ by (13); whereas $\xi = O(Te^{-T^2/2})$ when $T \to \infty$. We conclude that

$$0 < \xi < 1 \quad \text{and} \quad (1-\xi)^{-1} = O(1+T^2). \tag{15}$$

It then follows from (14) that $\frac{d}{dT}(2L_0 + T) = (1-\xi)/(1+\xi) > 0$. Coupled with (13) we find

$$(2L_0 + T)^{-1} = O(1+|T|). \tag{16}$$

In Figure 1, we show the values of L_0 and ξ as functions of T. We also show the functions F and H, where the formula for $G_{\le k}(n)$ in Theorem 1 is $2^{\binom{n}{2}} F(T)^n H(T)$.

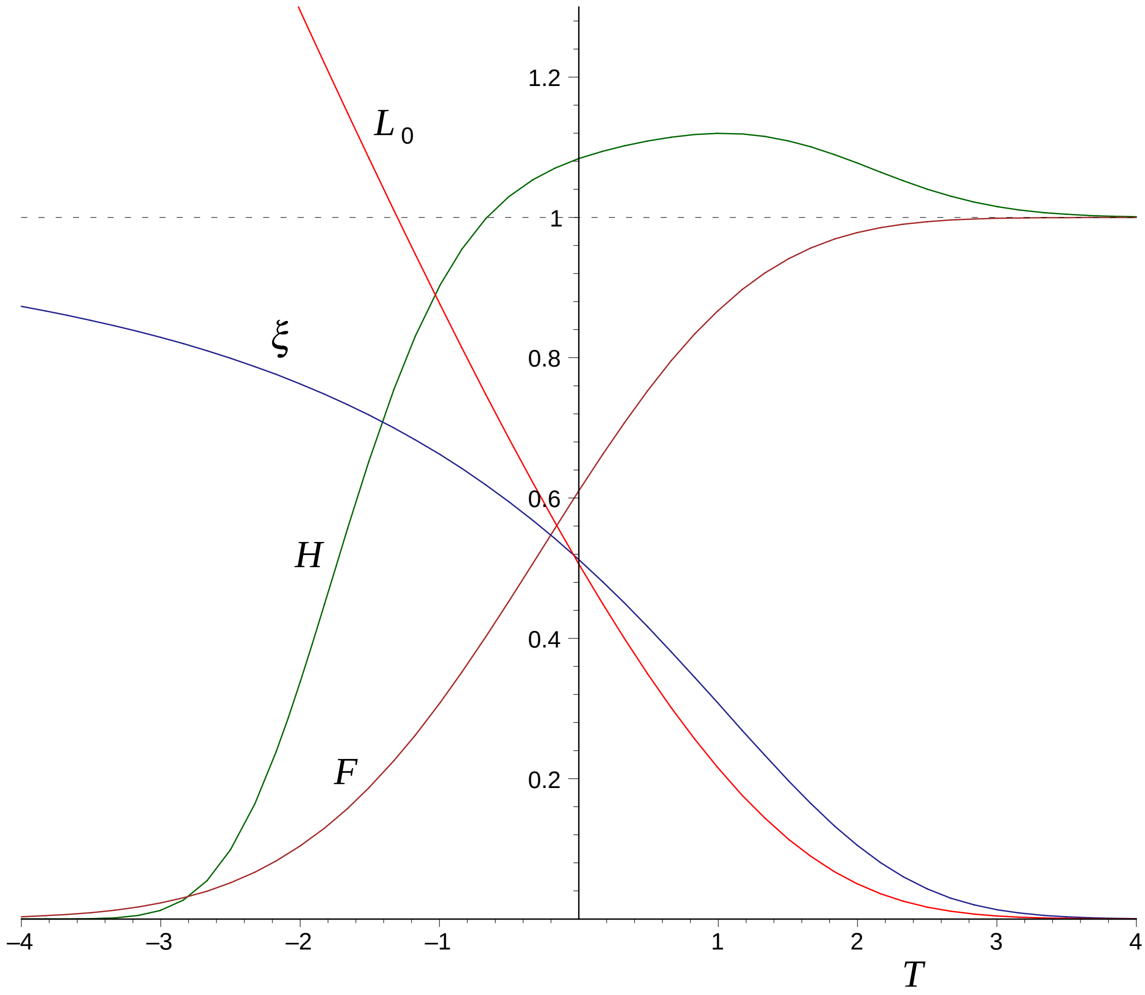

Figure 1. Some important functions of T.

6. The main calculation

As described earlier, we will count the required graphs according to m, the number of edges they contain. Let $p = m/N$ and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a concentrated degree sequence such that $\sum d_i = 2m$. Define I_0 to be the interval $[\frac{1}{2}N - n^{3/2+2\epsilon}, \frac{1}{2}kn]$. Our primary tool is the following result from [9, Thm 3], which relates the graph model to the independent binomials model for a range of m that includes I_0. Uniformly over all $\mathbf{d}$, provided $n \to \infty$,

$$\mathbf{P}_{D_p}(\mathbf{d}) \sim g(\gamma_2)\mathbf{P}_{B_p}(\mathbf{d}) \tag{17}$$

where

$$g(x) = \sqrt{2}\exp\left(\frac{1}{4} - \frac{x^2}{4p^2(1-p)^2}\right),$$

and

$$\gamma_2 = \frac{1}{(n-1)^2}\sum_{i=1}^{n}(d_i-\bar d)^2 = \frac{1}{(n-1)^2}\Big(\sum_{i=1}^{n} d_i^2 - n\bar d^2\Big). \tag{18}$$

Note that for $r \in (0,1)$

$$\begin{aligned}
\mathbf{P}_{B_p}(V_m^{\rm con}) &= \sum_{\mathbf{d}\in V_m^{\rm con}} \mathbf{P}_{B_p}(\mathbf{d}) = \sum_{\mathbf{d}\in V_m^{\rm con}} \mathbf{P}_{R_r}(\mathbf{d})\frac{\mathbf{P}_{B_p}(\mathbf{d})}{\mathbf{P}_{R_r}(\mathbf{d})}\\
&= \sum_{\mathbf{d}\in V_m^{\rm con}} \mathbf{P}_{R_r}(\mathbf{d})\frac{p^{2m}(1-p)^{2N-2m}\prod_i\binom{n-1}{d_i}}{r^{2m}(1-r)^{2N-2m}\prod_i\binom{n-1}{d_i}}\mathbf{P}_{B_r}(V)\\
&= \sum_{\mathbf{d}\in V_m^{\rm con}} \mathbf{P}_{R_r}(\mathbf{d})\frac{p^{2m}(1-p)^{2N-2m}}{r^{2m}(1-r)^{2N-2m}}\big(S(r,k)\big)^n\\
&= \mathbf{P}_{R_r}(V_m^{\rm con})\frac{p^{2m}(1-p)^{2N-2m}}{r^{2m}(1-r)^{2N-2m}}\big(S(r,k)\big)^n,
\end{aligned} \tag{19}$$

where $S(r,k)$ is defined in (1). Summing (17) over $V_m^{\rm con}$, for $m \in I_0$, gives

$$\begin{aligned}
\mathbf{P}_{D_{1/2}}(V_m^{\rm con}) &= \frac{(\frac{1}{2})^N}{p^m(1-p)^{N-m}}\mathbf{P}_{D_p}(V_m^{\rm con}) = 2^{-N}p^{-m}(1-p)^{m-N}\sum_{\mathbf{d}\in V_m^{\rm con}}\mathbf{P}_{D_p}(\mathbf{d})\\
&\sim 2^{-N}p^{-m}(1-p)^{m-N}\sum_{\mathbf{d}\in V_m^{\rm con}} g(\gamma_2)\mathbf{P}_{B_p}(\mathbf{d})\\
&= 2^{-N}p^{-m}(1-p)^{m-N}\mathbf{P}_{B_p}(V_m^{\rm con})\mathbf{E}_{R_m^{\rm con}}\big(g(\gamma_2)\big)\\
&= Q(r,m)\mathbf{P}_{R_r}(V_m^{\rm con})\mathbf{E}_{R_m^{\rm con}}\big(g(\gamma_2)\big),
\end{aligned} \tag{20}$$

where $Q(r,m) = 2^{-N}\big(S(r,k)\big)^n p^m(1-p)^{N-m}r^{-2m}(1-r)^{2m-2N}$, and we used (19) and the fact that the space B_p restricted to $V_m^{\rm con}$ is just $R_m^{\rm con}$, independently of p.

Given m, we would like to choose r such that $V_m^{\rm con}$ is not too small an event in R_r. Note that we can maximise $\mathbf{P}_{R_r}(V_m^{\rm con})$ by minimising $Q(r,m)$, since (19) implies that $\mathbf{P}_{R_r}(V_m^{\rm con})Q(r,m)$ is independent of r.

Now using (3),

$$\frac{1}{Q}\frac{\partial Q}{\partial r} = \frac{2rN-2m}{r(1-r)} + \frac{n}{S}\frac{dS}{dr} = \frac{2rN-2m}{r(1-r)} - \frac{n}{S}(n-1-k)Br^k(1-r)^{n-2-k}, \tag{21}$$

where $B = \binom{n-1}{k}$. Any local extrema of Q as a function of r satisfy $m = \alpha(r)$, where

$$\alpha(r) = rN - \frac{Bn(n-Tn^{1/2})}{4S(r,k)}r^{n/2+Tn^{1/2}/2}(1-r)^{n/2-Tn^{1/2}/2}. \tag{22}$$

What we actually want is to have m determine r, so we now prove that α^{-1} exists in the region of interest, which is the rectangle R where $0 \le m \le \frac{1}{2}kn$ and $0 < r < 1$.

Differentiating the logarithm of $rN - (22)$ using $m = \alpha(r)$ gives

$$\alpha'(r) = N - \frac{(Nr - m)(n^2 - 4m + Tn^{3/2} - 2nr)}{2nr(1 - r)}. \tag{23}$$

After simplification, the equation $\alpha'(r) = 0$ turns out to be linear in r. Its solution $\beta(m)$ is given by

$$\beta(m) = \frac{m(n^2 - 4m + Tn^{3/2})}{nN(n + Tn^{1/2} - 2) - m(4N - 2n)}.$$

Note that the denominator of this expression is monotonically decreasing in m and at $m = \frac{1}{2}kn$ it equals $\frac{1}{2}n^2(n + Tn^{1/2} - 2)$ which is positive. Hence $\beta(m)$ is differentiable inside R. Consider a point P of intersection when the two curves $m = \alpha(r)$ and $r = \beta(m)$ are plotted with r on the vertical axis. We have $\alpha'(r) = 0$, and $\beta'(m)$ is finite at P, so the $\alpha(r)$ curve crosses from below to above the $\beta(m)$ curve at P (as we increase r). In R both curves are continuous so it follows that they intersect at most once, that is $\alpha(r)$ has at most one stationary point. It is not hard to show that $\alpha(r) \to 0^+$ as $r \to 0^+$ and $\alpha(r) \to \frac{1}{2}kn^-$ as $r \to 1^-$. Hence $\alpha(r)$ is monotone, possibly with a single stationary point. We conclude that α^{-1} is defined on $(0, \frac{1}{2}kn)$. We can now deduce by taking $r \to 1^-$ and $r \to 0^+$ in (21) that, for a fixed $m \in (0, \frac{1}{2}kn)$, Q achieves its minimum $\hat{Q} = \hat{Q}(m)$ at some $r \in (0, 1)$ and this must be when $r = \alpha^{-1}(m)$. Henceforth we will choose r to take this value (it will become apparent from our subsequent calculations that we need not worry about the endpoints of the interval $[0, \frac{1}{2}kn]$). This leads by (22) to

$$\hat{Q} = 2^{-N}\left(\frac{Bn(n - Tn^{1/2})}{4(rN - m)}\right)^n p^m(1-p)^{N-m} r^{n^2/2 + Tn^{3/2}/2 - 2m}(1 - r)^{-n^2/2 - Tn^{3/2}/2 + 2m + n}. \tag{24}$$

We have now finished optimising over r and turn our attention to locating the peak of $\hat{Q}(m)$, with a view to summing over m. Differentiating (24) gives

$$\begin{aligned} \frac{d\hat{Q}}{dm} &= \frac{\partial \hat{Q}}{\partial m} + \frac{\partial \hat{Q}}{\partial r}\frac{dr}{dm} + \frac{1}{N}\frac{\partial \hat{Q}}{\partial p} \\ &= \hat{Q} \log \frac{(1 - r)^2 p}{r^2(1 - p)}, \end{aligned} \tag{25}$$

after simplification using (23). We deduce that local extrema of $\hat{Q}$ (with m regarded as a continuous variable) can only occur when $m = \mu(r)$, where

$$\mu(r) = \frac{Nr^2}{r^2 + (1 - r)^2} = rN - \frac{r(1 - r)(1 - 2r)N}{r^2 + (1 - r)^2}. \tag{26}$$

We would like to find the values of m for which $\hat{Q}$ can be large. By Lemma 5, we only have to consider $m = \alpha(r) \in I_0$. By (22), $\alpha(r) < rN$ and so this restricts r to

$r \geq \frac{1}{2} - O(n^{-1/2+2\epsilon})$. To locate the local extrema of $\hat{Q}$ we now turn our attention to solutions of $\alpha(r) = \mu(r)$ in the range I_0. From $\alpha(r) < rN$ and (26) it is immediate that $r < \frac{1}{2}$ for such extrema. Moreover, $\alpha(r) = rN - O(n^{-1/2+\epsilon})N$ by (8) and (22). It follows that $\alpha(r) = \mu(r) \in I_0$ can only hold when $r = \frac{1}{2} - O(n^{-1/2+\epsilon})$, and must hold at least once in this interval, by continuity. Let $m_0 = \alpha(r) = \mu(r)$ be a simultaneous solution of (22) and (26), in the required interval I_0. Later we will show that m_0 is uniquely determined and is a local maximum of $\hat{Q}$. In the meantime all statements we make about m_0 will hold for every local extremum in I_0. Hence we write $r = \frac{1}{2} - \frac{1}{2}Ln^{-1/2}$ and $m = \frac{1}{2}N - \frac{1}{2}\kappa n^{3/2}$ where

$$L = \kappa\bigl(1 + O(n^{-1+2\epsilon})\bigr) = O(n^{\epsilon}) \tag{27}$$

at the extremum, by (26). The last part of (27) comes from the estimate of r above. For this r we note that

$$r^{n/2+Tn^{1/2}/2}(1-r)^{n/2-Tn^{1/2}/2} = 2^{-n}\exp\bigl(-LT - \tfrac{1}{2}L^2 + O(n^{-1+4\epsilon})\bigr). \tag{28}$$

Next, we equate (26) with (22) and substitute (2), (28) and Lemma 2, resulting in an equation $F(L,T) = 0$ which determines the local extrema of $\hat{Q}$. Taking the series of F as $n \to \infty$ we find that

$$\frac{\phi(L+T)}{L} - \Phi(L+T) = o(n^{-1/2}).$$

The similarity of this equation to (12) motivates us to expand $F(L,T)$ around $L = L_0$, where L_0 is the solution of (12). (The existence and uniqueness of L_0 was shown in Section 5.) On performing this expansion, we find that for L corresponding to an extremum,

$$\frac{L}{L_0} = 1 - \frac{6L_0^4 + 5L_0^3T - L_0^2T^2 - 27L_0^2 + L_0T^3 + 5L_0T + T^4 + 6T^2 - 9 + O(n^{-\epsilon})}{12(2L_0^2 + L_0T + 1)n}. \tag{29}$$

Recall that $1 < 2L_0^2 + L_0T + 1 < 2$ as a result of (15). Next we take the logarithm of (24) with the value of B from (2) and eliminate $m = \mu(r)$ and $p = m/N$ using (26), to get

$$\log(\hat{Q}) = -\bigl(L^2 + LT + \tfrac{1}{2}T^2 + \log(L\sqrt{2\pi})\bigr)n + \tfrac{5}{2}L^2 - \tfrac{1}{2}T^2 - \tfrac{1}{3}L^3T - \tfrac{1}{12}T^4 + \tfrac{3}{4} + O(n^{-\epsilon}).$$

Then, by (29) we find the locally extremal value of $\hat{Q}$ is

$$\hat{Q}(m_0) = \left(\frac{\phi(L_0+T)e^{-L_0^2/2}}{L_0}\right)^n \exp\bigl(\tfrac{1}{12}L_0(6L_0^3 + L_0^2T - L_0T^2 + 3L_0 + T^3 + 5T) + o(1)\bigr). \tag{30}$$

Next we need to study behaviour near a local extremum m_0. For all m within $o(n^{3/2-5\epsilon})$ of m_0, equations (23) and (15) tell us that

$$\frac{dr}{dm} = \bigl((\tfrac{1}{2} - L_0^2 - \tfrac{1}{2}L_0T)n^2\bigr)^{-1}\bigl(1 + o(1)\bigr) = O(n^{-2+2\epsilon}) \tag{31}$$

and, by (27),

$$L = \kappa\bigl(1 + o(1)\bigr) = O(n^{\epsilon}). \tag{32}$$

Let $I_1 = [m_0 - n^{1+\epsilon}, m_0 + n^{1+\epsilon}]$. We show later that values of m outside the interval I_1 may be ignored for asymptotic purposes. For m inside I_1, differentiating (25) with respect to m gives,

$$\frac{d^2 \log \hat{Q}}{dm^2} = \frac{1}{p(1-p)N} - \frac{2}{r(1-r)}\frac{dr}{dm} = -8\frac{1 + 2L_0^2 + L_0 T}{1 - 2L_0^2 - L_0 T} n^{-2}\bigl(1 + o(1)\bigr) \tag{33}$$

by (31). Note that the second derivative (33) is negative in I_1 by (15), showing that every local extremum m_0 must be a maximum. The absence of minima in fact shows that m_0 is a unique maximum, which we can assume from now on. We can locate its value by (26) and (29):

$$m_0 = \tfrac{1}{2}N - \tfrac{1}{2}L_0 n^{3/2} + O(n^{1/2+5\epsilon}). \tag{34}$$

As an aside, this tells us that the boundary $\frac{1}{2}kn$ is far away from m_0. Indeed, by (16) $\frac{1}{2}kn - m_0 = \frac{1}{4}n^{3/2}(2L_0 + T) + O(n) > n^{3/2-2\epsilon}e^{O(1)}$ for sufficiently small ϵ.

Next we find that $\frac{d^3}{dm^3}\log \hat{Q} = O(n^{-7/2+7\epsilon})$ and hence from Taylor's theorem

$$\log \hat{Q} = \log \hat{Q}(m_0) + \frac{1}{2}\frac{d^2 \log \hat{Q}}{dm^2}(m - m_0)^2 + O(n^{-7/2+7\epsilon})(m - m_0)^3 \tag{35}$$

for all $m \in I_1$. Summing the contributions over I_1 using (33) gives

$$\sum_{m \in I_1} \hat{Q} = \hat{Q}(m_0)\sqrt{\frac{(1 - 2L_0^2 - L_0 T)\pi}{1 + 2L_0^2 + L_0 T}}\,\bigl(\tfrac{1}{2}n + o(n)\bigr). \tag{36}$$

For $m \notin I_1$, the contributions can be ignored. This is because for $m = m_0 \pm n^{1+\epsilon}$, (33) and (35) show that $\hat{Q}(m) = \hat{Q}(m_0)O(e^{-n^{\epsilon}})$. Together with the knowledge that $\hat{Q}$ is continuous with only one stationary point in I_0, we infer that

$$\hat{Q}(m) = \hat{Q}(m_0)O(e^{-n^{\epsilon}}) \tag{37}$$

for all $m \in I_0 \setminus I_1$.

The above calculation tells us how to sum (20) over m, assuming that the factors other than Q in the right hand side are basically independent of m when $m \in I_1$, and are not too much larger when $m \in I_0 \setminus I_1$. We show in the next section that this is the case.

7. Estimating γ_2 and $\mathbf{P}_{R_r}(V_m^{\mathrm{con}})$.

Our next task is to estimate the effect of γ_2 in equation (17). We will show that γ_2 is sharply concentrated in R_r for the important values of r and determine its expected value.

Throughout this section we consider r for which $m = \alpha(r) \in I_1$ as defined in the previous section, and recall the definition of L there. Hence by (32), $r = \frac{1}{2} - O(n^{-1/2+\epsilon})$ and moreover by (31), $r = \alpha^{-1}(m_0) + O(n^{-1+3\epsilon})$ and so by (29) and (15),

$$L = L_0 + O(n^{-1/2+3\epsilon}). \tag{38}$$

From the definition (18),

$$\tfrac{(n-1)^2}{n}\mathbf{E}(\gamma_2) = \mathbf{E}(d_1^2) - \mathbf{E}(\bar{d}^2) = \mathbf{E}(d_1^2) - \mathbf{E}(\bar{d})^2 - \mathrm{Var}(\bar{d}) = \mathbf{E}(d_1^2) - \mathbf{E}(d_1)^2 - \tfrac{1}{n}\mathrm{Var}(d_1),$$

from which we deduce that

$$\mathbf{E}_{R_r}(\gamma_2) = \tfrac{1}{n-1}\mathrm{Var}(d_1). \tag{39}$$

Now d_1 has distribution $\mathrm{Bin}_{\le k}(n-1, r)$. The probability at $k-j$ is proportional to s_j/s_0, defined in the proof of Lemma 2, which is $\exp\bigl(-2j^2n^{-1} + 2(L+T)jn^{-1/2} + O(n^{-1/2+2\epsilon})\bigr)$ for $j \le n^{1/2+2\epsilon}$, as in (5). We also showed there that $\mathbf{P}(k - d_1 > n^{1/2+2\epsilon}) = O(e^{-n^{\epsilon}})$. Therefore, using Lemma 1, $\mathbf{E}_{R_r}(\gamma_2) = \frac{1}{4}\sigma^2 + O(n^{-1/2+2\epsilon})$, where σ^2 is the variance of the standard normal distribution truncated at $L+T$. This gives

$$\mathbf{E}_{R_r}(\gamma_2) = \frac{1}{4} - \left(\frac{\phi(L+T)}{2\Phi(L+T)}\right)^2 - \frac{(L+T)\phi(L+T)}{4\Phi(L+T)} + O(n^{-1/2+2\epsilon}).$$

From (38), we have $\phi(L+T) = \phi(L_0+T)(1+O(n^{-1/2+4\epsilon}))$ and $\Phi(L+T) = \Phi(L_0+T) + O(n^{-1/2+3\epsilon})\phi(L_0+T)$. Recalling that $\phi(L_0+T) = L_0\Phi(L_0+T)$, we finally obtain

$$\mathbf{E}_{R_r}(\gamma_2) = \tfrac{1}{4} - \tfrac{1}{2}L_0^2 - \tfrac{1}{4}L_0T + O(n^{-1/2+6\epsilon}). \tag{40}$$

We borrow our next lemma directly from [11], and use it to prove that γ_2 is sharply concentrated around the value calculated in (40).

Lemma 7. *Let $X_0, X_1, \ldots$ be a martingale and $\beta \ge 0$, $0 < \rho < 1$. If for all i we have $|X_i - X_{i-1}| \le c$ with probability at least $1 - \upsilon$, and $|X_i - X_{i-1}| \le K$ always, then*

$$\mathbf{P}\bigl(|X_n - X_0| > \beta(c + K\rho)n^{1/2} + nK\rho\bigr) < n\upsilon(1 + 1/\rho) + 2e^{-\beta^2/2}.$$

Lemma 8. *For $\frac{1}{2} - O(n^{-1/2+\epsilon}) \le r < 1$, $\mathbf{P}_{R_r}\bigl(|\gamma_2 - \mathbf{E}_{R_r}(\gamma_2)| > n^{-1/3}\bigr) = O(e^{-n^{\epsilon}})$.*

Proof: Define $\nu_i = \frac{1}{2}n - d_i$ and $\nu_{\max} = \max_i \bigl\{|\nu_i|\bigr\}$. Let $Y = \sum(d_i - \bar{d})^2 = \sum d_i^2 - \frac{1}{n}(\sum d_i)^2$ and $X_j = \mathbf{E}(Y \mid d_1, d_2, \ldots, d_j)$ so that $\{X_0, X_1, \ldots, X_n\}$ is a martingale. Trivially, $0 \le$

$Y \le n^3$ so certainly $|X_j - X_{j-1}| \le n^3$ for each j. However, we know from the proof of Lemma 6 that $\nu_{\max} = O(n^{1/2+2\epsilon})$ with probability $1 - O(e^{-n^{3\epsilon}})$. Now,

$$\begin{aligned} X_j - X_{j-1} &= \tfrac{n-1}{n}\big(d_j^2 - \mathbf{E}(d_j^2)\big) - \tfrac{2}{n}\big(d_j - \mathbf{E}(d_j)\big)\Big(\sum_{i<j} d_i + \sum_{i>j} \mathbf{E}(d_i)\Big) \\ &= \tfrac{n-1}{n}\big(n\mathbf{E}(\nu_j) - n\nu_j + O(\nu_{\max}^2)\big) - 2\tfrac{n-1}{n}\big(\mathbf{E}(\nu_j) - \nu_j\big)\big(\tfrac{1}{2}n - O(\nu_{\max})\big) \\ &= O(n^{1+4\epsilon}), \end{aligned}$$

with probability $1-O(e^{-n^{3\epsilon}})$. We now invoke Lemma 7 with $K = n^3$, $c = n^{1+5\epsilon}$, $\upsilon = e^{-n^{2\epsilon}}$, $\beta = n^\epsilon$ and $\rho = 1/n^3$. It asserts that

$$\mathbf{P}\big(|X_n - X_0| > 2n^{3/2+6\epsilon}\big) = O(n^4 e^{-n^{2\epsilon}}).$$

Since $\gamma_2 = X_n/(n-1)^2$ and $\mathbf{E}(\gamma_2) = X_0/(n-1)^2$ the result follows. ⊙

Looking back at (20), we see that only $\mathbf{P}_{R_r}(V_m^{\text{con}})$ remains to be calculated.

Lemma 9. *Under the conditions stated at the beginning of this section,*

$$\mathbf{P}_{R_r}(V_m^{\text{con}}) \sim \big(\pi(\tfrac{1}{2} - L_0^2 - \tfrac{1}{2}L_0 T)\big)^{-1/2} n^{-1}.$$

Proof: Recall that R_r is a probability space consisting of sequences $Z_1, Z_2, \ldots, Z_n$ of independent copies of a random variable Z. Here Z has distribution $\text{Bin}_{\le k}(n-1, r)$. We are interested in $Y_n = \sum_i Z_i$. Recall that the probability r has been chosen to maximise $\mathbf{P}_{R_r}(Y_n = 2m)$.

We first note that Y_n obeys a central limit theorem – it is asymptotically normally distributed. This follows easily from the Berry-Esséen theorem, noting that the Z_i are i.i.d. random variables (although they do depend on n, which is why a simpler central limit theorem is inadequate).

Next we wish to use a result of Bender [4] which infers a local limit theorem from the corresponding central limit theorem. To do this we need to show that G, the probability generating function for Y_n, is log-concave. (A generating function is said to be log-concave if its sequence of coefficients is.) It is a classical result that the sequence of binomial coefficients is log-concave, and hence so is such a sequence when truncated and scaled. That is, the probability generating function for Z is log-concave. But this means G is also log-concave, since any product of log-concave polynomials is also log-concave. See, for example, Proposition 2 of [13]. Hence we can apply Lemma 2 of [4], to deduce that

$$\lim_{n\to\infty} \sigma \mathbf{P}_{R_r}(Y_n = 2m) = \frac{1}{\sqrt{2\pi}}$$

where σ is the standard deviation of Y_n. Now since variance is additive,

$$\sigma^2 = n\text{Var}(Z) = n(n-1)\mathbf{E}_{R_r}(\gamma_2) \sim (\tfrac{1}{4} - \tfrac{1}{2}L_0^2 - \tfrac{1}{4}L_0T)n^2$$

by (40). By (15) and since $T = O(n^\epsilon)$,

$$(\tfrac{1}{4} - \tfrac{1}{2}L_0^2 - \tfrac{1}{4}L_0T)^{-1} = O(n^{2\epsilon})$$

and so the estimate in the lemma is valid for $\mathbf{P}_{R_r}(Y_n = 2m)$. Also recall from Lemma 6 that a sequence in R_r is concentrated with probability $1 - O(e^{-n^{3\epsilon}})$. Therefore, since $\mathbf{P}_{R_r}(Y_n = 2m)$ is only polynomially small, non-concentrated sequences are also rare in R_m^{con}. Hence, $\mathbf{P}_{R_r}(V_m) \sim \mathbf{P}_{R_r}(V_m^{\text{con}})$ and we obtain the required estimate. ⊙

Equation (20) actually needs the expectation of $g(\gamma_2)$, but this is now easily inferred.

Lemma 10. *Under the conditions stated at the beginning of this section,*

$$\mathbf{E}_{R_m^{\text{con}}}(g(\gamma_2)) \sim g(\tfrac{1}{4} - \tfrac{1}{2}L_0^2 - \tfrac{1}{4}L_0T).$$

Proof: Our statements here will be true for sufficiently large n. From the definition of $g(x)$, we see that $|g'(x)| < 4$ for all x, since $p \sim \frac{1}{2}$. Also, Lemma 9 and (15) imply that $\mathbf{P}_{R_r}(V_m^{\text{con}}) > \frac{1}{2n}$. Therefore, from Lemma 8, we know that

$$\mathbf{P}_{R_m^{\text{con}}}\big(\big|g(\gamma_2) - g(\mathbf{E}_{R_r}(\gamma_2))\big| > 4n^{-1/3}\big) = O(ne^{-n^\epsilon}).$$

Furthermore, since the variance of a truncated binomial distribution is less than that of the original distribution [7], (39) implies that $g\big(E_{R_r}(\gamma_2)\big) > 1$. Therefore, since $g(x) < 2$ always,

$$\begin{aligned}\mathbf{E}_{R_m^{\text{con}}}(g(\gamma_2)) &= \big(g(\mathbf{E}_{R_r}(\gamma_2)) + O(n^{-1/3})\big)\big(1 - O(ne^{-n^\epsilon})\big) + O(ne^{-n^\epsilon}) \\ &\sim g(\mathbf{E}_{R_r}(\gamma_2)). \end{aligned} \tag{41}$$

The result now follows from (40) and the fact that $g'(x)$ is bounded. ⊙

8. Proofs of the Theorems

With all the groundwork done, it is just a matter of putting the pieces together.

Proof of Theorem 1: From Lemma 5, we know that $2^{-N}G_{\le k}(n) = \mathbf{P}_{D_{1/2}}(V) \sim \sum_{m\in I_0}\mathbf{P}_{D_{1/2}}(V_m^{\text{con}})$. Moreover, putting $r = \alpha^{-1}(m)$ in (20) for $m \in I_0$ gives

$$\mathbf{P}_{D_{1/2}}(V_m^{\text{con}}) \sim \mathbf{E}_{R_m^{\text{con}}}(g(\gamma_2))\hat{Q}(m)\mathbf{P}_{R_r}(V_m^{\text{con}}). \tag{42}$$

In the case that $m \in I_1$,

$$\mathbf{E}_{R_m^{\rm con}}(g(\gamma_2))\mathbf{P}_{R_r}(V_m^{\rm con}) \sim \frac{g(\frac14 - \frac12 L_0^2 - \frac14 L_0 T)}{\sqrt{\pi}(\frac12 - L_0^2 - \frac12 L_0 T)^{1/2}\, n},$$

by Lemmas 9 and 10. Since this value is independent of m, we can apply (30) and (36) to find that $2^N \sum_{m\in I_1} \mathbf{P}_{D_{1/2}}(V_m^{\rm con})$ asymptotically equals the value given in the theorem.

For $m \in I_0 \setminus I_1$, we can use the fact that $g(\gamma_2)$ and $\mathbf{P}_{R_r}(V_m^{\rm con})$ are uniformly bounded to infer from (37) and (42) that

$$\sum_{m\in I_0\setminus I_1} \mathbf{P}_{D_{1/2}}(V_m^{\rm con}) = O(n^2 e^{-n^\epsilon})\hat{Q}(m_0),$$

which is easily seen to be negligible in comparison to the sum over I_1. ⊙

Proof of Theorem 2: This follows from (33–35) and the assumptions listed at the end of Section 6 (which are justified in Section 7). ⊙

Proof of Theorem 3: If $m < \frac12 \min\{k, \frac12 n\}n - n^{3/2+\epsilon}$, all but a fraction $o(n^{-t})$ of graphs with n vertices and m edges have maximum degree less than k, so the theorem follows from [11, Thm 2.6(b)]. Hence, we can assume that $m \ge \frac12 N - O(n^{3/2+\epsilon})$.

The same argument that led to (20) also gives

$$\sum_{\mathbf{d}\in V_m^{\rm con}} F(\mathbf{d})\mathbf{P}_{D_{1/2}}(\mathbf{d}) \sim \mathbf{E}_{R_m^{\rm con}}\bigl(F(\mathbf{d})g(\gamma_2)\bigr)\mathbf{P}_{R_r}(V_m^{\rm con})Q(r,m).$$

Dividing by (20), we obtain

$$\mathbf{E}_{D_{1/2}}(F \mid V_m^{\rm con}) \sim \frac{\mathbf{E}_{R_m^{\rm con}}\bigl(F(\mathbf{d})g(\gamma_2)\bigr)}{\mathbf{E}_{R_m^{\rm con}}\bigl(g(\gamma_2)\bigr)}. \tag{43}$$

The two expressions on the right of (43) can be analysed similarly to the proof of Lemma 10. Since we are not restricted to $m \in I_1$, we will choose a different value of r. Define M_r to be the least of those M which maximise the probability that the sum of the components of a vector in R_r is M. Since the distribution of the sum is log-concave (see the proof of Lemma 10) and the probability of a given sum is a continuous function of r, the value $r = \inf\{r \mid M_r \ge 2m\}$ ensures that $2m$ is a most likely sum. It is easy to see that $\frac12 - O(n^{-1/2+\epsilon}) \le r < 1$. The probability of sum $2m$ will be at least n^{-2}, since there are at most n^2 possible sums. Using this bound in place of Lemma 9, the same argument as used in Lemma 10 yields the following in place of (41):

$$\begin{aligned}
&\mathbf{E}_{R_m^{\rm con}}(F(\mathbf{d})g(\gamma_2)) \\
&\quad = \mathbf{E}_{R_r}(F)\bigl(g(\mathbf{E}_{R_r}(\gamma_2)) + O(n^{-1/3})\bigr)\bigl(1 - O(n^2 e^{-n^\epsilon})\bigr) + O(n^2 e^{-n^\epsilon}) \max F(\mathbf{d}) \\
&\quad \sim \mathbf{E}_{R_r}(F) g(\mathbf{E}_{R_r}(\gamma_2)).
\end{aligned}$$

We have used the facts that $F(\mathbf{d})$ is at most $O(n^t)$ times $\mathbf{E}_{R_r}(F)$ (given) and that $g(\mathbf{E}_{R_r}(\gamma_2)) > 1$ (as in Lemma 10). The case $F = 1$ gives $\mathbf{E}_{R_m^{\text{con}}}(g(\gamma_2)) \sim g(\mathbf{E}_{R_r}(\gamma_2))$. Substituting these estimates into (43), we obtain

$$\mathbf{E}_{D_{1/2}}(F \mid V_m^{\text{con}}) \sim \mathbf{E}_{R_r}(F). \tag{44}$$

From Lemma 5, since the error term is much smaller than $O(n^{-t})$, we have that $\mathbf{E}_{D_{1/2}}(F \mid V_m^{\text{con}}) \sim \mathbf{E}_{D_{1/2}}(F \mid V_m)$. Substituting into (44) gives the theorem. ⊙

Proofs of the Corollaries: For Corollary 1 in the case $T > n^\epsilon$, it suffices to apply a crude tail estimate to each vertex. The case $T = O(n^\epsilon)$ follows from Theorem 1 if we note that $L_0 \sim \phi(T)$ as $T \to \infty$ and that, by Taylor's Theorem, $\Phi(L_0+T) = \Phi(T) + O\big(L_0\phi(T)\big)$.

For Corollary 2, expand the function $L_0(T)$ about the point $T = 0$ using the differential equation (14). ⊙

9. Numerical checks

The coefficient of $z_1^{d_1} z_2^{d_2} \cdots z_n^{d_n}$ in the generating function $\prod_{1\le j<k\le n}(1 + z_j z_k)$ is the number of graphs on n vertices with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$. Consequently $G_{\le k}(n)$ is the constant term in

$$\prod_{h=1}^{n} \big(1 + z_h^{-1} + \cdots + z_h^{-k}\big) \prod_{1\le i<j\le n} (1 + z_i z_j).$$

A process of summing a generating function over roots of unity to extract particular coefficients is given by McKay [8]. By these means, we have computed exact values of $G_{\le k}(n)$ for $0 \le k \le n - 1 \le 17$. The results were compared with the predictions of Theorem 1, and found to have very good agreement except for the smallest values of k. Data for $n = 17, 18$ is given in Table 1.

Acknowledgement.

A number of errors and obscurities in the original version of this paper were uncovered by the referee, who read the proofs with exceptional thoroughness. For this we are very grateful.

n	k	Actual probability	Estimated probability	Relative error
17	0	0.1148×10^{-40}	0.2427×10^{-40}	1.1
17	1	0.2431×10^{-32}	0.6528×10^{-32}	1.7
17	2	0.6402×10^{-25}	0.8267×10^{-25}	0.29
17	3	0.7106×10^{-19}	0.7100×10^{-19}	−0.00074
17	4	0.6006×10^{-14}	0.5687×10^{-14}	−0.053
17	5	0.5845×10^{-10}	0.5593×10^{-10}	−0.043
17	6	0.8765×10^{-7}	0.8561×10^{-7}	−0.023
17	7	0.2529×10^{-4}	0.2507×10^{-4}	−0.0090
17	8	0.001695	0.001691	−0.0025
17	9	0.03159	0.03155	−0.0014
17	10	0.1996	0.1991	−0.0026
17	11	0.5428	0.5412	−0.0029
17	12	0.8390	0.8382	−0.0010
17	13	0.9654	0.9664	0.0011
17	14	0.9956	0.9971	0.0015
17	15	0.9997	1.001	0.00087
17	16	1	1.000	0.00030
18	0	0.8758×10^{-46}	0.3406×10^{-45}	2.9
18	1	0.8735×10^{-37}	0.3420×10^{-36}	2.9
18	2	0.1010×10^{-28}	0.1545×10^{-28}	0.53
18	3	0.4195×10^{-22}	0.4463×10^{-22}	0.064
18	4	0.1168×10^{-16}	0.1121×10^{-16}	−0.040
18	5	0.3367×10^{-12}	0.3211×10^{-12}	−0.046
18	6	0.1365×10^{-8}	0.1324×10^{-8}	−0.030
18	7	0.9785×10^{-6}	0.9650×10^{-6}	−0.014
18	8	0.0001499	0.0001492	−0.0046
18	9	0.005837	0.005829	−0.0014
18	10	0.06915	0.06903	−0.0018
18	11	0.3071	0.3063	−0.0028
18	12	0.6575	0.6560	−0.0022
18	13	0.8939	0.8937	−0.00014
18	14	0.9792	0.9805	0.0013
18	15	0.9975	0.9988	0.0013
18	16	0.9999	1.001	0.00064
18	17	1	1.000	0.00020

Table 1. *Results for some typical small values. The third column gives the probability p_{act} of a random graph on n vertices having $\Delta \le k$. The fourth column shows an estimate p_{est} of the same probability, computed from Theorem 1. The last column gives the relative error $(p_{\text{est}} - p_{\text{act}})/p_{\text{act}}$.*

References.

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover publications, New York, 1965.

[2] N. Alon, Private communication, 1999.

[3] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley, New York, 1992.

[4] E. A. Bender, Central and local limit theorems applied to asymptotic enumeration, *J. Comb. Th. Ser. A* **15** (1973), 91–111.

[5] B. Bollobás, The distribution of the maximum degree of a random graph, *Discrete Math.* **32** (1980), 201–203.

[6] B. Bollobás, Private communication.

[7] E. Mailhot, Une propriété de la variance de certaines lois de probabilité réelles tronquées, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985) 241–244.

[8] B. D. McKay, Applications of a technique for labelled enumeration, *Congr. Numer.* **40** (1983), 207–221.

[9] B. D. McKay and N. C. Wormald, Asymptotic enumeration by degree sequence of graphs of high degree, *European J. Combin.* **11** (1990), 565–580.

[10] B. D. McKay and N. C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$, *Combinatorica* **11** (1991), 369–382.

[11] B. D. McKay and N. C. Wormald, The degree sequence of a random graph I: The models, *Random Struct. Alg.* **11** (1997), 97–117.

[12] O. Riordan and A. Selby, The maximum degree of a random graph, *Combinatorics, Probability and Computing* **9** (2000), 549–572.

[13] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics and geometry, *Ann. New York Acad. Sci.* **576** (1989), 500–535.

[14] R. Wong, *Asymptotic approximations of integrals*, Academic Press, Boston, 1989.